Research Article

On the Recursive Sequence $x_n = 1 + \sum_{i=1}^k \alpha_i x_{n-p_i} / \sum_{j=1}^m \beta_j x_{n-q_j}$

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We give a complete picture regarding the behavior of positive solutions of the following important difference equation: $x_n = 1 + \sum_{i=1}^k \alpha_i x_{n-p_i} / \sum_{j=1}^m \beta_j x_{n-q_j}$, $n \in \mathbb{N}_0$, where α_i , $i \in \{1, \dots, k\}$, and β_j , $j \in \{1, \dots, m\}$, are positive numbers such that $\sum_{i=1}^k \alpha_i = \sum_{j=1}^m \beta_j = 1$, and p_i , $i \in \{1, \dots, k\}$, and q_j , $j \in \{1, \dots, m\}$, are natural numbers such that $p_1 < p_2 < \dots < p_k$ and $q_1 < q_2 < \dots < q_m$. The case when $\gcd(p_1, \dots, p_k, q_1, \dots, q_m) = 1$ is the most important. For the case we prove that if all p_i , $i \in \{1, \dots, k\}$, are even and all q_j , $j \in \{1, \dots, m\}$, are odd, then every positive solution of this equation converges to a periodic solution of period two, otherwise, every positive solution of the equation converges to a unique positive equilibrium.

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1. Introduction and preliminaries

In [1], we studied the behavior of positive solutions of the recursive equation

$$y_n = 1 + \frac{y_{n-k}}{y_{n-m}}, \quad n \in \mathbb{N}_0, \quad (1.1)$$

with $y_{-s}, y_{-s+1}, \dots, y_{-1} \in (0, \infty)$ and $k, m \in \{1, 2, 3, 4, \dots\}$, where $s = \max\{k, m\}$. We proved that if 2^i is the highest power of 2 which divides m , then if $2^{i+1} \nmid k$, y_n tends to 2, exponentially, and otherwise every solution tends to a period t solution, with $t = 2 \gcd(k, m)$. The method we used in [1] is a little bit complicated and its idea essentially stems from the theory of nonexpansive metrics. Since the above result is formulated in number theoretic language, we expect that the result is a particular case of a more general result, which

motivates us to investigate the following somewhat natural generalization of (1.1):

$$x_n = 1 + \frac{\sum_{i=1}^k \alpha_i x_{n-p_i}}{\sum_{j=1}^m \beta_j x_{n-q_j}}, \quad n \in \mathbb{N}_0, \quad (1.2)$$

where $\alpha_i, i \in \{1, \dots, k\}$, and $\beta_j, j \in \{1, \dots, m\}$, are positive numbers such that $\sum_{i=1}^k \alpha_i = \sum_{j=1}^m \beta_j = 1$, and $p_i, i \in \{1, \dots, k\}$, and $q_j, j \in \{1, \dots, m\}$, are natural numbers such that $p_1 < p_2 < \dots < p_k$ and $q_1 < q_2 < \dots < q_m$.

Here, we give a complete picture regarding the asymptotic behavior of positive solutions of (1.2). For closely related results, see, for example, [1–16] and the references therein.

In the proof of the main result of this paper, we need the following result by Karakostas (see [8, 9]).

THEOREM 1.1. *Let J be some interval of real numbers, let $f \in C[J^2, J]$, and let $(x_n)_{n=-1}^\infty$ be a bounded solution of the difference equation*

$$x_{n+1} = f(x_n, x_{n-1}), \quad n \in \mathbb{N}_0, \quad (1.3)$$

with $I = \liminf_{n \rightarrow \infty} I_n$, $S = \limsup_{n \rightarrow \infty} x_n$ and with $I, S \in J$. Then there exist two solutions $(I_n)_{n=-\infty}^\infty$ and $(S_n)_{n=-\infty}^\infty$ of the difference equation

$$x_{n+1} = f(x_n, x_{n-1}) \quad (1.4)$$

which satisfy the equation for all $n \in \mathbb{Z}$, with $I_0 = I$, $S_0 = S$, $I_n, S_n \in [I, S]$ for all $n \in \mathbb{Z}$ and such that for every $N \in \mathbb{Z}$, I_N and S_N are limit points of $(x_n)_{n=-1}^\infty$. Furthermore, for every $m \leq -1$, there exist two subsequences (x_{r_n}) and (x_{l_n}) of the solution $(x_n)_{n=-1}^\infty$ such that the following are true:

$$\lim_{n \rightarrow \infty} x_{r_n+N} = I_N, \quad \lim_{n \rightarrow \infty} x_{l_n+N} = S_N \quad \text{for every } N \geq m. \quad (1.5)$$

The solutions $(I_n)_{n=-\infty}^\infty$ and $(S_n)_{n=-\infty}^\infty$ of (1.4) are called full limiting solutions of (1.4) associated with the solution $(x_n)_{n=-1}^\infty$ of (1.3).

2. Main results

First, we study the boundedness character of positive solutions of (1.2). For closely related results, see, for example, [4, 6, 12–14].

THEOREM 2.1. *Every positive solution of (1.2) is bounded.*

Proof. Assume that (x_n) is a positive solution of (1.2). Note that $x_n > 1$ for $n \geq 0$. Hence, it is possible to choose positive numbers l and L greater than one such that $lL = L + l$ and $l \leq x_i \leq L$ for $i \in \{0, 1, \dots, s-1\}$, where $s = \max\{p_k, q_m\}$. Employing (1.2), we obtain

$$l = 1 + \frac{l}{L} \leq x_s = 1 + \frac{\sum_{i=1}^k \alpha_i x_{s-p_i}}{\sum_{j=1}^m \beta_j x_{s-q_j}} \leq 1 + \frac{L}{l} = L. \quad (2.1)$$

By the induction, we obtain that $x_n \in [l, L]$ for every $n \in \mathbb{N}_0$, finishing the proof of the theorem. $\square$

We are now in a position to formulate and prove the main result of this paper.

THEOREM 2.2. *Consider (1.2). Assume that*

$$G := \gcd(p_1, \dots, p_k, q_1, \dots, q_m) = 1. \quad (2.2)$$

Then if all p_i , $i \in \{1, \dots, k\}$, are even and all q_j , $j \in \{1, \dots, m\}$, are odd, every positive solution of (1.2) converges to a periodic solution of period two. Otherwise, every positive solution of (1.2) converges to a unique positive equilibrium.

Proof. Let

$$\mathcal{P} = \{p_i \mid i = 1, \dots, k\}, \quad \mathcal{Q} = \{q_j \mid j = 1, \dots, m\}. \quad (2.3)$$

Assume first that $\mathcal{P} \cap \mathcal{Q} \neq \emptyset$. In view of Theorem 2.1, every positive solution (x_n) of (1.2) is bounded which implies that there are finite $\liminf_{n \rightarrow \infty} x_n = I$ and $\limsup_{n \rightarrow \infty} x_n = S$. Letting $n \rightarrow \infty$ in (1.2), we obtain

$$1 + \frac{I}{S} \leq I \leq S \leq 1 + \frac{S}{I}, \quad (2.4)$$

from which it follows that

$$SI = I + S. \quad (2.5)$$

Let $(L_{-i})_{i \in \mathbb{Z}}$ be a full limiting sequence of a solution (x_n) of (1.2), such that $L_0 = S$.

Since $(L_{-i})_{i \in \mathbb{Z}}$ is a solution of (1.2) belonging to the interval $[I, S]$, we have that

$$S = L_0 = 1 + \frac{\sum_{i=1}^k \alpha_i L_{-p_i}}{\sum_{j=1}^m \beta_j L_{-q_j}} \leq 1 + \frac{S}{I} = S. \quad (2.6)$$

From (2.6), it follows that $L_{-p_i} = S$ for every $i \in \{1, \dots, k\}$ and $L_{-q_j} = I$ for every $j \in \{1, \dots, m\}$. Employing assumption $\mathcal{P} \cap \mathcal{Q} \neq \emptyset$, we obtain $I = S$, from which the result follows in this case.

Now we assume that $\mathcal{P} \cap \mathcal{Q} = \emptyset$. Further, assume that there is $p_{i_0} \in \mathcal{P}$ which is odd. Let $p_{i_0} = 2s + 1$ and let q_{j_0} be an arbitrary element of $\mathcal{Q}$. Then, (1.2) can be written in the form

$$x_n = 1 + \frac{\alpha_{i_0} x_{n-(2s+1)} + \sum_{i=1, i \neq i_0}^k \alpha_i x_{n-p_i}}{\beta_{j_0} x_{n-q_{j_0}} + \sum_{j=1, j \neq j_0}^m \beta_j x_{n-q_j}}. \quad (2.7)$$

Let $(L_{-i})_{i \in \mathbb{Z}}$ be a full limiting sequence of a solution (x_n) of (1.2), such that $L_0 = S = \limsup_{n \rightarrow \infty} x_n$. From

$$S = L_0 = 1 + \frac{\alpha_{i_0} L_{-(2s+1)} + \sum_{i=1, i \neq i_0}^k \alpha_i L_{-p_i}}{\beta_{j_0} L_{-q_{j_0}} + \sum_{j=1, j \neq j_0}^m \beta_j L_{-q_j}}, \quad (2.8)$$

similar to (2.6), we obtain

$$L_{-(2s+1)} = S, \quad L_{-q_{j_0}} = I. \quad (2.9)$$

From (2.9) and since $(L_{-i})_{i \in \mathbb{Z}}$ is a solution of (2.7), it follows that

$$L_{-2(2s+1)} = S, \quad L_{-2q_{j_0}} = S. \quad (2.10)$$

Indeed, since

$$S = L_{-(2s+1)} = 1 + \frac{\alpha_{i_0} L_{-2(2s+1)} + \sum_{i=1, i \neq i_0}^k \alpha_i L_{-p_i - (2s+1)}}{\beta_{j_0} L_{-q_{j_0} - (2s+1)} + \sum_{j=1, j \neq j_0}^m \beta_j L_{-q_j - (2s+1)}} \leq 1 + \frac{S}{I} = S, \quad (2.11)$$

we obtain the first equality in (2.10). On the other hand, from

$$I = L_{-q_{j_0}} = 1 + \frac{\alpha_{i_0} L_{-q_{j_0} - (2s+1)} + \sum_{i=1, i \neq i_0}^k \alpha_i L_{-q_{j_0} - p_i}}{\beta_{j_0} L_{-2q_{j_0}} + \sum_{j=1, j \neq j_0}^m \beta_j L_{-q_{j_0} - q_j}} \geq 1 + \frac{I}{S} = I, \quad (2.12)$$

the second equality in (2.10) follows.

By induction we obtain

$$L_{-(2s+1)i} = S, \quad i \in \mathbb{N}, \quad (2.13)$$

$$L_{-q_{j_0}j} = \begin{cases} I, & j \text{ odd}, \\ S, & j \text{ even}. \end{cases} \quad (2.14)$$

If we take $i = q_{j_0}$ in (2.13) and $j = 2s + 1$ in (2.14), we obtain $I = L_{-(2s+1)q_{j_0}} = S$, as desired.

Now, assume that all $p_i \in \mathcal{P}$ are even, and $\mathcal{Q}$ has odd as well as even elements. Then, (1.2) can be written in the form

$$x_n = 1 + \frac{\sum_{i=1}^k \alpha_i x_{n-p_i}}{\beta_{j_0} x_{n-q_{j_0}} + \beta_{j_1} x_{n-q_{j_1}} + \sum_{j=1, j \neq j_0, j_1}^m \beta_j x_{n-q_j}}, \quad (2.15)$$

where $q_{j_0} = 2s$ and $q_{j_1} = 2t + 1$.

From a result in number theory [11], we know that the condition $G = 1$ implies that for each sufficiently large n , say, $n \geq n_0$, there are nonnegative numbers $d_i \in \mathbb{N}_0$, $i \in \{1, \dots, k + m\}$, such that

$$\sum_{i=1}^k p_i d_i + \sum_{j=1}^m q_j d_{k+j} = n. \quad (2.16)$$

From condition $G = 1$, by using (2.15) and (2.16), and employing the procedure described above for getting formulae (2.13) and (2.14), we obtain that the subsequence $(L_{-i})_{i \geq n_0}$ of the full limiting sequence $(L_i)_{i \in \mathbb{Z}}$ with $L_0 = S$ takes values I and S .

Now we prove that the sequence $(L_{-i})_{i \in \mathbb{N}}$ is eventually periodic with periods $p_1, p_2, \dots, p_k$ and also with periods $2q_1, \dots, 2q_m$. Indeed, if we replace n in (2.15) by $-n_0 - l$, $l \in \{0, 1, \dots, p_1 - 1\}$, we obtain that $L_{-n_0-l} = L_{-n_0-l-p_1 i}$ for every $i \in \mathbb{N}$ and each $l \in \{0, 1, \dots, p_1 - 1\}$, that is, $(L_{-i})_{i \in \mathbb{N}}$ is eventually periodic with period p_1 . Similarly it can be proven that $(L_{-i})_{i \in \mathbb{N}}$ is eventually periodic with periods $p_2, \dots, p_k$. The periodicity with periods $2q_1, \dots, 2q_m$ can be proven similar to (2.9) and (2.10) and by using induction.

Since all $p_i \in \mathcal{P}$ are even and $G = 1$, we have that

$$2 \leq \gcd(p_1, p_2, \dots, p_k, 2q_1, \dots, 2q_m) = 2 \gcd\left(\frac{p_1}{2}, \frac{p_2}{2}, \dots, \frac{p_k}{2}, q_1, \dots, q_m\right) \leq 2G = 2, \quad (2.17)$$

that is,

$$\gcd(p_1, p_2, \dots, p_k, 2q_1, \dots, 2q_m) = 2. \quad (2.18)$$

Hence, the sequence $(L_{-i})_{i \in \mathbb{N}}$ is eventually periodic with period two. Since $(L_i)_{i \in \mathbb{Z}}$ is a solution of (1.2), we obtain that $(L_i)_{i \in \mathbb{Z}}$ is also periodic with period two.

Assume now that

$$\dots, x, y, x, y, x, y, \dots, \quad (2.19)$$

is a two-periodic solution of (2.15). Then we have

$$x = 1 + \frac{x}{cx + (1-c)y}, \quad y = 1 + \frac{y}{cy + (1-c)x}, \quad (2.20)$$

for some $c \in (0, 1)$. Hence,

$$(c-1)xy = cx^2 - (c+1)x - (1-c)y = cy^2 - (c+1)y - (1-c)x, \quad (2.21)$$

from which it follows that $c(x-y)(x+y-2) = 0$. If $x+y = 2$ and $x \neq y$, then we have that x and y are different positive solutions of the equation

$$x = 1 + \frac{x}{cx + (1-c)(2-x)}, \quad (2.22)$$

which implies that $(2c-1)(x-1)^2 = 1$. Hence, if $c \leq 1/2$, then this equation does not have real roots. If $c > 1/2$, then $x = 1 \pm (1/(2c-1))^{1/2}$ are solutions. However, since $c \in (1/2, 1)$, the number $1 - (1/(2c-1))^{1/2}$ is negative. Therefore, it follows that $x = y$ as desired.

Assume now that the set $\mathcal{P}$ contains only even elements while $\mathcal{Q}$ contains only odd elements. Then, it is easy to see that (1.2) in this case has infinite prime two-periodic solutions of the form $x, y, x, y, \dots$, such that $xy = x + y$. Similar to (2.18), it can be proven that, in this case, the full limiting sequence $(L_i)_{i \in \mathbb{Z}}$, $L_0 = S$ is periodic with period two and that

$$L_{2i} = S, \quad L_{2i-1} = I, \quad i \in \mathbb{Z}. \quad (2.23)$$

Assume that $\varepsilon, \delta \in (0, S)$ are such that

$$(S - \varepsilon)(I + \delta) = (S - \varepsilon) + (I + \delta). \quad (2.24)$$

Then, for such chosen ε and δ , there is a $k_0 \in \mathbb{Z}$ such that

$$x_{k_0+2j} > S - \varepsilon, \quad x_{k_0+2j-1} < I + \delta, \quad (2.25)$$

for $j \in \{1, 2, \dots, [s/2] + 1\}$, where $s = \max\{p_k, q_m\}$.

From (1.2) and (2.25), we have that

$$\begin{aligned} x_{k_0+2[s/2]+3} &< 1 + \frac{I + \delta}{S - \varepsilon} = I + \delta, \\ x_{k_0+2[s/2]+4} &> 1 + \frac{S - \varepsilon}{I + \delta} = S - \varepsilon. \end{aligned} \quad (2.26)$$

By induction, we obtain

$$x_{k_0+2i+1} < I + \delta, \quad x_{k_0+2i} > S - \varepsilon, \quad (2.27)$$

for every $i \in \mathbb{N}$. From (2.27) and the fact that $\varepsilon \rightarrow 0$ implies $\delta \rightarrow 0$, it follows that $\lim_{n \rightarrow \infty} x_{2n} = S$ and $\lim_{n \rightarrow \infty} x_{2n-1} = I$, or $\lim_{n \rightarrow \infty} x_{2n} = I$ and $\lim_{n \rightarrow \infty} x_{2n-1} = S$, finishing the proof of the theorem. $\square$

Remark 2.3. Note that the case when all $p_i, i \in \{1, \dots, k\}$, and $q_j, j \in \{1, \dots, m\}$, are even is excluded from the consideration in Theorem 2.1 since we assume that $G = 1$. However, this case is reduced to the cases considered in Theorem 2.1. Indeed, let 2^s be the highest power of 2 which divides G , then (1.2) can be separated into 2^s different equations of the form

$$x_n^{(t)} = 1 + \frac{\sum_{i=1}^k \alpha_i x_{n-p_i/2^s}^{(t)}}{\sum_{j=1}^m \beta_j x_{n-q_j/2^s}^{(t)}}, \quad n \in \mathbb{N}_0, \quad (2.28)$$

where $t \in \{0, 1, \dots, 2^s - 1\}$. Note that by the definition of 2^s , it follows that at least one of the numbers $p_i/2^s, i \in \{1, \dots, k\}$, and $q_j/2^s, j \in \{1, \dots, m\}$, is odd. Hence, Theorem 2.1 can be applied to the equations in (2.28).

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