ON MONOTONE SOLUTIONS OF SOME CLASSES OF DIFFERENCE EQUATIONS

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We describe a method for finding monotone solutions of some classes of difference equations converging to the corresponding equilibria. The method enables us to confirm three conjectures posed by the present author in a talk, which are extensions of three conjectures by M. R. S. Kulenović and G. Ladas, Dynamics of Second Order Rational Difference Equations. With Open Problems and Conjectures. Chapman and Hall/CRC, 2002. It is interesting that the method, in some cases, can be applied also when the parameters are variable.

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1. Introduction

Recently there has been a great interest in studying nonlinear difference equations of order greater than one. Many of these equations stem from mathematical biology, economy, population dynamics, and so forth (see, e.g., [5, 7–9, 11, 14] and the references therein). An interesting problem in the theory of difference equations is finding monotone solutions. This paper is devoted to this problem.

Motivated by [8, Conjectures 5.4.6 and 6.10.3] in a talk (see, [16]) we posed the following three conjectures. The first one concerns a generalization of (1.2).

Conjecture 1.1. Show that for every $p > -1$, the following equation:

$$x_{n+1} = p + \frac{x_{n-k}}{\sum_{i=0}^{k-1} \alpha_i x_{n-i}}, \quad n = 0, 1, \ldots, \quad (1.1)$$

where $k \in \mathbb{N}$, $\alpha_i \geq 0$, $i = 0, \ldots, k - 1$, and $\sum_{i=0}^{k-1} \alpha_i = 1$, has a positive solution which remains above the equilibrium $\bar{x}_1 = p + 1$ for all $n \geq -k$. 
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In [6] DeVault et al. investigate the behavior of the positive solutions of the difference equation

\[ x_{n+1} = p + \frac{x_{n-k}}{x_n}, \quad n = 0, 1, \ldots, \]  (1.2)

where \( p > 0 \) and \( k \in \mathbb{N} \) is fixed. Among other things they have proved that all nonoscillatory solutions of (1.2) converge to the positive equilibrium \( \bar{x} = p + 1 \).

Based on this observation they have posed the following open problem.

Open problem 1.2. Do there exist nonoscillatory solutions of (1.2)?

The following conjectures are generalizations of [8, Conjectures 5.4.6 and 6.10.3].

Conjecture 1.3. Show that the following equation:

\[ x_{n+1} = 1 + \frac{x_{n-k}}{\sum_{i=0}^{k-1} \alpha_i x_{n-i}}, \quad n = 0, 1, \ldots, \]  (1.3)

where \( k \in \mathbb{N} \), \( \alpha_i \geq 0 \), \( i = 0, \ldots, k-1 \), and \( \sum_{i=0}^{k-1} \alpha_i = 1 \), has a nontrivial positive solution which decreases to the equilibrium \( x_2 = (1 + \sqrt{5})/2 \).

Conjecture 1.4. Show that the following equation:

\[ x_{n+1} = \frac{\alpha + x_{n-k}}{1 + \sum_{i=0}^{k-1} \alpha_i x_{n-i}}, \quad n = 0, 1, \ldots, \]  (1.4)

where \( k \in \mathbb{N} \), \( \alpha > 0 \), \( \alpha_i \geq 0 \), \( i = 0, \ldots, k-1 \), and \( \sum_{i=0}^{k-1} \alpha_i = 1 \), has a positive solution which decreases to the equilibrium \( x_3 = \sqrt{\alpha} \).

Our aim in this paper is to confirm the above mentioned conjectures.

The linearized equation for (1.1), respectively, (1.3) and (1.4), about the corresponding positive equilibrium \( \bar{x}_i \), \( i \in \{1, 2, 3\} \), is

\[ (p + 1) y_{n+1} + \alpha_0 y_n + \ldots + \alpha_{k-1} y_{n-k+1} - y_{n-k} = 0, \]  (1.5)

\[ \bar{x}_2 (y_{n+1} + \alpha_0 y_n + \ldots + \alpha_{k-1} y_{n-k+1}) - y_{n-k} = 0, \]  (1.6)

\[ (1 + \sqrt{\alpha}) y_{n+1} + \sqrt{\alpha} (\alpha_0 y_n + \ldots + \alpha_{k-1} y_{n-k+1}) - y_{n-k} = 0. \]  (1.7)

The characteristic polynomial associated with (1.5), respectively, (1.6) and (1.7), is

\[ p_1(t) = (p + 1) t^{k+1} + \alpha_0 t^k + \ldots + \alpha_{k-1} t - 1 = 0, \]  (1.8)

\[ p_2(t) = \bar{x}_2 (t^{k+1} + \alpha_0 t^k + \ldots + \alpha_{k-1} t) - 1 = 0, \]  (1.9)

\[ p_3(t) = (1 + \sqrt{\alpha}) t^{k+1} + \sqrt{\alpha} (\alpha_0 t^k + \ldots + \alpha_{k-1} t) - 1 = 0. \]  (1.10)

Since \( p_1(0) = -1 < 0 \), \( p_1(1) = p + 1 \), and \( p_1'(t) = (p + 1)(k + 1) t^k + \alpha_0 k t^{k-1} + \ldots + \alpha_{k-1} > 0 \) for \( t \in (0,1) \), it follows that for each \( p > -1 \), there is a unique positive root \( t_i \) of the polynomial (1.8) belonging to the interval \( (0,1) \).
Similarly, it can be shown that (1.9) and (1.10) have also a unique positive roots $t_2$ and $t_3$ in the interval $(0,1)$.

This fact motivated us to believe that there are solutions of (1.1), (1.3), and (1.4) which have the following asymptotics:

$$x_n = \bar{x} + at_i^n + o(t_i^n),$$

(1.11)

where $a \in \mathbb{R}$ and $t_i$, $i \in \{1, 2, 3\}$, are the above mentioned roots of polynomials (1.5), (1.6), and (1.7), respectively.

We solve the open problem, showing that such solutions exist, developing Berg’s idea in [2] which are based on asymptotics. Asymptotics for solutions of difference equations has been investigated for a long time by L. Berg and S. Stević, see, for example, [1–4, 10–15] and the reference therein. We solve it by constructing two appropriate sequences $y_n$ and $z_n$ with

$$y_n \leq x_n \leq z_n$$

(1.12)

for sufficiently large $n$. In [1, 2], some methods can be found for the construction of these bounds, see, also [3, 4].

From (1.11) and results in Berg’s paper [2], we expect that for $k \geq 2$ such solutions have the first four members in their asymptotics in the following form:

$$\varphi_n = \bar{x} + at^n + bt^{2n} + ct^{3n}.$$  

(1.13)

2. The inclusion theorem

We need the following result in the proof of the main theorem. The proof of the result is similar to that of [2, Theorem 1].

**Theorem 2.1.** Let $f : I^{k+2} \rightarrow I$ be a continuous and nondecreasing function in each argument on the interval $I \subset \mathbb{R}$, and let $(y_n)$ and $(z_n)$ be sequences with $y_n < z_n$ for $n \geq n_0$ and such that

$$y_{n-k} \leq f(n, y_{n-k+1}, \ldots, y_{n+1}), \quad f(n, z_{n-k+1}, \ldots, z_{n+1}) \leq z_{n-k}$$

(2.1)

for $n > n_0 + k - 1$.

Then there is a solution of the following difference equation:

$$x_{n-k} = f(n, x_{n-k+1}, \ldots, x_{n+1}),$$

(2.2)

with property (1.12) for $n \geq n_0$.

**Proof.** Let $N$ be an arbitrary integer such that $N > n_0 + k - 1$. The solution $(x_n)$ of (2.2) with given initial values $x_N, x_{N+1}, \ldots, x_{N+k}$ satisfying (1.12) for $n \in \{N, N+1, \ldots, N+k\}$ can be continued by (2.2) to all $n < N$. Inequalities (2.1) and the monotonic character of $f$ imply that (1.12) holds for all $n \in \{n_0, N+1, \ldots, N+k\}$. Let $A_N$ be the set of all $(k+1)$-tuples $(x_{m_0}, \ldots, x_{m_0+k})$ such that there exist solutions $(x_n)$ of (2.2) with these initial values satisfying (1.12) for all $n \in \{n_0, \ldots, N+k\}$. It is clear that $A_N$ is a closed nonempty set
for every $N > n_0 + k - 1$, and that $A_{N+1} \subset A_N$. It follows that the set $A = \cap_{n=n_0+k}^{\infty} A_N$ is a nonempty subset of $\mathbb{R}^{k+1}$ and that if $(x_{n_0}, \ldots, x_{n_0+k}) \in A$, then the corresponding solutions of (2.2) satisfy (1.12) for all $n \geq n_0$, as desired. \hfill \Box

3. The main result

In this section we prove the main result of this paper, which confirms Conjectures 1.1, 1.3, and 1.4.

**Theorem 3.1.** The following statements are true:
(a) let $\alpha_i \geq 0$, $i = 0, \ldots, k - 1$, $\sum_{i=0}^{k-1} \alpha_i = 1$, and $p > -1$. Then (1.1) has a positive solution which remains above the equilibrium $\bar{x}_1 = p + 1$;
(b) let $\alpha_i \geq 0$, $i = 0, \ldots, k - 1$, $\sum_{i=0}^{k-1} \alpha_i = 1$. Then (1.3) has a nontrivial positive solution which decreases to the equilibrium $\bar{x}_2$;
(c) let $\alpha > 0$, $\alpha_i \geq 0$, $i = 0, \ldots, k - 1$, $\sum_{i=0}^{k-1} \alpha_i = 1$. Then (1.4) has a nontrivial positive solution which decreases to the equilibrium $\bar{x}_3 = \sqrt{\alpha}$.

**Proof.** (a) Note that (1.2) can be written in the following equivalent form:

$$F(x_{n-k}, \ldots, x_n, x_{n+1}) = (x_{n+1} - p)(\alpha_0 x_n + \cdots + \alpha_{k-1} x_{n-k+1}) - x_{n-k} = 0. \quad (3.1)$$

We expect that solutions of (1.2) have asymptotic approximation (1.13). Thus, we calculate $F(\varphi_{n-k}, \ldots, \varphi_n, \varphi_{n+1})$. We have

$$F = (1 + at^{n+1} + bt^{2(n+1)} + ct^{3(n+1)})$$

$$\times (p + 1 + a\alpha_0 t^n + \cdots + a\alpha_{k-1} t^{n-k+1} + b\alpha_0 t^{2n} + \cdots + b\alpha_{k-1} t^{2(n-k+1)} + O(t^{3n}))$$

$$- (p + 1 + at^{n-k} + bt^{2(n-k)} + ct^{3(n-k)})$$

$$= at^n \left((p + 1)t + \alpha_0 + \cdots + \frac{\alpha_{k-1}}{t^{k-1}} - t^{-k}\right)$$

$$+ t^{2n} \left(b \left(\alpha_0 + \cdots + \frac{\alpha_{k-1}}{t^{2(k-1)}}\right) + a^2 t \left(\alpha_0 + \cdots + \frac{\alpha_{k-1}}{t^{k-1}}\right) + b(p + 1)t^2 - bt^{-2k}\right) + O(t^{3n}). \quad (3.2)$$

Let

$$D_1(t) = (p + 1)t + \alpha_0 + \cdots + \frac{\alpha_{k-1}}{t^{k-1}} - \frac{1}{t^k}. \quad (3.3)$$

Choose $t \in (0, 1)$ such that $D_1(t) = 0$, and $a, b \in \mathbb{R}$, $a \neq 0$, such that the coefficients in (3.2) are equal to zero. $D_1(t) = 0$ implies that $t = t_1$ (see, Section 1). Further we obtain

$$b = -\frac{a^2 t_1(\alpha_0 + \cdots + \alpha_{k-1} t_1^{-k+1})}{(p + 1)t_1^2 + \alpha_0 + \cdots + (\alpha_{k-1})/t_1^{2(k-1)} - t_1^{-2k}} = -\frac{a^2 t_1(\alpha_0 + \cdots + \alpha_{k-1} t_1^{-k+1})}{D_1(t_1^2)}. \quad (3.4)$$
If $\hat{\phi}_n = p + 1 + at_1^n + qt_1^{2n}$, we obtain

$$F(\hat{\phi}_{n-k}, \ldots, \hat{\phi}_n, \hat{\phi}_{n+1}) \sim (qD_1(t_1^2) + a^2t_1(\alpha_0 + \cdots + \alpha_{k-1}t_1^{-k+1}))t_1^{2n}. \quad (3.5)$$

Let

$$H_{t_1}(q) = qD_1(t_1^2) + a^2t_1(\alpha_0 + \cdots + \alpha_{k-1}t_1^{-k+1}). \quad (3.6)$$

We have

$$D_1'(t) = p + 1 + \frac{k}{t^{k+1}} - \frac{\alpha_1}{t^2} - \cdots - \frac{(k-1)\alpha_{k-1}}{t^k}. \quad (3.7)$$

Hence, when $t \in (0, 1)$, it follows that

$$D_1'(t) > p + 1 + \frac{k}{t^{k+1}} - \frac{\alpha_1 + \cdots + (k-1)\alpha_{k-1}}{t^{k+1}} > p + 1 + \frac{1}{t^{k+1}} > 0. \quad (3.8)$$

From this, since $D_1(t_1) = 0$, and $t_1^2 < t_1$, we have that $D_1(t_1^2) < 0$. Thus, we obtain that there are $q_1 < b$ and $q_2 > b$ such that $H_{t_1}(q_1) > 0$ and $H_{t_1}(q_2) < 0$.

With the notations

$$y_n = p + 1 + at_1^n + qt_1^{2n}, \quad z_n = p + 1 + at_1^n + qt_1^{2n}, \quad (3.9)$$

we get

$$F(y_{n-k}, \ldots, y_n, y_{n+1}) \sim (q_1D_1(t_1^2) + a^2t_1(\alpha_0 + \cdots + \alpha_{k-1}t_1^{-k+1}))t_1^{2n} > 0,$$

$$F(z_{n-k}, \ldots, z_n, z_{n+1}) \sim (q_2D_1(t_1^2) + a^2t_1(\alpha_0 + \cdots + \alpha_{k-1}t_1^{-k+1}))t_1^{2n} < 0. \quad (3.10)$$

These relations show that the inequalities in (1.12) are satisfied for sufficiently large $n$, where $f = F + x_{n-k}$ and $F$ is given by (3.1). Applying Theorem 2.1 it follows that there is a solution of (1.1) with the asymptotics $x_n = \hat{\phi}_n + o(t_1^{2n})$, in particular, the solution of (1.1) converges monotonically to the positive equilibrium $\bar{x}_1 = p + 1$, when $p > -1$ and $n \geq n_0$. Hence, the solution $x_{n+n_0+k}$ converges monotonically for $n \geq -k$.

(b) Equation (1.3) can be written in the following equivalent form:

$$F(x_{n-k}, \ldots, x_n, x_{n+1}) = x_{n+1}(\alpha_0x_n + \cdots + \alpha_{k-1}x_{n-k+1}) - (1 + x_{n-k}) = 0. \quad (3.11)$$
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Now we calculate $F(\varphi_{n-k}, \ldots, \varphi_n, \varphi_{n+1})$. We have

$$F = (\tilde{x}_2 + at^{n+1} + bt^{2(n+1)} + ct^{3(n+1)})$$

$$\times (\tilde{x}_2 + a\alpha t^n + \cdots + a\alpha_{k-1}t^{n-k+1} + b\alpha_0 t^{2n} + \cdots + b\alpha_{k-1}t^{2(n-k+1)} + \mathcal{O}(t^{3n}))$$

$$- (1 + \tilde{x}_2 + at^{n-k} + bt^{2(n-k)} + ct^{3(n-k)})$$

$$= at^n \left( \tilde{x}_2 t + \tilde{x}_2 \left( \alpha_0 + \cdots + \frac{\alpha_{k-1}}{t^{k-1}} \right) - \frac{1}{t^k} \right)$$

$$+ t^{2n} \left( \tilde{x}_2 b \left( \alpha_0 + \cdots + \frac{\alpha_{k-1}}{t^{2(k-1)}} \right) + a^2 t \left( \alpha_0 + \cdots + \frac{\alpha_{k-1}}{t^{k-1}} \right) + \tilde{x}_2 b t^2 - bt^{-2k} \right) + \mathcal{O}(t^{3n}).$$

(3.12)

Let

$$D_2(t) = \tilde{x}_2 t + \tilde{x}_2 \left( \alpha_0 + \cdots + \frac{\alpha_{k-1}}{t^{k-1}} \right) - t^{-k} = \frac{p_2(t)}{t^k}. \quad (3.13)$$

Choose $t \in (0, 1)$ such that $D_2(t) = 0$, and $a, b \in \mathbb{R}, a \neq 0$, such that the coefficients in (3.12) are equal to zero. Since $D_2(t) = 0$ is equivalent to $p_2(t) = 0$, we have that $t = t_2$, and consequently

$$b = -\frac{a^2 t_2 (\alpha_0 + \cdots + \alpha_{k-1}t^{-2k+1})}{\tilde{x}_2 t_2^2 + \tilde{x}_2 (\alpha_0 + \cdots + (\alpha_{k-1})/t_2^{2(k-1)}) - t_2^{-2k}} = -\frac{a^2 t_2 (\alpha_0 + \cdots + \alpha_{k-1}t^{-2k+1})}{D_2(t_2)}. \quad (3.14)$$

If $\hat{\varphi}_n = \tilde{x}_2 + at^n + qt_2^{2n}$, we obtain

$$F(\hat{\varphi}_{n-k}, \ldots, \hat{\varphi}_n, \hat{\varphi}_{n+1}) = (qD_2(t_2^2) + a^2 t_2 (\alpha_0 + \cdots + \alpha_{k-1}t_2^{-2k+1}))t_2^{2n}. \quad (3.15)$$

Let

$$H_{t_2}(q) = qD_2(t_2^2) + a^2 t_2 (\alpha_0 + \cdots + \alpha_{k-1}t_2^{-2k+1}). \quad (3.16)$$

Since

$$p'_2(t) = \tilde{x}_2 ((k+1)t^k + ka_0 t^{k-1} + \cdots + \alpha_{k-1}) > 0, \quad (3.17)$$

when $t \in (0, 1)$, and since $p_2(t_2) = 0$, and $t_2 < t_2$, we have that $p_2(t_2) < 0$, which implies $D_2(t_2) < 0$. Thus, we obtain that there are $q_3 < b$ and $q_4 > b$ such that $H_{t_2}(q_3) > 0$ and $H_{t_2}(q_4) < 0$.

With the notations

$$y_n = \tilde{x}_2 + at_2^n + q_3 t_2^{2n}, \quad z_n = \tilde{x}_2 + at_2^n + q_4 t_2^{2n}, \quad (3.18)$$

we get

$$F(y_{n-k}, \ldots, y_n, y_{n+1}) \sim H_{t_2}(q_3) t_2^{2n} > 0,$$

$$F(z_{n-k}, \ldots, z_n, z_{n+1}) \sim H_{t_2}(q_4) t_2^{2n} < 0. \quad (3.19)$$
These relations show that the inequalities in (1.12) are satisfied for sufficiently large \( n \), where \( f = F + x_{n-k} \) and \( F \) is given by (3.11). Applying Theorem 2.1 it follows that there is a solution of (1.3) with the asymptotics \( x_n = \hat{\phi}_n + o(t_{2n}^n) \). This solution obviously converges monotonically to the positive equilibrium \( \hat{x}_2 = (\sqrt{5} + 1)/2 \), for \( n \geq n_1 \). A suitable shift of \( x_n \) is decreasing for all \( n \geq -k \).

(c) Equation (1.4) can be written in the following equivalent form:

\[
F(x_{n-k}, \ldots, x_n, x_{n+1}) = x_{n+1}(1 + \alpha_0 x_n + \cdots + \alpha_{k-1} x_{n-k+1}) - (\alpha + x_{n-k}) = 0. \quad (3.20)
\]

We have

\[
F = (\hat{x}_3 + at^{n+1} + bt^{2n+1} + ct^{3n+1})
\times (1 + \hat{x}_3 + \alpha_0 t^n + \cdots + \alpha_{k-1} t^{n-k+1} + b\alpha_0 t^{2n} + \cdots + b\alpha_{k-1} t^{2(n-k+1)} + \mathcal{O}(t^{3n}))
\]

\[
- (\alpha + \hat{x}_3 + at^{n-k} + bt^{2(n-k)} + ct^{3(n-k)})
= at^n \left( (1 + \hat{x}_3) t + \hat{x}_3 \left( \alpha_0 + \cdots + \frac{\alpha_{k-1}}{t^{k-1}} \right) - t^{-k} \right)
\]

\[
+ t^n \hat{x}_3 b \left( \alpha_0 + \cdots + \frac{\alpha_{k-1}}{t^{2(k-1)}} \right) + a^2 t \left( \alpha_0 + \cdots + \frac{\alpha_{k-1}}{t^{k-1}} \right) + (1 + \hat{x}_3) bt^2 - bt^{-2k}
\]

\[
+ \mathcal{O}(t^{3n}). \quad (3.21)
\]

Let

\[
D_3(t) = (1 + \hat{x}_3) t + \hat{x}_3 \left( \alpha_0 + \cdots + \frac{\alpha_{k-1}}{t^{k-1}} \right) - t^{-k} = \frac{p_3(t)}{t^k}. \quad (3.22)
\]

Choose \( t \in (0,1) \) such that \( D_3(t) = 0 \), and \( a, b \in \mathbb{R}, \ a \neq 0 \), such that the coefficients in (3.21) are equal to zero.

Since

\[
p_3'(t) = (1 + \sqrt{\alpha})(k + 1)t^k + \sqrt{\alpha}(ka_0 t^{k-1} + \cdots + \alpha_{k-1}) > 0, \quad (3.23)
\]

when \( t \in (0,1] \), and \( D_3(t) = 0 \) is equivalent to \( p_3(t) = 0 \), we have that \( t = t_3 \). From this and (3.21) it follows that

\[
b = -\frac{a^2 t_3 (\alpha_0 + \cdots + \alpha_{k-1} t_3^{-k+1})}{(1 + \sqrt{\alpha} t_3^2 + \sqrt{\alpha}(\alpha_0 + \cdots + (\alpha_{k-1})/t_3^{2(k-1)}) - t_3^{-2k})} = -\frac{a^2 t_3 (\alpha_0 + \cdots + \alpha_{k-1} t_3^{-k+1})}{D_3(t_3^2)}. \quad (3.24)
\]

If \( \hat{\phi}_n = \sqrt{\alpha} + at^n + qt^{2n} \), we obtain

\[
F(\hat{\phi}_{n-k}, \ldots, \hat{\phi}_n, \hat{\phi}_{n+1}) - (qD_3(t_3^2) + a^2 t_3 (\alpha_0 + \cdots + \alpha_{k-1} t_3^{k+1})) t_3^n. \quad (3.25)
\]

Let

\[
H_3(q) = qD_3(t_3^2) + a^2 t_3 (\alpha_0 + \cdots + \alpha_{k-1} t_3^{k+1}). \quad (3.26)
\]
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Since $D_3(t_3) = 0$, and $t_3^2 < t_3$, we have that $D_3(t_3^2) < 0$. Thus, we obtain that there are $q_5 < b$ and $q_6 > b$ such that $H_{t_3}(q_5) > 0$ and $H_{t_3}(q_6) < 0$.

With the notations

$$y_n = \sqrt{\alpha + a t_3^n + q_5 t_3^{2n}}, \quad z_n = \sqrt{\alpha + a t_3^n + q_6 t_3^{2n}},$$ (3.27)

we get

$$F(y_{n-k}, \ldots, y_n, y_{n+1}) \sim H_{t_3}(q_5) t_3^{2n} > 0, \quad F(z_{n-k}, \ldots, z_n, z_{n+1}) \sim H_{t_3}(q_6) t_3^{2n} < 0. \quad (3.28)$$

These relations show that the inequalities in (1.12) are satisfied for sufficiently large $n$, where $f = F + x_{n-k}$ and $F$ is given by (3.20). Hence, there is a solution of (1.4) with the asymptotics $x_n = \hat{\varphi}_n + o(t_3^{2n})$. The result follows similarly to the above mentioned cases.

From Theorem 3.1(a) with $a_0 = 1$ and $\alpha_i = 0$, $i \neq 0$, we get the following corollary.

**Corollary 3.2.** There is a nonoscillatory solution of (1.2).

**Remark 3.3.** Since $a \in \mathbb{R} \setminus \{0\}$ is an arbitrary parameter, by Theorem 3.1 we find a set of nonoscillatory solutions of (1.1), (1.3), and (1.4) converging to the corresponding positive equilibria.

**Remark 3.4.** Note that using (1.13) better asymptotics for these solutions can be obtained, that is, $x_n = \varphi_n + o(t_3^{2n})$, $i \in \{1, 2, 3\}$, where $b$ is given by (3.4), (3.14), or (3.24), and $c$ can be found equating to zero the coefficient nearby $t_3^{3n}$.

**Remark 3.5.** From the proof of Theorem 3.1, we see that we can assume that the parameter $p$ in (1.1) can be replaced by a nondecreasing sequence with the following asymptotics: $p_n = p + o(t_3^{2n})$.

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