We establish existence, multiplicity, and nonexistence of periodic solutions for a class of first-order neutral difference equations modelling physiological processes and conditions. Our approach is based on a fixed point theorem in cones as well as some analysis techniques.

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1. Introduction

The existence of periodic solutions for difference equations has been extensively considered by many authors [1, 4, 8, 9, 12, 16]. Recently, existence of multiple solutions of functional differential equations has been studied and some results have been obtained [6, 14, 18]. Wang [14] investigated existence, multiplicity, and nonexistence of positive periodic solutions for the equation

$$\frac{dx(t)}{dt} = a(t)g(x(t))x(t) - \lambda b(t)f(x(t-\tau(t))),$$  \hspace{1cm} (1.1)

where $\lambda$ is a positive parameter. Chow [2], Smith and Kuang [13], and many others studied the type of equations or their generalized forms. This type of equations has been proposed as models for a variety of physiological processes and conditions including production of blood cells, respiration, and cardiac arrhythmias [11, 15].

To our best knowledge, few papers are on multiplicity of periodic solutions of neutral functional difference systems. In this paper, we consider the following first-order neutral difference equation:

$$\Delta(x(n) - cx(n-\delta)) = a(n)g(x(n))x(n) - \lambda b(n)f(x(n-\tau(n))), \hspace{1cm} n \in \mathbb{Z},$$  \hspace{1cm} (1.2)
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where \( \mathbb{Z} \) is the set of integers, \( \Delta x(n) = x(n + 1) - x(n) \), \( \lambda \) is a positive parameter, \( c \) is a constant, and \(|c| \neq 1\), \( \delta \) is a positive integer, \( a(n), b(n) \), and \( \tau(n) \) are positive \( T \)-periodic sequences, \( T \in \mathbb{N} \).

Let \( N^* = \{0, 1, 2, \ldots, T - 1\} \) and

\[
\begin{align*}
&f_0 = \lim_{u \to 0^+} \frac{f(u)}{u}, \quad f_\infty = \lim_{u \to \infty} \frac{f(u)}{u}, \\
i_0 = \text{number of zeros in the set } \{f_0, f_\infty\}, \\
i_\infty = \text{number of infinities in the set } \{f_0, f_\infty\}.
\end{align*}
\]

(1.3)

It is clear that \( i_0, i_\infty = 0, 1, \) or \( 2 \). Then we should show that (1.2) has \( i_0 \) or \( i_\infty \) periodic solution(s) for some certain \( \lambda \), respectively. In what follows, we set

\[
X = \{x \mid x(n), x(n + T) \equiv x(n), n \in \mathbb{Z}\}
\]

(1.4)

with the norm defined by \( \|x\|_X = \max\{|x(n)| : n \in N^*\} \). Then \( X \) is a Banach space. Let \( A : X \to X \) be defined by \( (Ax)(n) = x(n) - cx(n - \delta) \).

**Lemma 1.1.** If \(|c| \neq 1\), then \( A \) has continuous bounded inverse \( A^{-1} \) on \( X \) and for all \( x \in X \),

\[
(A^{-1}x)(n) = \begin{cases} 
\sum_{j=0}^{\infty} c^j x(n - j\delta), & \text{if } |c| < 1, \\
-\sum_{j=1}^{\infty} c^{-j} x(n + j\delta), & \text{if } |c| > 1, n \in \mathbb{Z},
\end{cases}
\]

(1.5)

\[
\|A^{-1}x\|_X \leq \frac{\|x\|_X}{1 - |c|}.
\]

**Proof.** According to [10, 17], we can get equality (1.5) and then verify the results of Lemma 1.1.

We consider the following assumptions.

\((E_1) a(n), b(n) \) are positive \( T \)-periodic sequences, \( \tau(n) \) is a positive \( T \)-periodic integer sequence.

\((E_2) f, g \in C([0, \infty), [0, \infty)) \) and there exist two positive constants \( l, L \) such that \( 0 < l \leq g(u) \leq L < +\infty \) for \( u \in \mathbb{R} \); \( f(u) > 0 \) for \( u > 0 \).

Define

\[
A_1 = \frac{1}{\prod_{r=n}^{n+T-1} [a(r)L + 1] - 1}, \quad B = \frac{\prod_{r=n}^{n+T-1} [a(r)L + 1]}{\prod_{r=n}^{n+T-1} [a(r)L + 1] - 1},
\]

(1.6)
and $\alpha = A_1/B$, for any $r > 0$, we denote
\[
M(r) = \max \left\{ f(t) : 0 \leq t \leq \frac{r}{1 - |c|} \right\},
\]
\[
m(r) = \min \left\{ f(t) : \frac{\alpha - |c|}{1 - c^2} r \leq t \leq \frac{r}{1 - |c|} \right\},
\]
\[
k = \min \left\{ \alpha, \frac{1}{1 + BL \Sigma_{s=0}^{r-1} a(s)} \right\}.
\]

We aim to establish existence, multiplicity, and nonexistence of positive $T$-periodic solutions for first-order neutral difference equation (1.2). Our approach is based on a fixed point theorem in cones as well as some analysis techniques which are used by Wang [14]. The rest of this paper is organized as follows. Section 2 is about statement of the method (a fixed point theorem in cones) and some lemmas which play important roles in proofs of main results; in Section 3, we establish our main results and give an example to illustrate the applicability of our results.

2. Preliminaries

We first state the following well-known result. For the proof, we refer to the classical works [3, 5, 7].

**Lemma 2.1** (Deimling [3], Guo and Lakshmikantham [5], and Krasnosel’skiı [7]). Let $E$ be a Banach space and $K$ a cone in $E$. For $r > 0$, define $K_r = \{ u \in K : ||u|| < r \}$. Assume that $T : K_r \to K$ is completely continuous such that $Tx \neq x$ for $x \in \partial K_r = \{ u \in K : ||u|| = r \}$.

(i) If $||Tx|| \geq ||x||$ for any $x \in \partial K_r$, then $i(T, K_r, K) = 0$.

(ii) If $||Tx|| \leq ||x||$ for any $x \in \partial K_r$, then $i(T, K_r, K) = 1$.

Next, we transfer existence of positive $T$-periodic solutions of (1.2) into existence of positive fixed points of some fixed point mapping.

In order to establish existence, multiplicity, and nonexistence of positive $T$-periodic solutions for (1.2), we first consider the following equation:
\[
\Delta y(n) = a(n)g((A^{-1}y)(n)) (A^{-1}y)(n) - \lambda b(n)f((A^{-1}y)(n - \tau(n))),
\]
where $A^{-1}$ is defined by (1.5). By Lemma 1.1 and the definition of $A$ and $A^{-1}$, we conclude the following.

**Lemma 2.2.** $y(n)$ is a $T$-periodic solution of (2.1) if and only if $(A^{-1}y)(n)$ is a $T$-periodic solution of (1.2).

Aiming to apply Lemma 2.1 to (2.1), we rewrite (2.1) as
\[
\Delta y(n) = a(n)g((A^{-1}y)(n)) y(n) - [a(n)G(y(n)) + \lambda b(n)f((A^{-1}y)(n - \tau(n)))],
\]
\[
(2.2)
\]
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where

\[ G(y(n)) = -cg((A^{-1}y)(n))(A^{-1}y)(n - \tau). \]  

(2.3)

A cone \( K \) in \( X \) is defined by

\[ K = \{ u \in X : u(n) \geq \alpha \| u \|_X, \ n \in \mathbb{Z} \}. \]

(2.4)

For \( r > 0 \), define \( \Omega_r \) by \( \Omega_r = \{ u \in K : \| u \|_X < r \} \) and \( \partial \Omega_r = \{ u \in K : \| u \|_X = r \} \). Let the operator \( Q : K \to X \) be defined by

\[ Qu(n) = \sum_{s=n}^{n+T-1} K_u(n, s)[a(s) G(u(s)) + \lambda b(s) f((A^{-1}u)(s - \tau(s)))], \quad n \in \mathbb{Z}, \]

(2.5)

where

\[ K_u(n, s) = \frac{\prod_{r=s}^{n+T-1} [a(r) g((A^{-1}u)(r)) + 1]}{\prod_{r=n}^{n+T-1} [a(r) g((A^{-1}u)(r)) + 1] - 1}, \quad n, s \in \mathbb{Z}, \ n \leq s \leq n + T - 1. \]

(2.6)

Assumption (E_2) implies that

\[ 0 < A_1 \leq K_u(n, s) \leq B, \quad n, s \in \mathbb{Z}, \ n \leq s \leq n + T - 1. \]

(2.7)

**Lemma 2.3.** The positive \( T \)-periodic solution of (2.1) is equivalent to the fixed point of \( Q \) in \( K \).

**Lemma 2.4.** If assumptions (E_1) and (E_2) hold, \( c \in (-\alpha, 0] \), and \( y \in K \), then

(a) \((\alpha - |c|)/(1 - c^2)) \| y \|_X \leq (A^{-1}y)(n) \leq (1/(1 - |c|)) \| y \|_X,

(b) \| |c|((\alpha - |c|)/(1 - c^2)) \| y \|_X \leq G(y(n)) \leq (L|c|/(1 - |c|)) \| y \|_X, \ n \in \mathbb{N}^*.

**Proof**

**Part (a).** Since \(-\alpha < c \leq 0\), it follows from Lemma 1.1 that

\[
(A^{-1}y)(n) = \sum_{j=0}^{\infty} c^j y(n - j\delta) = \sum_{j=0}^{\infty} c^{2j} y(n - 2j\delta) - \sum_{j=1}^{\infty} |c|^{2j-1} y(n - (2j - 1)\delta) \geq \frac{\alpha - |c|}{1 - c^2} \| y \|_X, \quad n \in \mathbb{N}^*,
\]

(2.8)

\[
(A^{-1}y)(n) \leq \frac{1}{1 - |c|} \| y \|_X.
\]

**Part (b).** From part (a) and the assumption (E_2), for any \( n \in \mathbb{Z} \), we get

\[
|c| \frac{\alpha - |c|}{1 - c^2} \| y \|_X \leq G(y(n)) \leq \frac{L|c|}{1 - |c|} \| y \|_X.
\]

(2.9)

\[ \square \]

**Lemma 2.5.** If assumptions (E_1) and (E_2) hold and \( c \in (-\alpha, 0] \), then \( Q(K) \subset K \) and \( Q : K \to K \) is completely continuous.
Proof. By Lemma 1.1, similar to the proof of Lemma 2.2 in [7], we can prove Lemma 2.5.

Lemma 2.6. If assumptions (E_1) and (E_2) hold and \( c \in (-\alpha, 0) \), then \( y(n) \) is the fixed point of \( Q \) in \( K \) if and only if \( (A^{-1}y)(n) \) is a positive T-periodic solution of (1.2).

Proof. If \( y(n) \) is the fixed point of \( Q \) in \( K \), \( y(n) \) is a positive T-periodic solution of (2.1) and \( y \in K \) by Lemma 2.3. It follows from Lemmas 2.2 and 2.4 that \((A^{-1}y)(n)\) is a T-periodic solution of (1.2) and \((A^{-1}y)(n) \geq ((\alpha - |c|)/(1 - \epsilon^2))\|y\|_X > 0\). Therefore, \((A^{-1}y)(n)\) is a positive T-periodic solution of (1.2).

If there exists \( y(n) \) such that \((A^{-1}y)(n)\) is a positive T-periodic solution of (1.2), then \( y(n) \) is a T-periodic solution of (2.1) by Lemma 2.2. From the definition of \( A^{-1} \) and \( c \in (-\alpha, 0) \), \( y(n) = (A^{-1}y)(n) - c(A^{-1}y)(n - \delta) > 0 \). Lemmas 2.3 and 2.5 imply that \( y(n) \) is the fixed point of \( Q \) in \( K \).

Lemma 2.7. Assumptions (E_1) and (E_2) hold and \( c \in (-\alpha, 0) \), \( \eta > 0 \). If \( f((A^{-1}y)(n - \tau(n))) \geq (A^{-1}y)(n - \tau(n))\eta \) for any \( y \in K \) and \( n \in \mathbb{Z} \), then

\[
\|Qy\|_X \geq \lambda A_1 \eta \sum_{s=0}^{T-1} b(s) \frac{\alpha - |c|}{1 - \epsilon^2} \|y\|_X. 
\] (2.10)

Proof. By Lemma 2.4, for any \( y \in K \) and \( n \in \mathbb{Z} \), \( G(y(n)) \geq 0 \) as \( c \in (-\alpha, 0) \). Therefore,

\[
Qy(n) \geq \lambda A_1 \sum_{s=0}^{n+T-1} b(s) f((A^{-1}y)(s - \tau(s))) = \lambda A_1 \sum_{s=0}^{T-1} b(s) f((A^{-1}y)(s - \tau(s)))
\]

\[
\geq \lambda A_1 \eta \sum_{s=0}^{T-1} b(s)(A^{-1}y)(s - \tau(s)) \geq \lambda A_1 \eta \sum_{s=0}^{T-1} b(s) \frac{\alpha - |c|}{1 - \epsilon^2} \|y\|_X.
\] (2.11)

That is,

\[
\|Qy\|_X \geq \lambda A_1 \eta \sum_{s=0}^{T-1} b(s) \frac{\alpha - |c|}{1 - \epsilon^2} \|y\|_X.
\] (2.12)

Lemma 2.8. Assumptions (E_1) and (E_2) hold and \( c \in (-\alpha, 0) \). For any \( n \in \mathbb{Z} \), if there exists \( \epsilon > 0 \) such that \( f((A^{-1}y)(n - \tau(n))) \leq (A^{-1}y)(n - \tau(n))\epsilon \), then

\[
\|Qy\|_X \leq \frac{B \sum_{s=0}^{T-1} [L|c|a(s) + \lambda \epsilon b(s)]}{1 - |c|} \|y\|_X.
\] (2.13)

Proof. From Lemmas 1.1 and 2.4, we have

\[
\|Qy\|_X \leq B \sum_{s=0}^{T-1} \left[ a(s)G(y(s)) + \lambda b(s) f((A^{-1}y)(s - \tau(s))) \right]
\]

\[
\leq \frac{B \sum_{s=0}^{T-1} [a(s)L|c| + \lambda \epsilon b(s)]}{1 - |c|} \|y\|_X + \lambda b(s)\epsilon(A^{-1}y)(s - \tau(s))
\] (2.14)

\[
\leq \frac{B \sum_{s=0}^{T-1} [L|c|a(s) + \lambda \epsilon b(s)]}{1 - |c|} \|y\|_X.
\]
Lemma 2.9. Assumptions \((E_1)\) and \((E_2)\) hold and \(c \in (-\alpha, 0]\). For \(y \in \partial \Omega_r, \ r > 0\), one can obtain

\[ \|Qy\|_X \geq \lambda A_1 m(r) \Sigma_{s=0}^{T-1} b(s). \]  \hspace{1cm} (2.15)

Proof. Since \(y \in \partial \Omega_r\), by Lemma 2.4, \(((\alpha - |c|)/(1 - c^2))r \leq (A^{-1}y)(n - \tau(n)) \leq r/(1 - |c|)\). So \(f((A^{-1}y)(n - \tau(n))) \geq m(r)\) for \(y \in \partial \Omega_r\) and \(n \in \mathbb{Z}\). Similar to the proof of Lemma 2.7, we can obtain Lemma 2.9.

Lemma 2.10. Assumptions \((E_1)\) and \((E_2)\) hold and \(c \in (-\alpha, 0]\). If \(y \in \partial \Omega_r, \ r > 0\), then

\[ \|Qy\|_X \leq B \Sigma_{s=0}^{T-1} \left[ \lambda b(s) M(r) + \frac{L |c| a(s) r}{1 - |c|} \right]. \]  \hspace{1cm} (2.16)

Proof. By \(y \in \partial \Omega_r\) and Lemma 1.1, \(0 \leq (A^{-1}y)(n - \tau(n)) \leq r/(1 - |c|)\). So \(f((A^{-1}y)(n - \tau(n))) \leq M(r)\) for any \(y \in \partial \Omega_r\) and \(n \in \mathbb{Z}\). From The proof of Lemma 2.8, we can similarly prove Lemma 2.10.

3. Main results

We state our main results as follows.

Theorem 3.1. Suppose that assumptions \((E_1)\), \((E_2)\) hold and \(-k < c \leq 0\).

(a) If \(i_0 = 1\) or \(2\), then \((1.2)\) has \(i_0\) positive \(T\)-periodic solution \(s)\) for \(\lambda > 1/A_1 m(1) \Sigma_{s=0}^{T-1} b(s) > 0\).

(b) If \(i_\infty = 1\) or \(2\), then \((1.2)\) has \(i_\infty\) positive \(T\)-periodic solution \(s)\) for \(0 < \lambda < (1 - |c| - BL|c| \Sigma_{s=0}^{T-1} a(s))/BM(1) \Sigma_{s=0}^{T-1} b(s)(1 - |c|)\).

(c) If \(i_\infty = 0\) or \(i_0 = 0\), then \((1.2)\) has no positive \(T\)-periodic solution for sufficiently small or large \(\lambda > 0\), respectively.

Theorem 3.2. Suppose that assumptions \((E_1)\), \((E_2)\) hold and \(-k < c \leq 0\).

(a) If there exists a constant \(c_1 > 0\) such that \(f(u) \geq c_1 u\) for \(u \in [0, +\infty)\), then \((1.2)\) has no positive \(T\)-periodic solution for \(\lambda > (1 - c^2)/A_1 c_1 (\alpha - |c|) \Sigma_{s=0}^{T-1} b(s)\).

(b) If there exists a constant \(c_2 > 0\) such that \(f(u) \leq c_2 u\) for \(u \in [0, +\infty)\), then \((1.2)\) has no positive \(T\)-periodic solution for \(0 < \lambda < (1 - |c| - BL|c| \Sigma_{s=0}^{T-1} a(s))/Bc_2 \Sigma_{s=0}^{T-1} b(s)\).

Theorem 3.3. Suppose that assumptions \((E_1)\), \((E_2)\) hold and \(-k < c \leq 0\). If \(i_0 = i_\infty = 0\) and

\[ \frac{1 - c^2}{\text{max} \left\{ f_\infty, f_0 \right\} A_1 (\alpha - |c|) \Sigma_{s=0}^{T-1} b(s)} < \lambda < \frac{1 - |c| - BL|c| \Sigma_{s=0}^{T-1} a(s)}{\text{min} \left\{ f_0, f_\infty \right\} B \Sigma_{s=0}^{T-1} b(s)}, \]  \hspace{1cm} (3.1)

then \((1.2)\) has one positive \(T\)-periodic solution.

Proof of Theorem 3.1

Part (a). Take \(r_1 = 1\) and \(\lambda_0 = 1/A_1 m(r_1) \Sigma_{s=0}^{T-1} b(s) > 0\). For any \(y \in \partial \Omega_{r_1}\) and \(\lambda > \lambda_0\), it follows from Lemma 2.9 that

\[ \|Qy\|_X > \|y\|_X, \quad y \in \partial \Omega_{r_1}. \]  \hspace{1cm} (3.2)

From Lemma 2.1, \(i(Q, \Omega_{r_1}, K) = 0\).

From Lemma 2.9, we obtain
Case 1. If \( f_0 = 0 \), then for any \( \epsilon > 0 \), we can choose \( 0 < \epsilon \leq \epsilon u \) for \( 0 \leq u \leq \epsilon \). Since \(-k < c \leq 0, 1 > BL|c| \frac{\sum_{n=0}^{T-1} a(s)}{1 - |c|}\). Take \( \epsilon > 0 \) satisfying
\[
\frac{\lambda Be \sum_{s=0}^{T-1} b(s)}{1 - |c|} < 1 - \frac{BL|c| \sum_{s=0}^{T-1} a(s)}{1 - |c|}.
\] (3.3)

Let \( r_2 = (1 - |c|) \epsilon \). If \( y \in \partial \Omega_{r_2} \), then \( 0 \leq (A^{-1} y)(n - \tau(n)) \leq 1/(1 - |c|) \| y \|_X \leq \epsilon \). So \( f((A^{-1} y)(n - \tau(n))) \leq \epsilon (A^{-1} y)(n - \tau(n)) \) for any \( y \in \partial \Omega_{r_2} \) and \( n \in \mathbb{Z} \). By Lemma 2.8 and inequality (3.3), for all \( y \in \partial \Omega_{r_2} \), we have
\[
\| Qy \|_X \geq \frac{\lambda Be \sum_{s=0}^{T-1} b(s) + BL|c| \sum_{s=0}^{T-1} a(s)}{1 - |c|} \| y \|_X < \| y \|_X.
\] (3.4)

Lemma 2.1 implies that \( i(Q, \Omega_{r_2}, K) = 1 \). Thus \( i(Q, \Omega_{r_1} \setminus \overline{\Omega}_{r_2}, K) = -1 \) and \( Q \) has a fixed point \( y(n) \) in \( \Omega_{r_1} \setminus \overline{\Omega}_{r_2} \). It follows from Lemma 2.6 that (1.2) has at least one positive \( T \)-periodic solution \((A^{-1} y)(n)\) for \( \lambda > \lambda_1 \).

Case 2. If \( f_\infty = 0 \), then there exists a constant \( \bar{H} > 0 \) for all \( \epsilon > 0 \) such that \( f(u) \leq \epsilon u \) for all \( u \geq \bar{H} \). \(-k < c \leq 0 \) shows that \( 1 > BL|c| \frac{\sum_{n=0}^{T-1} a(s)}{1 - |c|} \). So we can choose \( \epsilon > 0 \) satisfying inequality (3.3).

Take \( r_3 = \max \{ 2r_1, ((1 - c^2)/(\alpha - |c|)) \bar{H} \} \). For any \( y \in \partial \Omega_{r_3} \), since \( (A^{-1} y)(n - \tau(n)) \geq ((\alpha - |c|)/(1 - c^2)) \| y \|_X \geq \bar{H}, f((A^{-1} y)(n - \tau(n))) \leq \epsilon (A^{-1} y)(n - \tau(n)) \). From Lemma 2.8 and inequality (3.3), for each \( y \in \partial \Omega_{r_3} \), we get
\[
\| Qy \|_X \geq \frac{\lambda Be \sum_{s=0}^{T-1} b(s) + BL|c| \sum_{s=0}^{T-1} a(s)}{1 - |c|} \| y \|_X < \| y \|_X.
\] (3.5)

It follows from Lemma 2.1 that \( i(Q, \Omega_{r_3}, K) = 1 \). Therefore, \( i(Q, \Omega_{r_1} \setminus \overline{\Omega}_{r_3}, K) = 1 \) and \( Q \) has at least one fixed point \( y(n) \) in \( \Omega_{r_1} \setminus \overline{\Omega}_{r_3} \). By Lemma 2.6, we conclude that (1.2) has at least one positive \( T \)-periodic solution \((A^{-1} y)(n)\) for \( \lambda > \lambda_0 \).

Case 3. If \( f_\infty = f_0 = 0 \), from the above arguments, there exist \( r_1, r_2, r_3 \) with \( 0 < r_2 < r_1 < r_3 \) such that \( Q \) has fixed points \( y_1(n) \) and \( y_2(n) \) in \( \Omega_{r_1} \setminus \overline{\Omega}_{r_2} \) and \( \Omega_{r_3} \setminus \overline{\Omega}_{r_1} \), respectively. By Lemma 2.6, for any \( \lambda > \lambda_0 \), (1.2) has at least two positive \( T \)-periodic solutions \((A^{-1} y_1)(n)\) and \((A^{-1} y_2)(n)\).

Part (b). \(-k < c \leq 0 \) implies that \( 1 > BL|c| \frac{\sum_{n=0}^{T-1} a(s)}{1 - |c|} \). Let \( r_1 = 1 \) and \( \lambda_1 = (1 - |c| - BL|c| \frac{\sum_{n=0}^{T-1} a(s)}{BM(r_1) \sum_{s=0}^{T-1} b(s)}(1 - |c|) > 0 \). From Lemma 2.10, for any \( y \in \partial \Omega_{r_1} \) and \( 0 < \lambda < \lambda_1 \), we have
\[
\| Qy \|_X < \| y \|_X.
\] (3.6)

By Lemma 2.1, \( i(Q, \Omega_{r_1}, K) = 1 \).

Case 1. If \( f_0 = \infty \), then for any \( \eta > 0 \), there exists \( 0 < \epsilon \leq \epsilon \) such that \( f(u) \geq \eta u \) for each \( 0 \leq u \leq \epsilon \). Take \( \eta > 0 \) satisfying
\[
\lambda A_1 \eta \frac{\alpha - |c|}{1 - c^2} \sum_{s=0}^{T-1} b(s) > 1.
\] (3.7)
Let $r_2 = (1 - |c|)\bar{r}_2$. For any $y \in \partial \Omega_{r_2}$, $0 \leq (A^{-1}y)(n - \tau(n)) \leq (1/(1 - |c|))\|y\|_X \leq \bar{r}_2$. Thus $f((A^{-1}y)(n - \tau(n))) \geq \eta(A^{-1}y)(n - \tau(n))$ for $y \in \partial \Omega_{r_2}$ and $n \in \mathbb{Z}$. By Lemma 2.7 and inequality (3.7), for any $y \in \partial \Omega_{r_2}$, we get

$$\|Qy\|_X \geq \lambda A_1\eta \frac{\alpha - |c|}{1 - c^2} \sum_{s=0}^{T-1} b(s)\|y\|_X > \|y\|_X.$$  \hspace{1cm} (3.8)

Lemma 2.1 tells that $i(Q, \Omega_{r_2}, K) = 0$. So $i(Q, \Omega_{r_1} \setminus \Omega_{r_2}, K) = 1$ and $Q$ has at least one fixed point $y(n)$ in $\Omega_{r_1} \setminus \Omega_{r_2}$. From Lemma 2.6, $(A^{-1}y)(n)$ is a positive $T$-periodic solution of (1.2) for $\lambda \in (0, \lambda_1)$.

Case 2. If $f_{\infty} = \infty$, then for any $\eta > 0$, we can find $\bar{\lambda} > 0$ satisfying that $f(u) \geq \eta u$ for each $u \geq \bar{\lambda}$. Take $\eta > 0$ such that inequality (3.7) holds.

Let $r_3 = \max\{2r_1, (1 - c^2)/(\alpha - |c|)\bar{\lambda}\}$. As $y \in \partial \Omega_{r_3}$, $(A^{-1}y)(n - \tau(n)) \geq ((\alpha - |c|)/(1 - c^2))\|y\|_X \geq \bar{\lambda}$, then $f((A^{-1}y)(n - \tau(n))) \geq \eta(A^{-1}y)(n - \tau(n))$ for any $y \in \partial \Omega_{r_3}$. For any $y \in \partial \Omega_{r_3}$, it follows from Lemma 2.7 and inequality (3.7) that

$$\|Qy\|_X \geq \lambda A_1\eta \frac{\alpha - |c|}{1 - c^2} \sum_{s=0}^{T-1} b(s)\|y\|_X > \|y\|_X.$$  \hspace{1cm} (3.9)

By Lemma 2.1, we obtain $i(Q, \Omega_{r_3}, K) = 0$. Thus, $i(Q, \Omega_{r_3} \setminus \Omega_{r_2}, K) = -1$ and $Q$ has at least one fixed point $y(n)$ in $\Omega_{r_3} \setminus \Omega_{r_2}$. Lemma 2.6 shows that $(A^{-1}y)(n)$ is a positive $T$-periodic solution of (1.2) for $\lambda \in (0, \lambda_1)$.

Case 3. If $f_{\infty} = f_0 = \infty$, from the arguments of Cases 1 and 2 in Part (b), there exist constants $0 < r_2 < r_1 < r_3$ such that $Q$ has one fixed point in $\Omega_{r_1} \setminus \Omega_{r_2}$ and $\Omega_{r_2} \setminus \Omega_{r_3}$, respectively, denoting $y_1(n)$ and $y_2(n)$. That is, for any $\lambda \in (0, \lambda_1)$, (1.2) has at least two positive $T$-periodic solutions $(A^{-1}y_1)(n)$ and $(A^{-1}y_2)(n)$.

Part (c)

Case 1. If $i_0 = 0$, then $f_0 > 0$ and $f_{\infty} > 0$. Letting $c_1 = \min\{(f(u)/u) : u > 0\} > 0$, we have

$$f(u) \geq c_1u, \quad u \in [0, +\infty).$$  \hspace{1cm} (3.10)

Take $\lambda_2 = (1 - c^2)/(A_1c_1(\alpha - |c|)\sum_{s=0}^{T-1} b(s))$ and suppose that $u(n)$ is the positive $T$-periodic solution of (1.2) for $\lambda > \lambda_2$. For any $n \in \mathbb{Z}$, $f(A^{-1}u(n - \tau(n))) \geq c_1A^{-1}u(n - \tau(n)) \geq (c_1(\alpha - |c|)/(1 - c^2))\|u\|_X$ and $Qu(n) = u(n)$. From Lemma 2.7, for $\lambda > \lambda_2$, we obtain

$$\|u\|_X = \|Qu\|_X \geq \lambda A_1c_1\alpha - |c| \sum_{s=0}^{T-1} b(s)\|u\|_X > \|u\|_X,$$  \hspace{1cm} (3.11)

which is a contradiction. Thus, when $i_0 = 0$ and $\lambda > \lambda_2$, (1.2) has no positive $T$-periodic solution.

Case 2. $i_{\infty} = 0$ implies that $f_0 < \infty$ $f_{\infty} < \infty$. Since $-k < c \leq 0$, $1 - |c| > BL|c|\sum_{s=0}^{T-1} a(s)$. Letting $c_2 = \max\{(f(u)/u) : u > 0\} > 0$, we get

$$f(u) \leq c_2u, \quad u \in [0, +\infty).$$  \hspace{1cm} (3.12)

Take $\lambda_3 = (1 - |c| - BL|c|\sum_{s=0}^{T-1} a(s))/Bc_2\sum_{s=0}^{T-1} b(s)$. Suppose that $u(n)$ is the positive $T$-periodic solution of (1.2) corresponding to $\lambda \in (0, \lambda_3)$. For any $n \in \mathbb{Z}$, $f(A^{-1}u(n - \tau(n))) \geq c_2A^{-1}u(n - \tau(n)) \geq (c_2(1 - c^2)/(1 - c^2))\|u\|_X$ and $Qu(n) = u(n)$. From Lemma 2.7, for $\lambda > \lambda_3$, we obtain

$$\|u\|_X = \|Qu\|_X \geq \lambda A_1c_2\alpha - |c| \sum_{s=0}^{T-1} b(s)\|u\|_X > \|u\|_X,$$  \hspace{1cm} (3.13)

which is a contradiction. Thus, when $i_{\infty} = 0$ and $\lambda > \lambda_3$, (1.2) has no positive $T$-periodic solution.
\(\tau(n)) \leq c_2 A^{-1} u(n - \tau(n)) \leq (c_2/(1 - |c|))\|u\|_X\) and \(Qu(n) = u(n)\). Therefore, by Lemma 2.8, for \(\lambda \in (0, \lambda_3)\), we have

\[
\|u\|_X = \|Qu\|_X \leq \frac{\lambda B c_2 \sum_{s=0}^{T-1} b(s) + BL |c| \sum_{s=0}^{T-1} a(s)}{1 - |c|} \|u\|_X < \|u\|_X,
\]

which is a contradiction. So, When \(i_\infty = 0\), (1.2) has no positive \(T\)-periodic solution for any \(0 < \lambda < \lambda_3\).

Proof of Theorem 3.2. Following the proof of part (c) of Theorem 3.1, we can obtain this result immediately.

Proof of Theorem 3.3

Case 1. If \(f_0 \leq f_\infty\), then

\[
\frac{1 - c^2}{f_\infty A_1 (\alpha - |c|) \sum_{s=0}^{T-1} b(s)} < \lambda < \frac{1 - |c| - BL |c| \sum_{s=0}^{T-1} a(s)}{f_0 B \sum_{s=0}^{T-1} b(s)}.
\]

We can choose \(0 < \epsilon < f_\infty\) such that

\[
\frac{1 - c^2}{(f_\infty - \epsilon) A_1 (\alpha - |c|) \sum_{s=0}^{T-1} b(s)} < \lambda < \frac{1 - |c| - BL |c| \sum_{s=0}^{T-1} a(s)}{(f_0 + \epsilon) B \sum_{s=0}^{T-1} b(s)}.
\]

From the definition of \(f_0\), there exists \(r_1 > 0\) such that \(f(u) \leq (f_0 + \epsilon)u\) for any \(0 \leq u \leq r_1\). Take \(r_1 = (1 - |c|) \bar{r}_1\). For \(y \in \partial \Omega_{r_1}\), since \(0 \leq (A^{-1}y)(n - \tau(n)) \leq (1/(1 - |c|))\|y\|_X \leq \bar{r}_1\), then \(f((A^{-1}y)(n - \tau(n))) \leq (f_0 + \epsilon)(A^{-1}y)(n - \tau(n))\). By Lemma 2.8, for any \(y \in \partial \Omega_{r_1}\), we get

\[
\|Qu\|_X \leq \frac{B\lambda (f_0 + \epsilon) \sum_{s=0}^{T-1} b(s) + BL |c| \sum_{s=0}^{T-1} a(s)}{1 - |c|} \|y\|_X < \|y\|_X.
\]

On the other hand, we can choose \(\tilde{H} > 0\) such that \(f(u) \geq (f_\infty - \epsilon)u\) for \(u \geq \tilde{H}\). Let \(r_2 = \max\{2r_1, ((1 - c^2)/(\alpha - |c|))\tilde{H}\}\). If \(y \in \partial \Omega_{r_2}\), then \((A^{-1}y)(n - \tau(n)) \geq ((\alpha - |c|)/(1 - c^2))\|y\|_X \geq \tilde{H}\). So \(f((A^{-1}y)(n - \tau(n))) \geq (f_\infty - \epsilon)(A^{-1}y)(n - \tau(n))\) for any \(y \in \partial \Omega_{r_2}\).

From Lemma 2.7, for \(y \in \partial \Omega_{r_2}\), we have

\[
\|Qu\|_X \geq \lambda (f_\infty - \epsilon) A_1 \frac{\alpha - |c| \sum_{s=0}^{T-1} b(s)}{1 - c^2} \|y\|_X > \|y\|_X.
\]

It follows from Lemma 2.1 that

\[
i(Q, \Omega_{r_1}, K) = 1, \quad i(Q, \Omega_{r_2}, K) = 0, \quad i(Q, \Omega_{r_2} \setminus \overline{\Omega}_{r_1}, K) = -1.
\]

Then \(Q\) has at least one fixed point \(y(n)\) in \(\Omega_{r_2} \setminus \overline{\Omega}_{r_1}\). By Lemma 2.6, \((A^{-1}y)(n)\) is the positive \(T\)-periodic solution of (1.2).

Case 2. If \(f_0 > f_\infty\), then

\[
\frac{1 - c^2}{f_0 A_1 (\alpha - |c|) \sum_{s=0}^{T-1} b(s)} < \lambda < \frac{1 - |c| - BL |c| \sum_{s=0}^{T-1} a(s)}{f_\infty B \sum_{s=0}^{T-1} b(s)}.
\]
Thus, by Lemma 2.8, for any 0 ≤ u ≤ τ₁, f(u) ≥ (f₀ - ε)u. Let r₁ = (1 - |c|)τ₁. If y ∈ ∂Ω₁, then 0 ≤ (A⁻¹y)(n - τ(n)) ≤ (1/(1 - |c|))∥y∥₁ ≤ τ₁. So we have (A⁻¹y)(n - τ(n)) ≥ (f₀ - ε)(A⁻¹y)(n - τ(n)) for y ∈ ∂Ω₁. From Lemma 2.7, for any y ∈ ∂Ω₁, we obtain

\[ \|Qy\|_X \geq \lambda (f₀ - \epsilon) A₁ \Sigma_{s=0}^{T-1} b(s) \frac{\alpha - |c|}{1 - \epsilon^2} \|y\|_X > \|y\|_X. \]  

(3.21)

If 0 < f₀ < ∞, then there exists H > 0 satisfying for any u ≥ H, f(u) ≤ (f₀ + ε)u. Take r₂ = \max\{2r₁, ((1 - \epsilon^2)/(\alpha - |c|))H\}. y ∈ ∂Ω₂ tells that (A⁻¹y)(n - τ(n)) ≥ ((\alpha - |c|)/(1 - \epsilon^2))∥y∥₁ ≥ H. So f((A⁻¹y)(n - τ(n)) ≤ (f₀ + ε)(A⁻¹y)(n - τ(n)) for y ∈ ∂Ω₂. Thus, by Lemma 2.8, for y ∈ ∂Ω₂, we have

\[ \|Qy\|_X \leq \frac{\lambda B (f₀ + \epsilon) \Sigma_{s=0}^{T-1} b(s) + BL |c| \Sigma_{s=0}^{T-1} a(s)}{1 - |c|} \|y\|_X < \|y\|_X. \]  

(3.22)

It follows from Lemma 2.1 that

\[ i(Q, Ω₁, K) = 0, \quad i(Q, Ω₂, K) = 1. \]  

(3.23)

Therefore, \( i(Q, Ω₂ \setminus \overline{Ω}_₁, K) = 1 \) and Q has at least one fixed point y(n) in Ω₂ \ \overline{Ω}_₁. Lemma 2.6 shows that (A⁻¹y)(n) is a positive T-periodic solution of (1.2).

Our results are applicable to consider multiplicity of periodic solutions for many neutral difference equations.

**Example 3.4.** We consider the following neutral difference equation:

\[ \Delta \left[ u(n) + \frac{1}{3} u(n - 1) \right] = \frac{1}{4} u(n) - \lambda [1 - \sin \pi n] u^a(n - \tau(n)) e^{-u(n-\tau(n))}, \quad n ∈ ℤ, \]  

(3.24)

where λ and a are two positive parameters, τ(n + 2) = τ(n). Take τ = 1, c = -1/3, a(n) ≡ 1/4, b(n) = 1 - sin π n, g(u) ≡ 1, f(u) = u^a e^{-u}, L = l = 1. Then assumptions (E₁) and (E₂) hold, f₀ = 0, and max_{u∈[0,∞)} f(u) = f(a).

By direct computations, we have k = α = 2/5, f₀ = +∞ if a ∈ (0, 1), f₀ = 1 when a = 1, and f₀ = 0 as a > 1. Furthermore, let t₀ = min{a, (3/2)}, we have

\[ M(1) = \max \left\{ f(t) : 0 ≤ t ≤ \frac{3}{2} \right\} = f(t₀), \]  

\[ m(1) = \min \left\{ f(t) : \frac{3}{40} ≤ t ≤ \frac{3}{2} \right\} = \min \left\{ f \left( \frac{3}{2} \right), f \left( \frac{3}{40} \right) \right\} = r₀. \]  

(3.25)
Thus

\[
\lambda_0 = \frac{1}{A_1 m(1) \sum_{s=1}^{T-1} b(s)} = \frac{3}{4 r_0}, \quad \lambda_1 = \frac{1 - |c| - BL |c| \sum_{s=0}^{T-1} a(s)}{BM(1) \sum_{s=0}^{T-1} b(s)(1 - |c|)} = \frac{7}{40 f(t_0)}.
\]

(3.26)

Applying Theorem 3.1 to (3.24), we obtain the following results.

4. Conclusion

(a) If \( a \in (0, 1) \), then (3.24) has one positive two-periodic solution for \( \lambda > \frac{3}{4 r_0} > 0 \) or \( 0 < \lambda < 7/40 f(a) \).

(b) If \( a = 1 \), then (3.24) has one positive two-periodic solution for \( \lambda > \frac{3}{4 r_0} > 0 \).

(c) If \( a > 1 \), then (3.24) has two positive two-periodic solutions for \( \lambda > \frac{3}{4 r_0} > 0 \).

References


Two solutions of neutral difference equations


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