Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2007, Article ID 42540, 8 pages
doi:10.1155/2007/42540

Research Article

Stability Analysis of θ-Methods for Neutral Multidelay Integrodifferential System

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Received 12 June 2007; Revised 24 September 2007; Accepted 22 October 2007

This paper studies the stability of a class of neutral delay integrodifferential system. A necessary and sufficient condition of stability for its analytic solutions is considered. The improved θ-methods are developed. Some numerical stability properties are obtained and numerical experiments are given.

1. Introduction

Consider the neutral multidelay integrodifferential equations (NMDIDEs)

$$Au'(t)+Bu(t)+\sum_{j=1}^{M}\left[C_j u'(t-\tau_j)+D_j u(t-\tau_j)+G_j\int_{t-\tau_j}^{t}u(x)dx\right]=0, \tag{1.1}$$

where $A,B,C_j,D_j,G_j\in\mathbb{C}^{d\times d}$ for $j=1,\dots,M$ and $0<\tau_1\le\tau_2\le\cdots\le\tau_M$. The initial condition is $u(t)=\phi(t)$ for $-\tau_M\le t\le 0$. Particularly, when matrix A is singular, system (1.1) becomes the differential algebraic system.

Delay differential equations can be found in a wide variety of scientific and engineering fields such as biology, physics, ecology, and so on. Particularly, delay integrodifferential algebraic system plays an important role in modeling many phenomena of circuit analysis and chemical process simulation.

As for the linear delay integrodifferential system, there were some perfect results from Koto (cf. [1]). Recently, as for the linear neutral delay integrodifferential equation, the numerical stability of θ-methods and BDF methods can be referred to [2].

Although stability of numerical methods seems important for practical computation, there are few papers concerning this subject for NMDIDEs. Thus, this paper considers the asymptotic stability of analytic solutions and numerical solutions for system (1.1).

2. Asymptotic stability of NMDIDEs

When matrix A is nonsingular, the solvability of system (1.1) is obvious.

Definition 2.1. Matrices polynomials $f_1(\lambda)$ and $f_2(\lambda)$ are simultaneously regular if there exists $\lambda_0 \in \mathbb{C}$ such that $\det[f_1(\lambda_0)] \neq 0$ and $\det[f_2(\lambda_0)] \neq 0$.

Theorem 2.2. *System (1.1) with singular A is solvable if the matrices pencils $\lambda A + B$ and $\lambda^2 A + \lambda B + \sum_{j=1}^{M} G_j$ are simultaneously regular.*

Proof. It can be proved by [3, Theorem 2.1]. □

Definition 2.3. System (1.1) is said to be asymptotically stable if the exact solutions $u(t)$ satisfy $\lim_{t\to\infty} u(t) = 0$ for any continuous initial function.

To study the property of system (1.1), we consider characteristic polynomial

$$p(\zeta) = \zeta^{-d} \det[J(\zeta) + e^{-\tau_1\zeta} K(\zeta)], \tag{2.1}$$

where $J(\zeta) = A\zeta^2 + B\zeta + \sum_{j=1}^{M} G_j$, $K(\zeta) = \sum_{j=1}^{M} (C_j\zeta^2 + D_j\zeta - G_j) e^{\zeta(\tau_1 - \tau_j)}$.

Lemma 2.4 (cf. [4]). *If system (1.1) is asymptotically stable, then*

(S1) *all roots of the characteristic polynomial (2.1) have negative real parts.*

Lemma 2.5 (cf. [4]). *System (1.1) is asymptotically stable if*

(S2) *all roots of the characteristic polynomial (2.1) are uniformly bounded away from the imaginary axis in the left-half plane.*

For nonneutral delay differential equations, condition (S1) is also sufficient in Lemma 2.4. However, it is not true for the neutral case.

Lemma 2.6. *Let $H(\zeta) = -[J(\zeta)]^{-1} K(\zeta)$, then $\rho[H(\zeta)] < 1$ as $|\zeta| \to \infty$ if*

(C1) $|\langle \xi, A\xi \rangle| > \sum_{j=1}^{M} |\langle \xi, C_j \xi \rangle|$ *whenever* $|\langle \xi, A\xi \rangle| \neq 0$,

(C2) $\det[J(\zeta)] \neq 0$ *for any* $\Re\zeta \geq 0$ *and* $\zeta \neq 0$.

Lemma 2.7. *If system (1.1) is asymptotically stable, then $|\mu| \neq 1$ for any $\mu \in \sigma[H(\zeta)]$ with $\Re\zeta = 0$ and $\zeta \neq 0$.*

The above two conclusions are trivial, so we omit their proofs.

Lemma 2.8. *If system (1.1) is asymptotically stable, then one has (C2).*

Proof. If there exists ζ_1 with $\Re\zeta_1 > 0$ such that $\det[J(\zeta_1)] = 0$, then a positive oriented circle Σ centered at ζ_1 is found such that $\Re\zeta > 0$ and $\det[J(\zeta)] \neq 0$ when $\zeta \in \Sigma \backslash \{\zeta_1\}$. Thus, $\rho[e^{-\tau_1\zeta} H(\zeta)] < 1$ for sufficiently large τ_1.

Define $h_\alpha(\zeta) = \det[I - \alpha e^{-\zeta\tau_1}H(\zeta)]$ for $\alpha \in [0,1]$. Since $h_\alpha(\zeta) \neq 0$ for all $\zeta \in \Sigma\backslash\{\zeta_1\}$, then the change of argument along the curve $[\arg h_\alpha(\zeta)]_\Sigma = 2\pi m$ with $m \in \mathbb{Z}$. Notice that $[\arg h_\alpha(\zeta)]_\Sigma$ is uniformly continuous of α on $[0,1]$, then $[\arg h_1(\zeta)]_\Sigma = [\arg h_0(\zeta)]_\Sigma = 0$. Hence, we have $[\arg \zeta^d p(\zeta)]_\Sigma = [\arg \det J(\zeta)]_\Sigma$. According to the principle of argument (see [5]), there must exist $\tilde{\zeta}_1$ in the inferior of $\Sigma\backslash\{\zeta_1\}$ such that $p(\tilde{\zeta}_1) = 0$, which contradicts condition (S1).

If there exists ζ_2 with $\Re\zeta_2 = 0$ and $\zeta_2 \neq 0$ such that $\det[J(\zeta_2)] = 0$, then we can assume that $\det[J(\zeta)] \neq 0$ for any $\Re\zeta = 0$ and $\Im\zeta > \Im\zeta_2 > 0$. Thus, there exists a neighborhood U of ζ_2 such that $\det[J(\zeta)] \neq 0$ in $U\backslash\{\zeta_2\}$.

When the zeros $\mu_i(\zeta)$ $(1 \le i \le d)$ of $q(\mu,\zeta)$ are bounded near ζ_2, where $q(\mu,\zeta) = \det[\mu J(\zeta) + K(\zeta)]$, there must exist a neighborhood $V\backslash\{\zeta_2\}$ of ζ_2 and a constant $k > 0$ such that $|\mu_i(\zeta)| < k$ whenever $\zeta \in V \subset U$.

Notice that $\det[J(\zeta)] \neq 0$ for all $\zeta \in V\backslash\{\zeta_2\}$ and $q(\mu,\zeta) = \sum_{i=0}^{d} q_i(\zeta)\mu^{d-i}$, where $q_0(\zeta) = \det[J(\zeta)]$, $q_d(\zeta) = \det[K(\zeta)]$, and $q_i(\zeta)$ $(1 \le i \le d-1)$ are the polynomials of ζ. Therefore, $|q_d(\zeta)|\cdot|q_0(\zeta)|^{-1} = \prod_{i=1}^{d}|\mu_i(\zeta)| < k^d$ in $V\backslash\{\zeta_2\}$. Let $\zeta \to \zeta_2$, then $q_d(\zeta_2) = \cdots = q_1(\zeta_2) = 0$ since $q_0(\zeta_2) = 0$, that is, $q(\mu,\zeta_2) \equiv 0$. Choosing $\mu = e^{\zeta_2\tau_1}$ in $q(\mu,\zeta_2)$, we have $p(\zeta_2) = 0$, which contradicts (S1).

When the zeros $\mu_i(\zeta)$ $(1 \le i \le d)$ of $q(\mu,\zeta)$ cannot be bounded in a neighborhood of ζ_2, then $\rho(H(\zeta)) \to \infty$ as $\zeta \to \zeta_2$. According to Lemma 2.6, $\rho[H(\zeta)] < 1$ as $|\zeta| \to \infty$ and $\Re\zeta = 0$. Moreover, $\rho(H(\zeta))$ is continuous of ζ on $\{\zeta \in \mathbb{C} : \Re\zeta = 0, \Im\zeta > \Im\zeta_2\}$, so there must exist ζ_0 with $\Re\zeta_0 = 0$, $\Im\zeta_0 > \Im\zeta_2$ such that $\rho(H(\zeta_0)) = 1$, which contradicts Lemma 2.7. □

Theorem 2.9. *Under condition (C1), the system (1.1) is asymptotically stable if and only if it satisfies (C2) and*

(C3) $\rho(H(\zeta)) < 1$ *for any* $\Re\zeta = 0$ *and* $\zeta \neq 0$,

(C4) $\det[B + \sum_{j=1}^{M}(D_j + \tau_j G_j)] \neq 0$.

Proof. From (C2), if $\Re\zeta \ge 0$ and $\zeta \neq 0$, then $p(\zeta) = 0 \Leftrightarrow e^{\tau_1\zeta} \in \sigma[H(\zeta)]$. According to Lemma 2.6 and the maximum modulus principle, $\rho[H(\zeta)] < 1$. From (C4), we have $p(\zeta) \neq 0$. Denote $g(\zeta) = \zeta^2\hat{A} + \zeta\hat{B} + \sum_{j=1}^{M}[e^{-\zeta\tau_j}(\zeta^2\hat{C}_j + \zeta\hat{D}_j - \hat{G}_j) + \hat{G}_j]$, where $\hat{U} = \langle\xi, U\xi\rangle$ with $|\xi| = 1$.

Assuming (S2) does not hold, then there must exist a set $\{z_n\} \subset \mathbb{C}^-$ such that $p(z_n) = 0$ and z_n converges to a point in the imaginary axis. So, $g(z_n) = 0$. Let w_n be the imaginary part of z_n, then $\lim_{n\to\infty} g(iw_n) = \lim_{n\to\infty} g(z_n) = 0$. On the other hand, for sufficiently large n, we have $|g(iw_n)| > 0$, which contradicts the above analysis. According to Lemma 2.5, system (1.1) is asymptotically stable.

Conversely, if system (1.1) is asymptotically stable, then (C2), (C4), and (S1) hold according to Lemmas 2.4 and 2.8. If there exists ζ_3 with $\Re\zeta_3 = 0$ and $\zeta_3 \neq 0$ such that $\rho(H(\zeta_3)) \ge 1$, then from (C2) and Lemma 2.6, there must exist ζ_4 with $\Re\zeta_4 = 0$ and $\Im\zeta_4 \ge \Im\zeta_3$ such that $\rho(H(\zeta_4)) = 1$, which contradicts Lemma 2.7 Therefore, condition (C3) holds. □

From convenience, for $j = 1,\ldots,M$, we denote

(C5) for any complex number ζ, the matrices $(J(\zeta))^{-1}(C_j\zeta^2 + D_j\zeta - G_j)$ can be similarly transformed to upper-triangular matrices simultaneously;

(C6) the matrices $A^{-1}C_j$ can be similarly transformed to upper-triangular matrices simultaneously when A is invertible;

(C7) $\sum_{j=1}^{M}\rho[(J(\zeta))^{-1}(C_j\zeta^2+D_j\zeta-G_j)]<1$ for any $\Re\zeta\geq 0$ with $\zeta\neq 0$;

(C8) $\sum_{j=1}^{M}\rho[A^{-1}C_j]<1$ when matrix A is invertible.

$$\begin{aligned}\Omega &= \{(A,B,C_j,D_j,G_j):(\mathrm{C1}),(\mathrm{C2}),(\mathrm{C4}),(\mathrm{C5}),\text{ and }(\mathrm{C7})\text{ are satisfied}\},\\ \widetilde{\Omega} &= \{(A,B,C_j,D_j,G_j):(\mathrm{C1}),(\mathrm{C2}),\text{ and }(\mathrm{C4})-(\mathrm{C8})\text{ are satisfied}\}.\end{aligned} \tag{2.2}$$

Corollary 2.10. *System (1.1) is asymptotically stable if it satisfies conditions (C1), (C2), (C4), (C5), and (C7).*

3. Stability of θ-methods

For $u'(t)=f(t,u(t))$, the linear θ-method gives out the recurrence relation

$$u_{n+1}=u_n+h[\theta f(t_{n+1},u_{n+1})+(1-\theta)f(t_n,u_n)], \tag{3.1}$$

where $\theta\in[0,1]$, $t_n=nh$ with $n\in\mathbb{Z}^+$ and u_n are approximations to $u(t_n)$.

Applying the linear θ-method (3.1) to system (1.1), we have

$$L_1u_{n+1}+L_2u_n+L_3\sum_{i=1}^{m_j-3}u_{n-i}+L_4u_{n+2-m_j}+L_5u_{n+1-m_j}+L_6u_{n-m_j}=0, \tag{3.2}$$

where

$$\begin{aligned}&L_1=A+h\theta B+\theta^2L_3,\qquad L_2=-A+h(1-\theta)B+\theta(2-\theta)L_3,\qquad L_3=h^2\sum_{j=1}^{M}G_j,\\ &L_5=\sum_{j=1}^{M}\left[(1-2\delta_j)C_j+h(\theta+\delta_j-2\theta\delta_j)D_j+h^2(1-\theta^2-\delta_j^2-\theta\delta_j+2\theta\delta_j^2)G_j\right],\\ &L_6=\sum_{j=1}^{M}\left[-(1-\delta_j)C_j+h(1-\theta)(1-\delta_j)D_j+h^2(1-\theta)(1-\theta-\delta_j+\delta_j^2)G_j\right],\\ &L_4=\sum_{j=1}^{M}[\delta_jC_j+h\theta\delta_jD_j+h^2(1-\theta\delta_j^2)G_j].\end{aligned} \tag{3.3}$$

For system (1.1) with a single delay, the linear θ-method is no longer GP stable in [1]. So we focus on the step-size-dependent stability of (3.2).

On the other hand, let $v(t)=\int_{-m_Mh}^{t}u(x)dx$, then system (1.1) is written as

$$Au'(t)+Bu(t)+\sum_{j=1}^{M}\left[C_ju'(t-\tau_j)+D_ju(t-\tau_j)+G_j(v(t)-v(t-\tau_j))\right]=0. \tag{3.4}$$

Applying method (3.1) to system (3.4), we obtain the improved linear θ-method

$$L_1 u_{n+1} + L_2 u_n + L_3 \sum_{i=1}^{m_j-3} u_{n-i} + \hat{L}_4 u_{n+2-m_j} + \hat{L}_5 u_{n+1-m_j} + L_6 u_{n-m_j} = 0, \tag{3.5}$$

where

$$\begin{aligned} \hat{L}_5 &= \sum_{j=1}^{M} [(1-2\delta_j)C_j + h(\theta(1-\delta_j) + (1-\theta)\delta_j)D_j] + L_3(1-\theta)(1+\theta-2\theta\delta_j), \\ \hat{L}_4 &= \sum_{j=1}^{M} [\delta_j C_j + h\theta\delta_j D_j + h^2(1-\theta^2\delta_j)G_j]. \end{aligned} \tag{3.6}$$

Theorem 3.1. *The order of local truncation error for the improved linear θ-method (3.5) is $O(h^2)$. Especially, the order becomes $O(h^3)$ if $\theta = 1/2$.*

Proof. For method (3.5), the local truncation error $\tilde{T}_{n+1} = (\theta - 1/2)h^2[Bu'(t_n) + \sum_{j=1}^{M} D_j u'(t_{n-m_j})] + O(h^3)$. □

Theorem 3.2. *For $\theta \in [0,1]$, the improved linear θ-method (3.5) is convergent of order 1. Especially, the convergent order is 2 if $\theta = 1/2$.*

Proof. The proof is similar to that of [6, Theorem 1.2]. □

Since (3.5) and (3.2) are the same for $\delta_j = 0$ or $\delta_j = \theta$, then it only needs to consider the characteristic polynomial

$$\begin{aligned} D(z) &= \left[\frac{h^2(\theta z + 1 - \theta)^2}{z-1}\right]^d \det\left[P(z)z^{m_M} + Q(z)\right] \quad \text{for } \theta z + 1 - \theta \neq 0,\ z \neq 1, \\ P(z) &= \left(\frac{z-1}{h(\theta z + 1 - \theta)}\right)^2 A + \left(\frac{z-1}{h(\theta z + 1 - \theta)}\right) B + \sum_{j=1}^{M} G_j, \\ Q(z) &= \sum_{j=1}^{M} \left[\left(\frac{z-1}{h(\theta z + 1 - \theta)}\right)^2 C_j + \left(\frac{z-1}{h(\theta z + 1 - \theta)}\right) D_j - G_j\right] z^{m_M - m_j}(\delta_j z + 1 - \delta_j). \end{aligned} \tag{3.7}$$

Let $S_{\theta,m_1,\ldots,m_M}^{(\delta_1,\ldots,\delta_M)} = \{(A,B,C_j,D_j,G_j) : \text{all roots of } D(z) \text{ satisfy } |z| < 1\}$, $S_{\theta}^{(\delta_1,\ldots,\delta_M)} = \bigcap_{\delta_1,\ldots,\delta_M \in [0,1)} S_{\theta,m_1,\ldots,m_M}^{(\delta_1,\ldots,\delta_M)}$ for $j = 1,\ldots,M$.

Theorem 3.3. *For (1.1) with singular A, if $\theta \in (1/2,1]$, then $\Omega \subset S_{\theta}^{(\delta_1,\ldots,\delta_M)}$.*

Proof. It is sufficient to prove that $D(z)$ is a Schur polynomial if $\theta \in (1/2,1]$. A polynomial is said to be of Schur type if all of its roots are less than 1 in modulus (see [7]). From (C4), we know that $z = 1$ is not the root of $D(z)$. So it only needs to prove that $D(z) \neq 0$ when $|z| \geq 1$ and $z \neq 1$.

If $\theta \in (1/2,1]$, then from (C2), $P(z)$ is invertible. Hence, we have

$$D(z) = \left[\frac{h^2(\theta z+1-\theta)^2}{z-1}\right]^d \det[P(z)z^{m_M}]\det[I_d + P^{-1}(z)Q(z)z^{-m_M}]. \tag{3.8}$$

Under (C5) and (C7), we obtain $\rho[P^{-1}(z)Q(z)z^{-m_M}] < 1$ for $|z| \geq 1$ and $z \neq 1$. □

Theorem 3.4. *For (1.1) with nonsingular A, if $\theta \in [1/2,1]$, then $\widetilde{\Omega} \subset S_\theta^{(\delta_1,\ldots,\delta_M)}$.*

Proof. To prove $D(z)$ is a Schur polynomial, similar to Theorem 3.3, we have that $z = 1$ is not the root of $D(z)$ and $D(z) \neq 0$ for $|z| \geq 1$ when $\theta z + 1 - \theta \neq 0$.

If $\theta z+1-\theta=0$, then $\det[I+\sum_{j=1}^{M}(-1)^{-m_j}A^{-1}C_j(1-2\delta_j)]=0$, provided that $D(z) = 0$. From (C6) and (C8), $\rho[\sum_{j=1}^{M}(-1)^{-m_j}A^{-1}C_j(1-2\delta_j)] < 1$. So $\det[I+\sum_{j=1}^{M}(-1)^{-m_j}A^{-1}C_j(1-2\delta_j)] \neq 0$, which makes a contradiction. □

Corollary 3.5. *For system (1.1) with singular A, if $\theta \in (1/2,1]$, then $\Omega \subset S_\theta^{(0,\ldots,0)}$ or $\Omega \subset S_\theta^{(\theta,\ldots,\theta)}$.*

Corollary 3.6. *For (1.1) with nonsingular A, if $\theta \in [1/2,1]$, then $\widetilde{\Omega} \subset S_\theta^{(0,\ldots,0)}$ or $\widetilde{\Omega} \subset S_\theta^{(\theta,\ldots,\theta)}$.*

Remark 3.7. Improved linear θ-method (3.5) with $\theta \in (1/2,1]$ (or $\theta \in [1/2,1]$) can possess a similar stability property to GP stability with respect to NMDIDEs with singular (or nonsingular) A.

4. Numerical experiments

Example 4.1. Consider (1.1) with $M = 2$, $\tau_1 = 1$, $\tau_2 = 2$, and $\phi(t) = (\cos(t),\sin(t),\cos(t))^T$ for $t \in [-2,0]$, where $C_1 = -0.3A$, $C_2 = -0.5A$, and

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1.5 & 0 & 0 \\ 0 & 0 & -0.4 \\ 0 & 0.8 & 0.5 \end{bmatrix},$$

$$D_1 = \begin{bmatrix} 0.2 & 0.4 & 0 \\ 0 & 0 & 0.1 \\ 0 & 0.2 & -0.2 \end{bmatrix}, \qquad D_2 = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0 & 0.1 \\ 0 & 0.15 & 0.8 \end{bmatrix}, \tag{4.1}$$

$$G_1 = \begin{bmatrix} 0.2 & 0 & 0 \\ 0 & 0 & -0.2 \\ 0 & 0.1 & 0.25 \end{bmatrix}, \qquad G_2 = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0 & -0.2 \\ 0 & 0.05 & 0.25 \end{bmatrix}.$$

Figure 4.1 is in agreement with the conclusions in the paper. To compare the improved linear θ-method (3.5) with the linear θ-method presented in [1], we only consider the case of $\theta \in [1/2,1]$. We know that these two methods can both possess a similar stability property to P stability if $\theta \in [1/2,1]$.

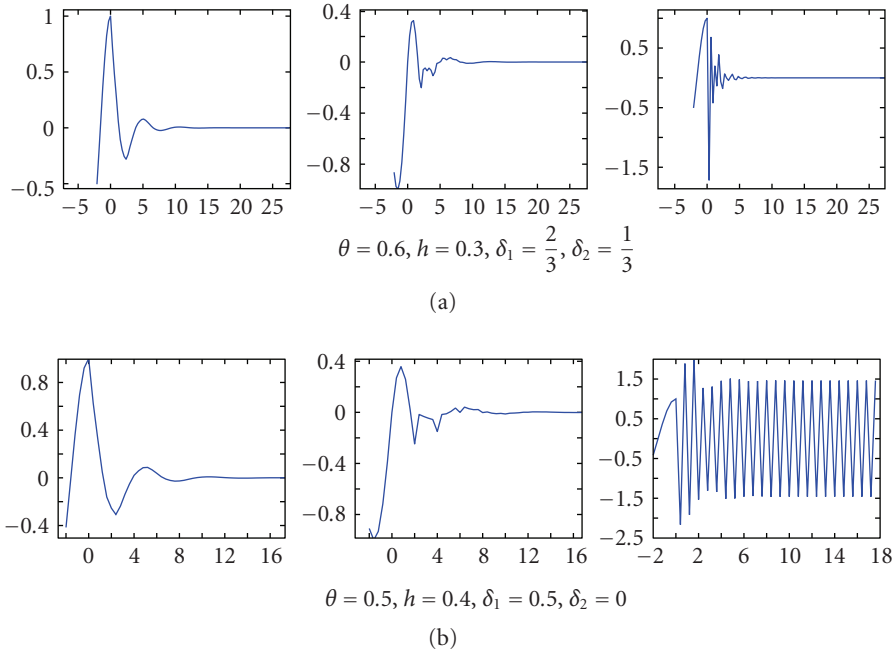

FIGURE 4.1. Improved linear θ-methods (3.5) for Example 4.1 with singular A.

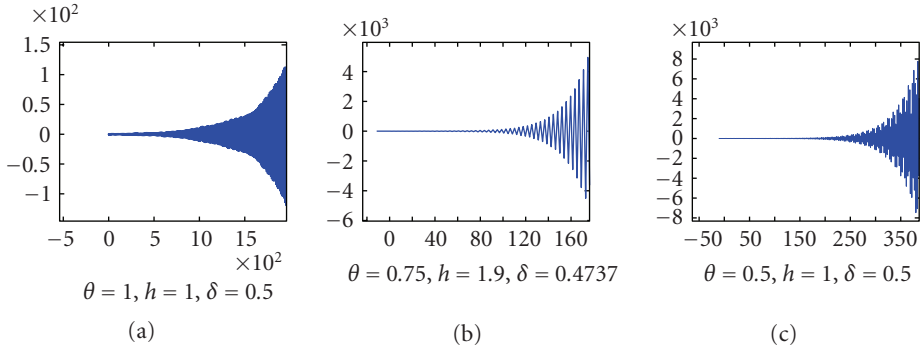

FIGURE 4.2. Linear θ-method in [1] for Example 4.2.

Example 4.2. Consider $u'(t) = -10.1u(t) + u(t - 10.5) - 50\int_{t-10.5}^{t} u(\sigma)d\sigma$, where $u(t) = \exp(t)$ for $-10.5 \le t \le 0$.

Note that the linear θ-method in [1] cannot possess a similar property to *GP* stability in Figure 4.2, but the improved linear θ-method (3.5) preserves this property in Figure 4.3. It is shown that both Theorem 3.4 in this paper and [1, Theorem 3] are valid from the above numerical experiment.

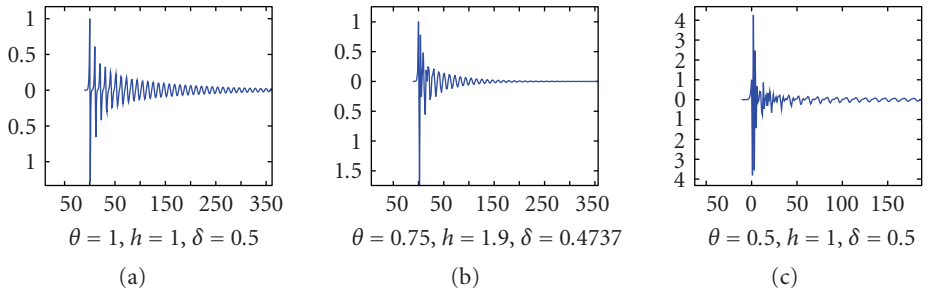

FIGURE 4.3. Improved linear θ-method (3.5) for Example 4.2.

Acknowledgments

The authors are grateful to the anonymous referees for their valuable comments. This paper was supported by the Natural Science Foundation of Province (Grant no. A200602), and the projects HITQNJS2006053 and HITC200710 of Science Research Foundation in HIT, China.

References

[1] T. Koto, "Stability of θ-methods for delay integro-differential equations," *Journal of Computational and Applied Mathematics*, vol. 161, no. 2, pp. 393–404, 2003.

[2] J. J. Zhao, Y. Xu, and M. Z. Liu, "Stability analysis of numerical methods for linear neutral Volterra delay-integro-differential system," *Applied Mathematics and Computation*, vol. 167, no. 2, pp. 1062–1079, 2005.

[3] W. Zhu and L. R. Petzold, "Asymptotic stability of linear delay differential-algebraic equations and numerical methods," *Applied Numerical Mathematics*, vol. 24, no. 2-3, pp. 247–264, 1997.

[4] J. X. Kuang and Y. H. Cong, *Stability of Numerical Methods for Delay Differential Equations*, Science Press, Beijing, China, 2005.

[5] L. V. Ahlfors, *Complex Analysis*, McGraw-Hill, Beijing, China, 2004.

[6] A. Iserles, *A First Course in the Numerical Analysis of Differential Equations*, Cambridge Texts in Applied Mathematics, Cambridge University Press, Cambridge, UK, 1996.

[7] Y. K. Liu, "Stability analysis of θ-methods for neutral functional-differential equations," *Numerische Mathematik*, vol. 70, no. 4, pp. 473–485, 1995.

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