We present a Razumilchin-type theorem for stochastic delay difference equation, and use it to investigate the mean square exponential stability of a kind of nonautonomous stochastic difference equation which may also be viewed as an approximation of a nonautonomous stochastic delay integro-differential equations (SDIDEs), and of a difference equation arises from some of the earliest mathematical models of the macroeconomic “trade cycle” with the environmental noise.

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1. Introduction

The problem of stability of stochastic difference equation has been investigated in a number of papers. We refer the readers to [2, 3, 13–17]. Some results on the asymptotic behavior of the moments were obtained in [18]. But very few results on the Razumilchin-type theorem for stochastic delay difference equation have been published. In this paper, we present a Razumilchin-type theorem for stochastic delay difference equation, and use it to investigate the mean square exponential stability of a kind of nonautonomous stochastic difference equation.

We consider the equation

\[ X_{n+1} = a(n, X_{n-m}, \ldots, X_n) + b(n, X_{n-m}, \ldots, X_n) \triangle \mu_n, \quad n \in \mathbb{Z}^+, \]

where \( a \in \mathbb{R}, b \in \mathbb{R}, a(n, 0, \ldots, 0) = b(n, 0, \ldots, 0) = 0, \mathbb{Z}^+ = \{0, 1, \ldots\} \), and \( X_n = \xi_n \) \( (n \in I = \{-m, -m + 1, \ldots, 1, 0\}) \) is the initial segment to be \( \mathcal{F}_0 \)-measurable. \( \triangle \mu_n \) are independent \( N(0,1) \)-distributed Gaussian random variables.

We denote by \( (\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n \in \mathbb{N}}, \mathbb{P}) \) a complete filtered probability space, where filtration \( \{\mathcal{F}_n\}_{n \in \mathbb{N}} \) is naturally generated: \( \mathcal{F}_{n+1} = \sigma \{\triangle \mu_{i+1} : i = 0, 1, \ldots, n\} \). Among all the sequences \( \{X_n\}_{n \in \mathbb{N}} \) of the random variables, we distinguish those for which \( X_n \) are \( \mathcal{F}_n \)-measurable for all \( n \in \mathbb{N} \).
2 Exponential stability of stochastic difference equations

2. Main result

Definition 2.1. The stochastic difference equation (1.1) is said to be $p$th moment exponentially stable if there are positive constants $\gamma$ and $N$ such that with initial data $\xi_n$, $n \in I$,

$$E|X_n|^p \leq NE\|\xi\|^p e^{-\gamma n} \quad \text{on } n \in \mathbb{Z}^+,$$

(2.1)

where $\|\xi\| = \max_{n \in I} |\xi_n|$.

Theorem 2.2. Let all $\lambda$, $p$, $c_1$, $c_2$ be positive numbers and $q > 1$, suppose there exists a Lyapunov function $V : I \cup \mathbb{Z}^+ \times \mathbb{R} \rightarrow \mathbb{R}^+$ such that

$$c_1|X|^p \leq V(t,X) \leq c_2|X|^p \quad \forall (n,X) \in I \cup \mathbb{Z}^+ \times \mathbb{R}$$

(2.2)

and for all $n \in \mathbb{Z}^+$,

$$E\Delta V(n,X_n) = E(V(n+1,X_{n+1}) - V(n,X_n)) \leq -\lambda EV(n+1,X_{n+1})$$

(2.3)

if

$$EV(n+s,X_{n+s}) < qEV(n+1,X_{n+1}) \quad \text{on } s \in I.$$

(2.4)

Then for all $\xi_n$, $n \in I$, $\|\xi\| = \max_{n \in I} |\xi_n|$,

$$E|X_n|^p \leq \frac{c_2}{c_1}E\|\xi\|^p e^{-\gamma n} \quad \text{on } n \in \mathbb{Z}^+,$$

(2.5)

where $0 < \gamma = \min\{\log(1+\lambda), \log q/(1+m)\}$.

Proof. Let

$$U(n) = \sup_{s \in I} \{e^{\gamma(n+s)}EV(n+s,X_{n+s})\}, \quad n \in \mathbb{Z}^+.$$  

(2.6)

For any $n \in \mathbb{Z}^+$, we affirm that

$$\Delta U(n) = U(n+1) - U(n) \leq 0.$$  

(2.7)

Otherwise, there exists an $n \in \mathbb{Z}^+$ such that

$$e^{\gamma(n+1)}EV(n+1,X_{n+1}) > U(n),$$

(2.8)

that is,

$$e^{\gamma(n+1)}EV(n+1,X_{n+1}) > e^{\gamma(n+s)}EV(n+s,X_{n+s}), \quad s \in I.$$  

(2.9)

For any $s \in I$, the inequality (2.4) implies

$$EV(n+s,X_{n+s}) < e^{\gamma(m+1)}EV(n+1,X_{n+1}) < qEV(n+1,X_{n+1})$$  

(2.10)
since $\gamma < \log q/(1 + m)$. Thus, by condition (2.3),

$$E\Delta V(n, X_n) = E(V(n + 1, X_{n+1}) - V(n, X_n)) \leq -\lambda EV(n + 1, X_{n+1}).$$

(2.11)

We obtain that

$$EV(n + 1, X_{n+1}) \leq \frac{1}{1 + \lambda} EV(n, X_n).$$

(2.12)

Multiplying both sides of above inequality by $e^{\gamma(n+1)}$ and noting that $\gamma \leq \log(1 + \lambda)$, we can get

$$e^{\gamma(n+1)} EV(n + 1, X_{n+1}) \leq e^{\gamma(n+1)} \frac{1}{1 + \lambda} EV(n, X_n) \leq e^{\gamma n} EV(n, X_n).$$

(2.13)

From the definition of $U(n)$, we have

$$e^{\gamma(n+1)} EV(n + 1, X_{n+1}) \leq U(n).$$

(2.14)

It is a contradiction with assumption (2.8), therefore, (2.7) holds. Thus, it follows from (2.7) immediately that

$$U(n) \leq U(0), \quad \forall n \in \mathbb{Z}^+.$$  

(2.15)

From the definition of $U(n)$ and the condition (2.2), we obtain that the inequality (2.5) holds.

\[\square\]

3. Application

3.1. A numerical approximation to SDIDEs. We consider the stochastic delay difference equation

$$X_{n+1} = X_n + h\left[a(n)X_n + h \sum_{i=1}^{m} K(n,i)X_{n-i}\right] + \sqrt{h}c(n)X_{n-m} \Delta \mu_n, \quad n \in \mathbb{Z}^+,$$

where $h > 0$ is a nonrandom parameter. For the functions $a(\cdot), b(\cdot), K(\cdot, \cdot)$, suppose that

$$A(m, n, h) = 2a(n) + a^2(n)h + m \left| 1 + a(n)h \right| h^2,$$

$$B(m, n, h) = c^2(n) + h(mh^2 + \left| 1 + a(n)h \right|) \sum_{i=1}^{m} K^2(n, i)$$

(3.2)
satisfy the conditions (H1) and (H2) as the following:

(H1) $A(m,n,h) + B(m,n,h) \leq -\delta(m,h) < 0$ for $n \in \mathbb{Z}^+$;

(H2) $-1 < h\mu(m,h) \leq hA(m,n,h)$.

Let

$$q(m,h) = \inf_{n \in \mathbb{Z}^+} \left\{ \frac{B(m,nh) - A(m,n,h)}{2B(m,n,h)} \right\},$$

$$\lambda(m,h) = -\inf_{n \in \mathbb{Z}^+} \left\{ \frac{A(m,n,h) + B(m,n,h)}{2(1 + hA(m,n,h))} \right\}.$$

From the conditions (H1) and (H2), we know that

$$q(m,h) > 1, \quad \lambda(m,h) > 0. \quad (3.4)$$

Equation (3.1) may also be viewed as an approximation of the stochastic delay integrodifferential equation

$$dX(t) = \left[ a(t)X(t) + \int_{t-\tau}^{t} K(t,t-\theta)X(\theta)d\theta \right]dt + c(t)X(t-\tau)dW(t), \quad t > 0,$$

$$X(t) = \xi(t), \quad t \in [-\tau,0], \quad (3.5)$$

where $W(t)$ is a standard Brownian motion. Here, setting $h = \tau/m$ and approximating the differential part of (3.5) with the Euler-Maruyama method and the integral part with composite left-side rectangle rule [12], $t_n = nh$, write $X_n$ for an approximate value to $X(nh)$, and use $X_{n-m}$ to approximate the delayed argument $X(t_n - \tau)$. When $n \in I = \{-m,-m+1,\ldots,-1,0\}$, we have $X_n = \xi(t_n)$. Moreover, the increments $\sqrt{h}\triangle \mu_n := W(t_{n+1}) - W(t_n)$ are independent $N(0,h)$-distributed Gaussian random variables, so $\triangle \mu_n$ are independent $N(0,1)$-distributed Gaussian random variables. We assume $X_n$ to be $\mathcal{F}_n$-measurable at the mesh-points $t_n$. It is therefore to be hoped for $h$ sufficiently small that solutions of (3.1) have similar asymptotic properties to those of (3.5). A statement of these asymptotic results for stochastic delay differential equations can be found in, for example, [1, 4–11].

Here, we use the above Razumilchin-type Theorem 2.2 to study the moment exponential stability of (3.1).

**Theorem 3.1.** Assume conditions (H1) and (H2) are satisfied, the solution sequence $\{X_n\}$ produced by the difference equation (3.1) satisfies

$$E|X_n|^2 \leq E\|\xi\|^2 e^{-\gamma(m,h)nh} \quad \text{on} \quad n \in \mathbb{Z}^+, \quad (3.6)$$

where $0 < \gamma(m,h) = \min\{\log(1 + h\lambda(m,h))/h, \log q(m,h)/h(1 + m)\}$. 

Proof. Define a Lyapunov function $V(n,X) = |X|^2$. Clearly, the condition (2.2) in Theorem 2.2 is satisfied naturally with $c_1 = c_2 = 1, p = 2$. We see from (3.1) that

$$
X_{n+1}^2 = (1 + a(n)h)^2 X_n^2 + h^4 \sum_{i=1}^{m} K(n,i)X_{n-i}^2 + (\sqrt{hc}(n) \triangle \mu_n)^2 X_{n-m}^2
$$

$$
+ 2(1 + a(n)h) h^2 \sum_{i=1}^{m} K(n,i)X_{n-i}X_n + 2(1 + a(n)h) \sqrt{hc}(n) \triangle \mu_n X_n X_{n-m}
$$

$$
+ 2h^2 \sqrt{hc}(n) \triangle \mu_n \sum_{i=1}^{m} K(n,i)X_{n-i}X_{n-m}
$$

$$
\leq (1 + a(n)h)^2 X_n^2 + mh^4 \sum_{i=1}^{m} K^2(n,i)X_{n-i}^2 + (\sqrt{hc}(n) \triangle \mu_n)^2 X_{n-m}^2
$$

$$
+ |1 + a(n)h|h^2 \sum_{i=1}^{m} (K^2(n,i)X_{n-i}^2 + X_n^2) + 2(1 + a(n)h) \sqrt{hc}(n) \triangle \mu_n X_n X_{n-m}
$$

$$
+ 2h^2 \sqrt{hc}(n) \triangle \mu_n \sum_{i=1}^{m} K(n,i)X_{n-i}X_{n-m}
$$

$$
= [(1 + a(n)h)^2 + m|1 + a(n)h|h^2] X_n^2
$$

$$
+ (\sqrt{hc}(n) \triangle \mu_n)^2 X_{n-m}^2 + \sum_{i=1}^{m} [mh^4 + |1 + a(n)h|h^2] K^2(n,i)X_{n-i}^2
$$

$$
+ 2(1 + a(n)h) \sqrt{hc}(n) \triangle \mu_n X_n X_{n-m} + 2h^2 \sqrt{hc}(n) \triangle \mu_n \sum_{i=1}^{m} K(n,i)X_{n-i}X_{n-m}.
$$

(3.7)

Note that $E(\triangle \mu_n) = 0, E[(\triangle \mu_n)^2] = 1$ and $X_n, X_{n-k}$ are $\mathcal{F}_n$-measurable, Hence

$$
E(\triangle \mu_n X_{n-k} X_{n-m}) = E(X_{n-k}X_{n-m}E(\triangle \mu_n | \mathcal{F}_n)) = 0,
$$

$$
E(\triangle \mu_n X_{n-m}^2) = E(X_{n-m}^2E(\triangle \mu_n | \mathcal{F}_n)) = 0,
$$

$$
E(\triangle \mu_n^2 X_{n-m}^2) = E(X_{n-m}^2E(\triangle \mu_n^2 | \mathcal{F}_n)) = E(X_{n-m}^2).
$$

(3.8)

From (3.7), (3.8) we get that

$$
EV(n+1, X_{n+1}) \leq [(1 + a(n)h)^2 + m|1 + a(n)h|h^2] EV(n, X_n)
$$

$$
+ (\sqrt{hc}(n))^2 EV(n-m, X_{n-m})
$$

$$
+ h^2 (mh^2 + |1 + a(n)h|) \sum_{i=1}^{m} K^2(n,i)EV(n-i, X_{n-i}).
$$

(3.9)
Let $q = q(m, h)$, from assumption (2.4), we also have

$$
EV(n + 1, X_{n+1}) \leq \left[ (1 + a(n)h)^2 + m \| 1 + a(n)h \| h^2 \right] EV(n, X_n)
$$

$$
+ hq(m, h) \left[ c^2(n) + h(mh^2 + \| 1 + a(n)h \|) \sum_{i=1}^{m} K^2(n, i) \right] EV(n + 1, X_{n+1})
$$

$$
= (1 + hA(m, n, h)) EV(n, X_n) + hq(m, h)B(m, n, h)EV(n + 1, X_{n+1})
$$

$$
\leq (1 + hA(m, n, h)) EV(n, X_n) + h \frac{B(m, n, h) - A(m, n, h)}{2} EV(n + 1, X_{n+1}),
$$

(3.10)

that is,

$$
E \triangle V(n, X_n) \leq h \frac{B(m, n, h) - A(m, n, h)}{2(1 + hA(m, n, h))} EV(n + 1, X_{n+1}).
$$

(3.11)

By the definition of $\lambda(m, h)$ and (3.4), we get

$$
E \triangle V(n, X_n) \leq -h\lambda(m, h)EV(n + 1, X_{n+1}).
$$

(3.12)

Therefore, the inequality (3.6) holds by Theorem 2.2.

\[\square\]

3.2. Models of macroeconomics. Consider the following nonlinear delay difference equation:

$$
x(n + 1) = cx(n) + f(x(n) - x(n - m)) + \epsilon x(n) \triangle \mu_n,
$$

(3.13)

where $c \in [0, 1)$ and $\epsilon$ are constants, $m$ is a positive integer, $\triangle \mu_n$ are independent $N(0, 1)$-distributed Gaussian random variables. We assume that $x(n)$ are $\mathcal{F}_n$-measurable for all $n \in \mathbb{N}$, and we have $x(n) = \xi_n$ when $n \in I$. $f: \mathbb{R} \to \mathbb{R}$ satisfies $f(0) = 0$, $f(u) \neq 0$ for $u \neq 0$, and there exists a constant $\alpha$ such that

$$
| f(u) | \leq \alpha | u |.
$$

(3.14)

Such equation arises from some of the earliest mathematical models of the macroeconomic “trade cycle” with the environmental noise.

Theorem 3.2. Assume that the conditions (3.14) and

$$
0 \leq c < -2\alpha + \sqrt{2\alpha^2 - \epsilon^2 - 2\alpha + 1}
$$

(3.15)

are satisfied. Then there exists positive constants $\gamma$ such that with initial data $\xi_n$, $n \in I$,

$$
E | X(n) |^2 \leq E \| \xi \|^2 e^{-\gamma n} \quad \text{on } n \in \mathbb{Z}^+.
$$

(3.16)
Proof. Also define a Lyapunov function \( V(n,x) = |x|^2 \). Similar to the proof of Theorem 3.1, let

\[
q = \frac{1 - c^2 - 2ac - 2\alpha^2 - \epsilon^2 + 2\alpha}{2\alpha(2 + c)} ,
\]

\[
\lambda = \frac{[2(2 + c)\alpha - 1](c^2 + 2\alpha^2 + \epsilon^2 + 3ca) + 1}{2(2 + c)\alpha(2\alpha^2 + \epsilon^2 + 3ca)} ,
\]

the inequality (3.16) with \( \gamma = \min\{\log(1 + \lambda), \log q/(1 + k)\} \) can be completed by Theorem 2.2 easily.

\( \Box \)

4. Examples

4.1. A numerical approximation to autonomous SDIDEs. Consider the autonomous stochastic delay integro-differential equation as (3.5),

\[
dX(t) = \left[aX(t) + b\int_{t-\tau}^t X(\theta)d\theta \right]dt + \sqrt{h}cX(t-\tau)dW(t), \quad t > 0, \\
X(t) = \xi(t), \quad t \in [-\tau,0].
\]

(4.1)

Approximating the differential part of (3.5) with the Euler-Maruyama method and the integral part with composite left-side rectangle rule, we get the difference equation as follows:

\[
X_{n+1} = X_n + \left(aX_n + bh \sum_{k=1}^m X_{n-k} \right)h + \sqrt{h}cX_{n-m} \Delta \mu_n, \quad n \in \mathbb{Z}^+. 
\]

(4.2)

Here, \( h = \tau/m \) and

\[
A(m,n,h) = 2a + a^2 h + \tau h |1 + ah| , \\
B(m,n,h) = c^2 + b^2 \tau (\tau h + |1 + ah|). 
\]

(4.3)

Since

\[
1 + hA(m,n,h) = (1 + ah)^2 + \tau h |1 + ah| > 0, 
\]

(4.4)

so that the (H2) is satisfied, and if

\[
2a + c^2 + b^2 \tau + (a^2 + \tau + b^2 \tau^2 - ab^2 \tau) h - \alpha h^2 < 0, 
\]

(4.5)

then (H1) also is satisfied.

By letting

\[
q(m,h) = \frac{B(m,nh) - A(m,n,h)}{2B(m,n,h)} , \\
\lambda(m,h) = -\frac{A(m,n,h) + B(m,n,h)}{2(1 + hA(m,n,h))}. 
\]

(4.6)

From Theorem 3.1 we know that the inequality (3.6) holds.
From the above analysis and (4.5), we can get the theorem as follows.

**Theorem 4.1.** Assume condition $2a + c^2 + b^2\tau < 0$ is satisfied, then there exists an $h^* > 0$ such that for all $0 < h < h^*$, the solution sequence $\{X_n\}$ produced by (4.2) satisfies

$$E|X_n|^2 \leq E\|\xi\|^2 e^{-\gamma(m,h)nh} \quad \text{on } n \in \mathbb{Z}^+,$$

where $0 < \gamma(h) = \min\{\log(1 + \lambda h), \log q/(h + \tau)\}$, and

$$h^* = \frac{a^2 + \tau + b^2\tau^2 - ab\tau - \sqrt{(a^2 + \tau + b^2\tau^2 - ab\tau)^2 + 4a\tau(2a + c^2 + b^2\tau)}}{2a\tau}.$$  (4.8)

**Corollary 4.2.** Assume condition $2a + c^2 + b^2\tau < 0$ is satisfied, then for any given $\varepsilon > 0$, there exists an $h(\varepsilon) > 0$ such that for all $0 < h < h(\varepsilon)$, the numerical solution sequence $\{X_n\}$ produced by the numerical scheme (4.2) satisfies

$$E|X_n|^2 \leq E\|\xi\|^2 e^{-(\gamma - \varepsilon)nh} \quad \text{on } n \in \mathbb{Z}^+,$$

where $\gamma = \min\{\lambda, \log q/\tau\}$ with $\lambda = -(a + (1/2)(c^2 + b^2\tau))$ and $q = 1/2 - a/(c^2 + b^2\tau)$.

**Proof.** From Theorem 4.1, the corollary is a consequence of the fact that

$$\frac{\log(1 + h\lambda(h))}{h} = \lambda + O(h),$$  \quad (4.10)

$$\frac{\log q(h)}{(h + \tau)} = \frac{\log q}{\tau} + O(h).$$

\[]

**4.2. An example of macroeconomics models.** Consider the difference equation

$$x(n + 1) = cx(n) + 0.25 \sin(x(n) - x(n - 3)) + 0.25x(n)\Delta \mu_n,$$  (4.11)

we can get that if the condition

$$0 \leq c \leq 0.25$$  (4.12)

is satisfied, then the inequality (3.6) holds, with $\gamma = \min\{\log(1 + \lambda), \log q/4\}$. Here

$$\lambda = \frac{21/8 - 2c^2 - c}{2 + c},$$  \quad (4.13)

$$q = \frac{2 + c(c^2 + (3/4)c + 3/16)}{(2 + c)(c^2 + (3/4)c + 3/16)}.$$

**Acknowledgment**

This work is supported by the NSF of China (no.10271036) and of HIT(200518).
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