We show that the difference equation \( x_n = f_3(x_{n-1}) f_2(x_{n-2}) f_1(x_{n-3}), \ n \in \mathbb{N}_0, \) where \( f_i \in C([0,\infty), (0,\infty]), i \in \{1,2,3\}, \) is periodic with period 4 if and only if \( f_i(x) = c_i/x \) for some positive constants \( c_i, i \in \{1,2,3\} \) or if \( f_i(x) = c_i/x \) when \( i = 2 \) and \( f_i(x) = c_i x \) if \( i \in \{1,3\}, \) with \( c_1 c_2 c_3 = 1. \) Also, we prove that the difference equation \( x_n = f_4(x_{n-1}) f_3(x_{n-2}) f_2(x_{n-3}) f_1(x_{n-4}), \ n \in \mathbb{N}_0, \) where \( f_i \in C([0,\infty), (0,\infty]), i \in \{1,2,3,4\}, \) is periodic with period 5 if and only if \( f_i(x) = c_i/x, \) for some positive constants \( c_i, i \in \{1,2,3,4\}. \)

Copyright © 2007 Stevo Stević. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

1. Introduction

The study of the periodic character of solutions of rational and nonlinear difference equations has recently attracted attention; see, for example, [1–12] and the references therein. For some classical results see [13–17].

**Definition 1.1.** Let \( f \) be a real valued function defined on a subset of \( \mathbb{R}^n. \) Say that the difference equation

\[
x_n = f(x_{n-1}, \ldots, x_{n-k}), \quad n \in \mathbb{N}_0,
\]

where \( k \in \mathbb{N}, \) is periodic if every solution of (1.1) is periodic.

It is easy to see that every solution of the difference equation

\[
x_n = \frac{C}{x_{n-1} x_{n-2} \cdots x_{n-k}}, \quad n \in \mathbb{N}_0,
\]

is periodic with period \((k + 1).\)
We also know that every solution of the difference equation

\[ x_n = \frac{x_{n-1} \cdots x_{n-(2k+1)}}{x_{n-2} \cdots x_{n-2k}}, \quad n \in \mathbb{N}_0, \quad (1.3) \]

is periodic with period \(2k + 2\). Indeed, from (1.3) it follows that

\[ x_n x_{n-2} \cdots x_{n-2k} = 1, \quad n \in \mathbb{N}_0. \quad (1.4) \]

Using the changes \( y_n = x_{2n} \) and \( z_n = x_{2n+1} \), the last equation is reduced to (1.2), from which the statement follows.

In [18] we studied the global periodicity of (1.1) with \( k = 2 \). Among other results it was proved that if \( f \) separates the variables, that is, if

\[ f(x, y) = f_2(x) f_1(y), \quad (1.5) \]

then every solution of (1.1) is periodic with period 3 if and only if \( f(x, y) = c/xy \) where \( c \) is a positive constant.

Motivated by the method used in paper [18], in this paper we investigate the global periodicity of the positive solutions of the difference equation

\[ x_n = f_k(x_{n-1}) \cdots f_1(x_{n-k}), \quad n \in \mathbb{N}_0, \quad (1.6) \]

where \( k \in \{3,4\} \), \( f_i \in C((0, \infty), (0, \infty)) \), \( i = 1, \ldots, k \).

We prove the following result.

**Theorem 1.2.** Consider (1.6). Then the following statements hold true.

(a) Assume that \( k = 3 \). Then, every positive solution of (1.6) is periodic with period 4 if and only if \( f_i(x) = c_i/x \) for some positive constants \( c_i, i \in \{1,2,3\} \), or if \( f_i(x) = c_i/x \) when \( i = 2 \) and \( f_i(x) = c_i x \) if \( i \in \{1,3\} \), with \( c_1 c_2 c_3 = 1 \).

(b) Assume that \( k = 4 \). Then, every positive solution of (1.6) is periodic with period 5 if and only if \( f_i(x) = c_i/x \) for some positive constants \( c_i, i \in \{1,2,3,4\} \).

2. Auxiliary results

Before we give a proof of Theorem 1.2, we will prove some auxiliary results which are incorporated in the following lemmas. We say that for a mapping \( f : X \to X \), \( (f^p)_p \in \mathbb{N} \cup \{0\} \) denotes the sequence of iterates of \( f \), that is, \( f^{[0]} = I \), the identity function on \( X \), \( f^{[1]} = f \) and generally \( f^{[p+1]} = f \circ f^{[p]} \) for any \( p \in \mathbb{N} \).
The following lemma is folklore and can be found, for example, in [19] (see also [20]). We give a proof of the lemma for the benefit of the reader.

**Lemma 2.1.** Assume that $f : I \to I$ is a continuous function on the open (or closed) interval $I \subset \mathbb{R}$ satisfying the equation

$$f^{[p]}(x) = x, \quad x \in I,$$

for some $p \in \mathbb{N}$. Then $f(x) \equiv x, x \in I$ or $f^{[2]}(x) = x$.

**Proof.** Assume that $f \in C[I,I]$ is such that $f^{[p]}(x) = x$ for every $x \in I$. Then, if $f(x) = f(y)$, it follows that

$$x = f^{[p]}(x) = f^{[p]}(y) = y$$

which implies that the function $f$ must be $1 - 1$. Since $f$ is a continuous function, we have that $f$ must be strictly monotone.

First assume that $f$ is strictly increasing. If there is a point $x_0 \in I$ such that $x_0 < f(x_0)$, then by the monotonicity of $f$ we have

$$x_0 < f(x_0) < f^{[2]}(x_0) < \cdots < f^{[p]}(x_0) = x_0$$

which is a contradiction. If $x_0 > f(x_0)$, then we have

$$x_0 > f(x_0) > f^{[2]}(x_0) > \cdots > f^{[p]}(x_0) = x_0$$

arriving again at a contradiction.

From this it follows that $f(x) = x$ for every $x \in I$.

Assume now that $f$ is strictly decreasing. Then the function $g(x) = f^{[2]}(x)$ is strictly increasing and according to the first case we have that

$$g^{[p]}(x) = (f^{[2]})^{[p]}(x) = (f^{[p]})^{[2]}(x) = x \circ x = x,$$

that is, $f^{[2]}(x) \equiv x$, finishing the proof of the lemma. $\square$

**Lemma 2.2.** Assume that $f$ is a decreasing continuous function which maps the interval $(0, \infty)$ into itself, and satisfies the following conditions

$$\lim_{z \to +0} f(z) = \infty, \quad \lim_{z \to +\infty} f(z) = 0,$$

$$f(z) f\left(\frac{1}{z}\right) = 1, \quad z \in (0, \infty),$$

$$f(z) = f^{-1}(z) \quad z \in (0, \infty).$$

Then $f(z) = 1/z$. 
Proof. Assume that \( f(z) \neq 1/z, z \in (0, \infty) \), then there is a \( z_0 \in (0, \infty) \) such that \( f(z_0) > 1/z_0 \) or \( f(z_0) < 1/z_0 \). From (2.6) and positivity of the function \( f \) it follows that \( f(1) = 1 \). Hence \( z_0 \neq 1 \).

First, assume that \( f(z_0) > 1/z_0 \) and \( z_0 < 1 \). From this and (2.6) it follows that

\[
f\left(\frac{1}{z_0}\right) = \frac{1}{f(z_0)} < z_0 < 1 < \frac{1}{z_0} < f(z_0).
\]  

On the other hand, the point \((f(1/z_0), 1/z_0)\) belongs to the graph of the curve \( y = f(z) \), since \( f \) is self-invertible. Hence the points \((f(1/z_0), 1/z_0), (1, 1)\), and \((z_0, f(z_0))\) belong to the graph of the curve \( y = f(z) \). We know that \( f \) is decreasing and from (2.8) we have \( f(1/z_0) < z_0 < 1 \), thus we obtain \( f(f(1/z_0)) > f(z_0) > f(1) \), that is, \( 1/z_0 > f(z_0) > 1 \). The last statement contradicts (2.8).

Now, assume that \( f(z_0) > 1/z_0, 1 < z_0, \) and \( f(z_0) < 1 \). Note that the points \((1/z_0, f(1/z_0)), (1, 1)\), and \((f(z_0), z_0)\) are on the graph of \( f \). Since \( 1/z_0 < f(z_0) < 1 \) and \( f \) is decreasing it follows that \( f(1/z_0) > f(f(z_0)) > f(1) \), that is, \( 1/z_0 > f(z_0) > 1 \), which is a contradiction.

Assume that \( f(z_0) > 1/z_0, 1 < z_0 \) and \( f(z_0) > 1 \). In the case the points \((z_0, f(z_0)), (1, 1)\) and \((f(z_0), z_0)\) are on the graph of \( f \). If \( 1 < z_0 < f(z_0) \), then we obtain that \( 1 > f(z_0) > z_0 \), a contradiction. If \( 1 < f(z_0) \leq z_0 \), then it follows that \( 1 > z_0 \geq f(z_0) \), which is again a contradiction.

The case \( f(z_0) < 1/z_0 \) can be treated similarly so we omit the proof of this part of the lemma. \( \square \)

3. Proof of the main result

In this section we give a proof of Theorem 1.2. Before this we present some formulae which are of some interest not only for these two cases in Theorem 1.2, but also for all \( k \geq 3 \).

Hence, assume that all positive solutions of (1.1) are periodic with period \((k + 1)\). Then for every \( x_1, \ldots, x_k \in (0, \infty) \) we have that the following system of functional relationships holds:

\[
\begin{align*}
  u &= f_k(x_k) f_{k-1}(x_{k-1}) \cdots f_2(x_2) f_1(x_1), \\
  x_1 &= f_k(u) f_{k-1}(x_k) \cdots f_2(x_3) f_1(x_2), \\
  x_2 &= f_k(x_1) f_{k-1}(u) \cdots f_2(x_4) f_1(x_3), \\
  &\vdots \\
  x_k &= f_k(x_{k-1}) f_{k-1}(x_{k-2}) \cdots f_2(x_1) f_1(u).
\end{align*}
\]
From (3.1) it follows that

\[ x_1 = f_k \left( \prod_{j=1}^{k} f_j(x_j) \right) f_{k-1}(x_k) \cdots f_2(x_3) f_1(x_2), \]
\[ x_2 = f_k(x_1) f_{k-1} \left( \prod_{j=1}^{k} f_j(x_j) \right) \cdots f_2(x_4) f_1(x_3), \]
\[ x_3 = f_k(x_2) f_{k-1}(x_1) f_{k-2} \left( \prod_{j=1}^{k} f_j(x_j) \right) \cdots f_2(x_5) f_1(x_4), \]
\[ \vdots \]
\[ x_{k-1} = f_k(x_{k-2}) f_{k-1}(x_{k-3}) \cdots f_2 \left( \prod_{j=1}^{k} f_j(x_j) \right) f_1(x_k), \]
\[ x_k = f_k(x_{k-1}) f_{k-1}(x_{k-2}) \cdots f_2(x_1) f_1 \left( \prod_{j=1}^{k} f_j(x_j) \right). \]  

(3.2)

In each of the \( k \) equations in (3.2) we choose that all variables, except the \( j \)th which is arbitrary, are equal to 1, and use the changes

\[ g_j(x) = f_j(x) \prod_{i=1, i \neq j}^{k} f_i(1), \quad j = 1, \ldots, k. \]  

(3.3)

Then, we obtain

\[ g_k(g_1(z)) = z, \quad g_k(g_j(z))g_{j-1}(z) = C, \quad 2 \leq j \leq k; \]
\[ g_{k-1}(g_1(z))g_k(z) = C, \quad g_{k-1}(g_2(z)) = z, \quad g_{k-1}(g_j(z))g_{j-2}(z) = C, \quad 3 \leq j \leq k; \]
\[ g_{k-2}(g_j(z))g_{j+k-2}(z) = C, \quad j = 1, 2, \]
\[ g_{k-2}(g_3(z)) = z, \quad g_{k-2}(g_j(z))g_{j-3}(z) = C, \quad 4 \leq j \leq k; \]
\[ \vdots \]
\[ g_2(g_j(z))g_{j+2}(z) = C, \quad 1 \leq j \leq k-2, \quad g_2(g_{k-1}(z)) = z, \quad g_2(g_k(z))g_1(z) = z, \]
\[ g_1(g_j(z))g_{j+1}(z) = C, \quad 1 \leq j \leq k-1, \quad g_1(g_k(z)) = z, \]

(3.4)

where \( C = \prod_{i=1}^{k} f_i(1) \).

From (3.4) it follows that

\[ g_j \circ g_{k+1-j}(z) = z, \quad j = 1, \ldots, k, \]
\[ g_j \circ g_i(z) = g_i \circ g_j(z), \]

(3.5)

(3.6)
if $i \neq j$ and $i + j \neq k + 1$, and

$$g_k^2(z)g_{k-1}(z) = C, \quad g_{k-1}^2(z)g_{k-2}(z) = C, \quad g_{k-2}^2(z)g_{k-3}(z) = C,$$

\[ \vdots \]

$$g_2^2(z)g_4(z) = C, \quad g_1^2(z)g_2(z) = C. \quad (3.7)$$

**Proof of Theorem 1.2.** The sufficiency part of the theorem follows from (1.3) and (1.2). Hence, we need only prove the necessity.

First, assume that $k = 3$. Then (3.5)–(3.7) are

$$g_3(g_1(z)) = z, \quad g_3(g_2(z))g_1(z) = C, \quad g_3(g_3(z))g_2(z) = C,$$

$$g_2(g_1(z))g_3(z) = C, \quad g_2(g_2(z)) = z, \quad g_2(g_3(z))g_1(z) = C,$$

$$g_1(g_1(z))g_2(z) = C, \quad g_1(g_2(z))g_3(z) = C, \quad g_1(g_3(z)) = z. \quad (3.8)$$

From (3.8) we have

$$g_1(g_3(z)) = g_3(g_1(z)) = z, \quad g_2(g_2(z)) = z, \quad (3.9)$$

which implies that

$$g_3(z) = g_1^{-1}(z), \quad g_2(z) = g_2^{-1}(z), \quad (3.10)$$

and that the functions $g_1, g_2, g_3$ map the interval $(0, \infty)$, “$1 − 1$” and onto itself.

Further, from the third and seventh identity in (3.8) we have that

$$g_1(g_1(z)) = g_3(g_3(z)). \quad (3.11)$$

From (3.10) and (3.11) it follows that

$$g_1^{[4]}(z) = z. \quad (3.12)$$

Lemma 2.1 implies that

$$g_1(z) = z \quad \text{or} \quad g_1^{[2]}(z) = z. \quad (3.13)$$

If $g_1(z) = z$, then (3.10) implies $g_3(z) = z$, from this and the second identity in (3.8) we obtain that $g_2(z) = C/z$. Hence, the equation becomes

$$x_n = C \frac{x_{n-1}x_{n-3}}{x_{n-2}}. \quad (3.14)$$

By some simple calculations it is shown that $C$ must be equal to 1 in order that all solutions of the equation are periodic with period four, from which the result follows in this case.

If $g_1^{[2]}(z) = z$, then

$$g_1(z) = g_1^{-1}(z). \quad (3.15)$$
Equations (3.10) and (3.15) imply that $g_1 = g_3$. From this and the sixth identity in (3.8) it follows that

$$g_2(g_1(z)) = \frac{C}{g_1(z)}$$

(3.16)

and by the change $g_1(z) \to z$, we have that

$$g_2(z) = \frac{C}{z}.$$  

(3.17)

Substituting (3.17) into the eighth identity in (3.8) we obtain

$$g_1(z)g_1\left(\frac{C}{z}\right) = C.$$  

(3.18)

Using the change $h_1(z) = (1/\sqrt{C})g_1(\sqrt{C}z)$ we see that the function $h_1$ satisfies the following relationships:

$$h_1(z)h_1\left(\frac{1}{z}\right) = 1, \quad h_1(z) = h_1^{-1}(z).$$  

(3.19)

From this we see that the function $h_1$ satisfies the conditions of Lemma 2.2, which implies that $h_1(z) = 1/z$. Hence $g_1(z) = C/z$ and consequently

$$g_3(z) = \frac{C}{z}, \quad g_2(z) = \frac{C}{z},$$  

(3.20)

form which the result follows.

Assume now that $k = 4$. Then (3.5)–(3.7) are

$$g_4(g_1(z)) = z, \quad g_4(g_2(z))g_1(z) = C, \quad g_4(g_3(z))g_2(z) = C, \quad g_4(g_4(z))g_3(z) = C,$$

$$g_3(g_1(z))g_4(z) = C, \quad g_3(g_2(z)) = z, \quad g_3(g_3(z))g_1(z) = C, \quad g_3(g_4(z))g_2(z) = C,$$

$$g_2(g_1(z))g_3(z) = C, \quad g_2(g_2(z))g_4(z) = C, \quad g_2(g_3(z)) = z, \quad g_2(g_4(z))g_1(z) = C,$$

$$g_1(g_1(z))g_2(z) = C, \quad g_1(g_2(z))g_3(z) = C, \quad g_1(g_3(z))g_4(z) = C, \quad g_1(g_4(z)) = z.$$  

(3.21)

From (3.21) we have

$$g_1(g_4(z)) = g_4(g_1(z)) = z, \quad g_2(g_3(z)) = g_3(g_2(z)) = z,$$  

(3.22)

which implies

$$g_4(z) = g_1^{-1}(z), \quad g_3(z) = g_2^{-1}(z),$$  

(3.23)
and consequently that the functions \( g_1, g_2, g_3, \) and \( g_4 \) map the interval \((0, \infty), \) “1–1” and onto itself. Also, we have

\[
\begin{align*}
g_i(g_j(z)) &= g_j(g_i(z)), \quad \text{when } i + j \neq 5, \ i \neq j, \\
g_4(g_4(z))g_3(z) &= C, \quad g_3(g_3(z))g_1(z) = C, \\
g_2(g_2(z))g_4(z) &= C, \quad g_1(g_1(z))g_2(z) = C.
\end{align*}
\]

From (3.24) and (3.25), it follows that

\[
g_4^{[2]} \circ g_2(z) = g_3^{[2]} \circ g_4(z) = g_1^{[2]} \circ g_4(z) = g_3^{[2]} \circ g_4(z) = \frac{C}{z}. \tag{3.26}
\]

From (3.24) and (3.26), it follows that

\[
g_3^{[3]}(z) = g_4(z), \quad g_4^{[3]}(z) = g_2(z), \quad g_1^{[3]}(z) = g_3(z), \quad g_2^{[3]}(z) = g_1(z). \tag{3.27}
\]

For example, if we replace in the first equality in (3.26) \( z \) by \( g_3(z) \) and use (3.22) and (3.24), we obtain

\[
g_4^{[2]}(z) = g_4^{[2]} \circ g_2 \circ g_3(z) = g_1^{[2]} \circ g_3 \circ g_3(z) = g_3^{[2]} \circ g_4(z), \tag{3.28}
\]

from which it follows that

\[
g_3^{[3]}(z) = g_4(z). \tag{3.29}
\]

Using (3.27), we obtain that

\[
g_i^{[81]}(z) = g_i(z), \quad i \in \{1, 2, 3, 4\}, \tag{3.30}
\]

and consequently

\[
g_i^{[80]}(z) = z, \quad i \in \{1, 2, 3, 4\}. \tag{3.31}
\]

By Lemma 2.1, we have that

\[
g_i(z) = z \quad \text{or} \quad g_i^{[2]}(z) = z. \tag{3.32}
\]

If \( g_1(z) = z \), then \( g_4(z) = z \). From this and the fourth equality in (3.21), it follows that

\[
g_3(z) = \frac{C}{z}. \tag{3.33}
\]

On the other hand, from (3.33) and the seventh equality in (3.21), it follows that

\[
g_3\left(\frac{C}{z}\right) = \frac{C}{z}. \tag{3.34}
\]
which implies that \( g_3(z) = z \), a contradiction. Similar, if \( g_i(z) = z \) for some \( i \in \{2, 3, 4\} \), we obtain a contradiction.

Hence, \( g_i^{[2]}(z) = z \), for every \( i \in \{1, 2, 3, 4\} \). From this and (3.25) it follows that

\[
g_1(z) = g_2(z) = g_3(z) = g_4(z) = \frac{C}{z}, \tag{3.35}
\]

finishing the proof of the theorem. \(\Box\)

**Remark 3.1.** We believe that Theorem 1.2 can be extended in a natural way for every \( k \geq 2 \), and that the proof of the corresponding result can be obtained by some modifications of the proof of Theorem 1.2. We leave the solution of the problem to the reader.

**References**


[18] K. Berenhaut and S. Stević, “The behaviour of the positive solutions of the difference equation $x_n = f(x_{n-2})/g(x_{n-1})$,” to appear in *Dynamics of Continuous Discrete and Impulsive Systems: Series A - Mathematical Analysis*.


Stevo Stević: Mathematical Institute of the Serbian Academy of Science, Knez Mihailova 35/I, 11000 Beograd, Serbia

*Email addresses:* sstevic@ptt.yu; sstevo@matf.bg.ac.yu
Submit your manuscripts at
http://www.hindawi.com