ON THE GLOBAL BEHAVIOR OF THE NONLINEAR DIFFERENCE EQUATION $x_{n+1} = f(p_n, x_{n-m}, x_{n-t(k+1)+1})$

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We consider the following nonlinear difference equation: $x_{n+1} = f(p_n, x_{n-m}, x_{n-t(k+1)+1})$, $n = 0, 1, 2, \dots$, where $m \in \{0, 1, 2, \dots\}$ and $k, t \in \{1, 2, \dots\}$ with $0 \leq m < t(k+1) - 1$, the initial values $x_{-t(k+1)+1}, x_{-t(k+1)+2}, \dots, x_0 \in (0, +\infty)$, and $\{p_n\}_{n=0}^\infty$ is a positive sequence of the period $k+1$. We give sufficient conditions under which every positive solution of this equation tends to the period $k+1$ solution.

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1. Introduction

Nonlinear difference equations of order greater than one are of paramount importance in applications. Such equations also appear naturally as discrete analogues and as numerical solutions of differential and delay differential equations which model various diverse phenomena in biology, ecology, physiology, physics, engineering, and economics. Some nonlinear difference equations, especially the boundedness, global attractivity, oscillatory, and some other properties of second-order nonlinear difference equations have been investigated by many authors, see [2, 3, 5].

Amleh et al. [1] studied the characteristics of the difference equation

$$x_{n+1} = p + \frac{x_{n-1}}{x_n}, \quad n = 0, 1, 2, \dots, \quad (1.1)$$

and obtained that the solutions of (1.1) with positive initial conditions are globally asymptotically stable provided that $p > 1$.

Kulenović et al. [4] investigated the boundedness, the periodicity, and the global asymptotic stability of solutions of the nonautonomous difference equation

$$x_{n+1} = p_n + \frac{x_{n-1}}{x_n}, \quad n = 0, 1, 2, \dots, \quad (1.2)$$

where the initial values $x_{-1}, x_0 \in (0, +\infty)$ and $\{p_n\}_{n=0}^\infty$ is a positive sequence of period 2.

2 Global behavior of a difference equation

Recently, Papaschinopoulos and Schinas [6] have studied the boundedness, the periodicity, and the attractivity of the positive solutions of the nonautonomous difference equation

$$x_{n+1} = p_n + \frac{x_{n-k}}{x_n}, \quad n = 0, 1, 2, \dots, \quad (1.3)$$

where the initial values $x_{-k}, x_{-k+1}, \dots, x_0 \in (0, +\infty)$ and $\{p_n\}_{n=0}^{\infty}$ is a positive sequence of period $k+1$.

In this paper, we consider the more general equation

$$x_{n+1} = f(p_n, x_{n-m}, x_{n-t(k+1)+1}), \quad n = 0, 1, 2, \dots, \quad (1.4)$$

where $m \in \{0, 1, 2, \dots\}$ and $k, t \in \{1, 2, \dots\}$ with $0 \leq m < t(k+1) - 1$, the initial values $x_{-t(k+1)+1}, x_{-t(k+1)+2}, \dots, x_0 \in (0, +\infty)$ and $\{p_n\}_{n=0}^{\infty}$ is a positive sequence of period $k+1$, and f satisfies the following hypotheses.

- (H₁) For every $i \in \{0, 1, \dots, k\}$, $f(p_i, u, v) \in C(E \times E, (0, +\infty))$ and $a_i = \inf_{(u,v) \in E \times E} f(p_{i-1}, u, v) \in E$, where $p_{-1} = p_k$ and $E \in \{[0, \infty), (0, +\infty)\}$.
- (H₂) For every $i \in \{0, 1, \dots, k\}$, $f(p_i, u, v)$ is decreasing in u and increasing in v .
- (H₃) Let $a_{n+k+1} = a_n$ for every integer n . For every $i \in \{0, 1, \dots, k\}$, $f(p_i, a_{i-m}, x)$ has only one fixed point in the interval $(a_{i+1}, +\infty)$, denoted by A_{i+1} .
- (H₄) For every $i \in \{0, 1, \dots, k\}$ and every $y \in E$, $f(p_i, y, x)/x$ is nonincreasing in x in $(0, +\infty)$.
- (H₅) Let $y_{n+k+1} = y_n$ for every integer n . Equation

$$\begin{aligned} y_1 &= f(p_0, y_{-m}, y_1), \\ y_2 &= f(p_1, y_{1-m}, y_2), \\ &\vdots \\ y_k &= f(p_{k-1}, y_{k-1-m}, y_k), \\ y_0 &= f(p_k, y_{k-m}, y_0) \end{aligned} \quad (1.5)$$

has a unique positive solution $(y_0, y_1, \dots, y_k) = (\bar{y}_0, \bar{y}_1, \dots, \bar{y}_k)$.

2. Main results

For every integer n , write

$$p_{n+k+1} = p_n, \quad a_{n+k+1} = a_n, \quad A_{n+k+1} = A_n, \quad \bar{y}_{n+k+1} = \bar{y}_n. \quad (2.1)$$

THEOREM 2.1. Assume that (H₁)–(H₅) hold and $\{x_n\}_{n=-t(k+1)+1}^{\infty}$ is a positive solution of (1.4), then there exists a positive integer N such that for all $n \geq N$ and all $i \in \{0, 1, \dots, k\}$,

$$f(p_{i-1}, A_{i-1-m}, a_i) \leq x_{n(k+1)+i} \leq A_i. \quad (2.2)$$

Proof. By (H_1) , (H_2) , and (H_5) , we know that for every $0 \leq i \leq k$,

$$\bar{y}_i = f(p_{i-1}, \bar{y}_{i-1-m}, \bar{y}_i) > f(p_{i-1}, \bar{y}_{i-1-m} + 1, \bar{y}_i) \geq a_i. \quad (2.3)$$

Claim 1. $f(p_{i-1}, A_{i-1-m}, a_i) < \bar{y}_i < A_i$ for every $0 \leq i \leq k$.

Proof of Claim 1. If $\bar{y}_i \geq A_i$ for some $0 \leq i \leq k$, then it follows from (H_2) – (H_5) that

$$\begin{aligned} A_i &= f(p_{i-1}, a_{i-1-m}, A_i) > f(p_{i-1}, \bar{y}_{i-1-m}, A_i) \\ &= \frac{f(p_{i-1}, \bar{y}_{i-1-m}, A_i)}{A_i} \times A_i \geq \frac{f(p_{i-1}, \bar{y}_{i-1-m}, \bar{y}_i)}{\bar{y}_i} \times A_i = A_i. \end{aligned} \quad (2.4)$$

This is a contradiction. Therefore, $\bar{y}_i < A_i$ for every $0 \leq i \leq k$. Moreover, we have

$$A_i > \bar{y}_i = f(p_{i-1}, \bar{y}_{i-1-m}, \bar{y}_i) > f(p_{i-1}, A_{i-1-m}, a_i). \quad (2.5)$$

Claim 1 is proven. □

Claim 2. Let $0 \leq i \leq k$ and $n \geq 0$.

- (1) If $x_{n(k+1)+i} \leq A_i$, then $x_{(n+t)(k+1)+i} \leq A_i$.
- (2) If $x_{n(k+1)+i} > A_i$, then $x_{(n+t)(k+1)+i} \leq x_{n(k+1)+i}$.

Proof of Claim 2. Let $0 \leq i \leq k$ and $n \geq 0$.

If $x_{n(k+1)+i} \leq A_i$, then it follows from (H_1) – (H_3) that

$$\begin{aligned} x_{(n+t)(k+1)+i} &= f(p_{i-1}, x_{(n+t)(k+1)+i-1-m}, x_{n(k+1)+i}) \\ &\leq f(p_{i-1}, a_{i-1-m}, x_{n(k+1)+i}) \leq f(p_{i-1}, a_{i-1-m}, A_i) = A_i. \end{aligned} \quad (2.6)$$

If $x_{n(k+1)+i} > A_i$, then it follows from (H_1) – (H_4) that

$$\begin{aligned} x_{(n+t)(k+1)+i} &= f(p_{i-1}, x_{(n+t)(k+1)+i-1-m}, x_{n(k+1)+i}) \\ &\leq f(p_{i-1}, a_{i-1-m}, x_{n(k+1)+i}) \\ &= \frac{f(p_{i-1}, a_{i-1-m}, x_{n(k+1)+i})}{x_{n(k+1)+i}} x_{n(k+1)+i} \\ &\leq \frac{f(p_{i-1}, a_{i-1-m}, A_i)}{A_i} x_{n(k+1)+i} = x_{n(k+1)+i}. \end{aligned} \quad (2.7)$$

Claim 2 is proven. □

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Claim 3. For every $0 \leq i \leq k$, there exists a positive integer N_i such that $x_{n(k+1)+i} \leq A_i$ for all $n > N_i$.

Proof of Claim 3. Assume on the contrary that there exists some $0 \leq i \leq k$ such that Claim 3 does not hold. Then it follows from Claim 2 that there exists positive integer p such that $x_{(p+nt)(k+1)+i} \geq x_{(p+(n+1)t)(k+1)+i} > A_i$ for every $n \geq 1$.

Let $\lim_{n \rightarrow +\infty} x_{(p+nt)(k+1)+i} = B_i$, then $B_i \geq A_i$. By using Claim 2, we know that $\{x_{(p+nt)(k+1)+i-1-m}\}_{n=0}^{+\infty}$ is bounded. Let

$$\begin{aligned} \limsup_{n \rightarrow +\infty} x_{(p+nt)(k+1)+i-1-m} &= C_{i-1-m}, \\ \lim_{l \rightarrow +\infty} x_{(p+n_l t)(k+1)+i-1-m} &= C_{i-1-m}. \end{aligned} \quad (2.8)$$

By (1.4) we have that

$$x_{(p+n_l t)(k+1)+i} = f(p_{i-1}, x_{(p+n_l t)(k+1)+i-1-m}, x_{(p+(n_l-1)t)(k+1)+i}), \quad (2.9)$$

from which it follows that

$$\begin{aligned} B_i &= f(p_{i-1}, C_{i-1-m}, B_i) \leq f(p_{i-1}, a_{i-1-m}, B_i) \\ &= \frac{f(p_{i-1}, a_{i-1-m}, B_i)}{B_i} \times B_i \leq \frac{f(p_{i-1}, a_{i-1-m}, A_i)}{A_i} \times B_i = B_i. \end{aligned} \quad (2.10)$$

Thus $C_{i-1-m} = a_{i-1-m}$, which implies $\lim_{n \rightarrow +\infty} x_{(p+nt)(k+1)+i-1-m} = a_{i-1-m}$.

Again by Claim 2 we know that $\{x_{(p+nt)(k+1)+i-2-2m}\}_{n=0}^{+\infty}$ is bounded. Let

$$\begin{aligned} \limsup_{n \rightarrow +\infty} x_{(p+nt)(k+1)+i-2-2m} &= C_{i-2-2m}, \\ \lim_{s \rightarrow +\infty} x_{(p+r_s t)(k+1)+i-2-2m} &= C_{i-2-2m}. \end{aligned} \quad (2.11)$$

Again by (1.4) we have that

$$x_{(p+r_s t)(k+1)+i-1-m} = f(p_{i-2-m}, x_{(p+r_s t)(k+1)+i-2-2m}, x_{(p+(r_s-1)t)(k+1)+i-1-m}), \quad (2.12)$$

from which it follows that

$$a_{i-1-m} = f(p_{i-2-m}, C_{i-2-2m}, a_{i-1-m}) > f(p_{i-2-m}, C_{i-2-2m} + 1, a_{i-1-m}) \geq a_{i-1-m}, \quad (2.13)$$

a contradiction. Claim 3 is proven. $\square$

Let $N = \max\{N_0, N_1, \dots, N_k\} + (t(k+1))^2$, then for all $n \geq N$ and all $i \in \{0, 1, \dots, k\}$,

$$f(p_{i-1}, A_{i-1-m}, a_i) \leq x_{n(k+1)+i} \leq A_i. \quad (2.14)$$

Theorem 2.1 is proven. $\square$

THEOREM 2.2. For every $0 \leq i \leq k$ and every integer j , write $I_i = [m_i, M_i]$ and $I_{j(k+1)+i} = I_i$. Assume that $f_i \in C(I_{i-m-1} \times I_i, I_i)$ satisfies the following statements.

- (L₁) $f_i(u, v)$ is decreasing in u and increasing in v for every $0 \leq i \leq k$.
 (L₂) If $\{d_i\}_{i=-\infty}^{+\infty}$ and $\{D_i\}_{i=-\infty}^{+\infty}$ are sequences of period $k+1$ with $d_i, D_i \in I_i$ satisfying the following equations:

$$\begin{aligned} D_0 &= f_0(d_{-1-m}, D_0), \\ D_1 &= f_1(d_{-m}, D_1), \\ &\vdots \\ D_k &= f_k(d_{k-1-m}, D_k), \\ d_0 &= f_0(D_{-1-m}, d_0), \\ d_1 &= f_1(D_{-m}, d_1), \\ &\vdots \\ d_k &= f_k(D_{k-1-m}, d_k), \end{aligned} \quad (2.15)$$

then $d_i = D_i$ for every $0 \leq i \leq k$.

Then the system

$$\begin{aligned} x_{n(k+1)+1} &= f_1(x_{n(k+1)-m}, x_{(n-t)(k+1)+1}), \\ x_{n(k+1)+2} &= f_2(x_{n(k+1)-m+1}, x_{(n-t)(k+1)+2}), \\ &\vdots \\ x_{n(k+1)+k} &= f_k(x_{n(k+1)-m+k-1}, x_{(n-t)(k+1)+k}), \\ x_{(n+1)(k+1)} &= f_0(x_{n(k+1)-m+k}, x_{(n+1-t)(k+1)}), \end{aligned} \quad (2.16)$$

$n = 0, 1, \dots$, has a unique equilibrium $(\bar{z}_0, \bar{z}_1, \dots, \bar{z}_k)$ and every solution $\{x_n\}_{n=-t(k+1)+1}^{+\infty}$ of (2.16) with the initial values $x_{l(k+1)+i} \in I_i$ ($1 \leq i \leq k+1$, $-t \leq l \leq -1$) converges to $(\bar{z}_0, \bar{z}_1, \dots, \bar{z}_k)$.

Proof. Write $M_{j(k+1)+i}^0 = M_i^0 = M_i$ and $m_{j(k+1)+i}^0 = m_i^0 = m_i$ for every $0 \leq i \leq k$ and every integer j . For any positive integer n , we define

$$\begin{aligned} M_i^n &= f_i(m_{i-1-m}^{n-1}, M_i^{n-1}), \\ m_i^n &= f_i(M_{i-1-m}^{n-1}, m_i^{n-1}), \end{aligned} \quad (2.17)$$

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where $M_{j(k+1)+i}^n = M_i^n$ and $m_{j(k+1)+i}^n = m_i^n$ for every $0 \leq i \leq k$ and every integer j . It follows from (L_1) that for every $0 \leq i \leq k$,

$$m_i = m_i^0 \leq m_i^1 = f_i(M_{i-1-m}^0, m_i^0) \leq f_i(m_{i-1-m}^0, M_i^0) = M_i^1 \leq M_i^0, \quad (2.18)$$

$$\begin{aligned} m_i^1 &= f_i(M_{i-1-m}^0, m_i^0) \leq f_i(M_{i-1-m}^1, m_i^1) = m_i^2 \\ &\leq f_i(m_{i-1-m}^1, M_i^1) = M_i^2 \leq f_i(m_{i-1-m}^0, M_i^0) = M_i^1. \end{aligned} \quad (2.19)$$

By induction we have that for every $0 \leq i \leq k$,

$$m_i = m_i^0 \leq m_i^1 \leq m_i^2 \leq \dots \leq m_i^n \leq \dots \leq M_i^n \leq \dots \leq M_i^2 \leq M_i^1 \leq M_i^0 = M_i. \quad (2.20)$$

Let

$$\lim_{n \rightarrow +\infty} m_i^n = d_i, \quad \lim_{n \rightarrow +\infty} M_i^n = D_i. \quad (2.21)$$

Since $x_{l(k+1)+i} \in [m_i, M_i]$ ($1 \leq i \leq k+1, -t \leq l \leq -1$), it follows that

$$m_i^1 = f_i(M_{i-1-m}^0, m_i^0) \leq x_i = f_i(x_{i-m-1}, x_{i-t(k+1)}) \leq f_i(m_{i-1-m}^0, M_i^0) = M_i^1. \quad (2.22)$$

By induction we obtain that for $n = 0, 1, 2, \dots$,

$$m_i^{n+1} \leq x_{n(k+1)+i} \leq M_i^{n+1}. \quad (2.23)$$

Let $d_{j(k+1)+i} = d_i$, $D_{j(k+1)+i} = D_i$ for every $0 \leq i \leq k$ and every integer j . By the continuity of f_i , it follows from (2.17) and (2.21) that

$$\begin{aligned} D_0 &= f_0(d_{-1-m}, D_0), \\ D_1 &= f_1(d_{-m}, D_1), \\ &\vdots \\ D_k &= f_k(d_{k-1-m}, D_k), \\ d_0 &= f_0(D_{-1-m}, d_0), \\ d_1 &= f_1(D_{-m}, d_1), \\ &\vdots \\ d_k &= f_k(D_{k-1-m}, d_k). \end{aligned} \quad (2.24)$$

Using assumption (L_2) , it follows from (2.23) that for $0 \leq i \leq k$,

$$\lim_{n \rightarrow +\infty} x_{n(k+1)+i} = d_i = D_i = \bar{z}_i. \quad (2.25)$$

Theorem 2.2 is proven. □

By Theorems 2.1 and 2.2, we obtain the following theorem.

THEOREM 2.3. *Consider (1.4). If (H_1) – (H_5) hold and the system*

$$\begin{aligned}
 D_0 &= f(p_k, d_{k-m}, D_0), \\
 D_1 &= f(p_0, d_{-m}, D_1), \\
 &\vdots \\
 D_k &= f(p_{k-1}, d_{k-1-m}, D_k), \\
 d_0 &= f(p_k, D_{k-m}, d_0), \\
 d_1 &= f(p_0, D_{-m}, d_1), \\
 &\vdots \\
 d_k &= f(p_{k-1}, D_{k-1-m}, d_k)
 \end{aligned} \tag{2.26}$$

has the unique solution $d_i = D_i = \bar{y}_i \in [f(p_{i-1}, A_{i-1-m}, a_i), A_i]$ ($0 \leq i \leq k$), where $d_{j(k+1)+i} = d_i$ and $D_{j(k+1)+i} = D_i$ for every $0 \leq i \leq k$ and every integer j . Then every solution of (1.4) converges to the period $k+1$ solution $\bar{y}_1, \dots, \bar{y}_k, \bar{y}_0, \bar{y}_1, \dots, \bar{y}_k, \bar{y}_0, \dots$

3. Examples

To illustrate the applicability of Theorems 2.1–2.3, we present the following examples.

Example 3.1. Consider equation

$$x_{n+1} = p_n + \frac{x_{n-t(k+1)+1}}{x_{n-m}}, \quad n = 0, 1, 2, \dots, \tag{3.1}$$

where $m \in \{0, 1, 2, \dots\}$ and $k, t \in \{1, 2, \dots\}$ with $0 \leq m < t(k+1) - 1$, the initial values $x_{-t(k+1)+1}, x_{-t(k+1)+2}, \dots, x_0 \in (0, +\infty)$ and $\{p_n\}_{n=0}^\infty$ is a positive sequence of period $k+1$ with $p_n > 1$ for every $n \geq 0$, then every positive solution of (3.1) converges to the unique period $k+1$ solution.

Proof. Let $E = (0, +\infty)$. It is easy to verify that (H_1) – (H_4) hold for (3.1). In the following we show that (H_5) and (L_2) hold for (3.1).

For any $0 \leq i \leq k$ and any integer j , write

$$\begin{aligned}
 p_{j(k+1)+i} &= p_i, \\
 y_{j(k+1)+i} &= y_i.
 \end{aligned} \tag{3.2}$$

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Consider the system

$$\begin{aligned} y_0 &= p_k + \frac{y_0}{y_{k-m}}, \\ y_1 &= p_0 + \frac{y_1}{y_{-m}}, \\ &\vdots \\ y_k &= p_{k-1} + \frac{y_k}{y_{k-1-m}}. \end{aligned} \tag{3.3}$$

It follows from (3.2) and (3.3) that for any integer j ,

$$y_j = \frac{p_{j-1}y_{j-(m+1)}}{y_{j-(m+1)} - 1}. \tag{3.4}$$

Let $A_1^j = 1$ and $A_n^j = A_{n-1}^j p_{j-1-(n-1)(m+1)} + (-1)^{n-1}$ for any positive integer $n \geq 2$. It is easy to verify that $A_n^j > 0$ for any positive integer $n \geq 1$. By (3.4) we know that for any integer j ,

$$\begin{aligned} y_j &= \frac{p_{j-1}p_{j-1-(m+1)} \cdots p_{j-1-k(m+1)}y_{j-(k+1)(m+1)}}{A_{k+1}^j y_{j-(k+1)(m+1)} + (-1)^{k+1}} \\ &= \frac{p_{j-1}p_{j-1-(m+1)} \cdots p_{j-1-k(m+1)}y_j}{A_{k+1}^j y_j + (-1)^{k+1}}, \end{aligned} \tag{3.5}$$

from which it follows that the system (3.3) has a unique positive solution

$$y_j = \bar{y}_j = \frac{p_{j-1}p_{j-1-(m+1)} \cdots p_{j-1-k(m+1)} + (-1)^k}{A_{k+1}^j}, \quad j = 0, 1, \dots, k, \tag{3.6}$$

which implies that (H₅) holds for (3.1).

For any $0 \leq i \leq k$ and any integer j , write

$$d_{j(k+1)+i} = d_i, \quad D_{j(k+1)+i} = D_i. \tag{3.7}$$

Consider equations

$$\begin{aligned} D_0 &= p_k + \frac{D_0}{d_{k-m}}, \\ D_1 &= p_0 + \frac{D_1}{d_{-m}}, \\ &\vdots \\ D_k &= p_{k-1} + \frac{D_k}{d_{k-1-m}}, \end{aligned}$$

$$\begin{aligned}
d_0 &= p_k + \frac{d_0}{D_{k-m}}, \\
d_1 &= p_0 + \frac{d_1}{D_{-m}}, \\
&\vdots \\
d_k &= p_{k-1} + \frac{d_k}{D_{k-1-m}}.
\end{aligned} \tag{3.8}$$

It follows from (3.7) and (3) that for any integer j ,

$$D_j = \frac{d_{j-(m+1)}}{d_{j-(m+1)} - 1} p_{j-1}, \quad d_j = \frac{D_{j-(m+1)}}{D_{j-(m+1)} - 1} p_{j-1}. \tag{3.9}$$

That is,

$$D_j = \frac{p_{j-1} p_{j-1-(m+1)} D_{j-2(m+1)}}{(p_{j-1-(m+1)} - 1) D_{j-2(m+1)} + 1}, \quad d_j = \frac{p_{j-1} p_{j-1-(m+1)} d_{j-2(m+1)}}{(p_{j-1-(m+1)} - 1) d_{j-2(m+1)} + 1}. \tag{3.10}$$

By (3.10) we see that for any integer j ,

$$\begin{aligned}
D_j &= \frac{p_{j-1} p_{j-1-(m+1)} \cdots p_{j-1-(2k+1)(m+1)} D_{j-2(k+1)(m+1)}}{A_{2k+2}^j D_{j-2(k+1)(m+1)} + 1} \\
&= \frac{p_{j-1} p_{j-1-(m+1)} \cdots p_{j-1-(2k+1)(m+1)} D_j}{A_{2k+2}^j D_j + 1},
\end{aligned} \tag{3.11}$$

$$\begin{aligned}
d_j &= \frac{p_{j-1} p_{j-1-(m+1)} \cdots p_{j-1-(2k+1)(m+1)} d_{j-2(k+1)(m+1)}}{A_{2k+2}^j d_{j-2(k+1)(m+1)} + 1} \\
&= \frac{p_{j-1} p_{j-1-(m+1)} \cdots p_{j-1-(2k+1)(m+1)} d_j}{A_{2k+2}^j d_j + 1}.
\end{aligned} \tag{3.12}$$

Thus

$$D_j = d_j = \frac{p_{j-1} p_{j-1-(m+1)} \cdots p_{j-1-(2k+1)(m+1)} - 1}{A_{2k+2}^j}, \quad j = 0, 1, \dots, k, \tag{3.13}$$

from which with (3.3) and (3) it follows that the system (3) has a unique solution

$$D_j = d_j = \bar{y}_j, \quad j = 0, 1, \dots, k, \tag{3.14}$$

which implies that (L_2) holds for (3.1). It follows from Theorems 2.1–2.3 that every positive solution of (3.1) converges to the period $k+1$ solution $\bar{y}_1, \dots, \bar{y}_k, \bar{y}_0, \bar{y}_1, \dots, \bar{y}_k, \bar{y}_0, \dots$. $\square$

Remark 3.2. Example 3.1 includes [6, Proposition 2.2].

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Example 3.3. Consider equation

$$x_{n+1} = \frac{q + x_{n-t(k+1)+1}}{p_n + x_{n-m}}, \quad (3.15)$$

where $m \in \{0, 1, 2, \dots\}$ and $k, t \in \{1, 2, \dots\}$ with $0 \leq m < t(k+1) - 1$, the initial values $x_{-t(k+1)+1}, x_{-t(k+1)+2}, \dots, x_0 \in (0, +\infty)$, $q \in (0, +\infty)$, and $\{p_n\}_{n=0}^{\infty}$ is a positive sequence of period $k+1$ with $p_n > 1$ for every $n \geq 0$, then every positive solution of (3.15) converges to the period $k+1$ solution.

Proof. Let $E = [0, +\infty)$. It is easy to verify that (H_1) – (H_4) hold for (3.15). In the following we show that (H_5) and (L_2) hold for (3.15).

For any $0 \leq i \leq k$ and any integer j , write

$$p_{j(k+1)+i} = p_i, \quad y_{j(k+1)+i} = y_i. \quad (3.16)$$

Consider equations

$$\begin{aligned} y_0 &= \frac{q + y_0}{p_k + y_{k-m}}, \\ y_1 &= \frac{q + y_1}{p_0 + y_{-m}}, \\ &\vdots \\ y_k &= \frac{q + y_k}{p_{k-1} + y_{k-1-m}}. \end{aligned} \quad (3.17)$$

It follows from (3.16) and (3.17) that for any integer j ,

$$y_j = \frac{q}{p_{j-1} - 1 + y_{j-(m+1)}}. \quad (3.18)$$

Let $B_j = p_{j-1} - 1$, $a_1^j = q$, $b_1^j = 0$, $c_1^j = B_j$, $d_1^j = 1$. For any positive integer $n \geq 2$, let

$$\begin{aligned} a_n^j &= a_{n-1}^j B_{j-(n-1)(1+m)} + b_{n-1}^j q, \\ b_n^j &= a_{n-1}^j, \\ c_n^j &= c_{n-1}^j B_{j-(n-1)(1+m)} + d_{n-1}^j q, \\ d_n^j &= c_{n-1}^j. \end{aligned} \quad (3.19)$$

It is easy to verify that $a_n^j > 0$, $b_n^j > 0$, $c_n^j > 0$, $d_n^j > 0$ for any positive integer $n \geq 2$. By (3.18) we see that for any integer j ,

$$y_j = \frac{a_{k+1}^j + b_{k+1}^j y_{j-(k+1)(m+1)}}{c_{k+1}^j + d_{k+1}^j y_{j-(k+1)(m+1)}} = \frac{a_{k+1}^j + b_{k+1}^j y_j}{c_{k+1}^j + d_{k+1}^j y_j}, \quad (3.20)$$

from which it follows that the system (3.17) has a unique positive solution

$$y_j = \bar{y}_j = \frac{b_{k+1}^j - c_{k+1}^j + \sqrt{(b_{k+1}^j - c_{k+1}^j)^2 + 4a_{k+1}^j d_{k+1}^j}}{2d_{k+1}^j}, \quad j = 0, 1, \dots, k, \quad (3.21)$$

which implies that (H₅) holds for (3.15).

For any $0 \leq i \leq k$ and any integer j , write

$$y_{j(k+1)+i} = y_i, \quad Y_{j(k+1)+i} = Y_i. \quad (3.22)$$

Consider equations

$$\begin{aligned} Y_0 &= \frac{q + Y_0}{p_k + y_{k-m}}, \\ Y_1 &= \frac{q + Y_1}{p_0 + y_{-m}}, \\ &\vdots \\ Y_k &= \frac{q + Y_k}{p_{k-1} + y_{k-1-m}}, \\ y_0 &= \frac{q + y_0}{p_k + Y_{k-m}}, \\ y_1 &= \frac{q + y_1}{p_0 + Y_{-m}}, \\ &\vdots \\ y_k &= \frac{q + y_k}{p_{k-1} + Y_{k-1-m}}. \end{aligned} \quad (3.23)$$

It follows from (3.22) and (3.23) that for any integer j ,

$$y_j = \frac{q}{p_{j-1} - 1 + Y_{j-(1+m)}}, \quad Y_j = \frac{q}{p_{j-1} - 1 + y_{j-(1+m)}}. \quad (3.24)$$

That is,

$$Y_j = \frac{a_2^j + b_2^j Y_{j-2(1+m)}}{c_2^j + d_2^j Y_{j-2(1+m)}}, \quad y_j = \frac{a_2^j + b_2^j y_{j-2(1+m)}}{c_2^j + d_2^j y_{j-2(1+m)}}. \quad (3.25)$$

By (3.25) we see that for any integer j ,

$$\begin{aligned} y_j &= \frac{a_{2k+2}^j + b_{2k+2}^j y_{j-2(k+1)(1+m)}}{c_{2k+2}^j + d_{2k+2}^j y_{j-2(k+1)(1+m)}} = \frac{a_{2k+2}^j + b_{2k+2}^j y_j}{c_{2k+2}^j + d_{2k+2}^j y_j}, \\ Y_j &= \frac{a_{2k+2}^j + b_{2k+2}^j Y_{j-2(k+1)(1+m)}}{c_{2k+2}^j + d_{2k+2}^j Y_{j-2(k+1)(1+m)}} = \frac{a_{2k+2}^j + b_{2k+2}^j Y_j}{c_{2k+2}^j + d_{2k+2}^j Y_j}. \end{aligned} \quad (3.26)$$

Thus

$$Y_j = y_j = \frac{b_{2k+2}^j - c_{2k+2}^j + \sqrt{(b_{2k+2}^j - c_{2k+2}^j)^2 + 4a_{2k+2}^j d_{2k+2}^j}}{2d_{2k+2}^j}, \quad j = 0, 1, \dots, k, \quad (3.27)$$

from which with (3.17) and (3.23) we know that the system (3.23) has a unique positive solution

$$Y_j = y_j = \bar{y}_j, \quad j = 0, 1, \dots, k, \quad (3.28)$$

which implies that (L_2) holds for (3.15). It follows from Theorems 2.1–2.3 that every positive solution of (3.15) converges to the period $k + 1$ solution $\bar{y}_1, \dots, \bar{y}_k, \bar{y}_0, \bar{y}_1, \dots, \bar{y}_k, \bar{y}_0, \dots$ $\square$

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