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Research Article

Asymptotics of Some Classes of Higher-Order Difference Equations

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We present some methods for finding asymptotics of some classes of nonlinear higher-order difference equations. Among others, we confirm a conjecture posed by S. Stević (2005). Monotonous solutions of the equation $y_n = A + (y_{n-k}/\sum_{j=1}^{m}\beta_j y_{n-q_j})^p$, $n \in \mathbb{N}_0$, where $p, A \in (0,\infty)$, $k, m \in \mathbb{N}$, q_j, $j \in \{1,\dots,m\}$, are natural numbers such that $q_1 < q_2 < \cdots < q_m$, $\beta_j \in (0,+\infty)$, $j \in \{1,\dots,m\}$, $\sum_{j=1}^{m}\beta_j = 1$, and $y_{-s}, y_{-s+1},\dots,y_{-1} \in (0,\infty)$, where $s = \max\{k, q_m\}$, are found. A new inclusion theorem is proved. Also, some open problems and conjectures are posed.

1. Introduction

Recently, there has been a great interest in studying properties of rational and nonlinear difference equations (c.f. [1–40] and the references therein). Those equations which model some real-life situations in population biology and ecology are of a particular interest.

This paper, which has some elements of a survey, presents some methods for finding asymptotics of some classes of nonlinear higher-order difference equations and applies them in solving several problems. Some basics on asymptotics for first-order difference equations can be found in the classical book [18, Chapter 5] (we strongly encourage the reader to read the chapter). If we are able to find asymptotics of a solution (or family of solutions) of a difference equation, we are in a good position to determine some other properties of the solution, for example, monotonicity, nontriviality, and so forth.

The paper is organized as follows. In Section 2, we quote an inclusion theorem due to Berg (Theorem 2.1) and give an application of the result (Example 2.2). Section 3 is devoted to finding an asymptotic series of some solutions of the difference equation

$x_{n+1} = x_{n-1}/(p + x_{n-1} + x_n)$, while in Section 4 we extend some results regarding positive solutions of the equation to a generalization of this one. Existence of some nonoscillatory solutions of a class of difference equations is given in Section 5. A new inclusion theorem (Theorem 6.1) is presented in Section 6. An interesting open problem on finding asymptotics of some unbounded solutions of a difference equation is given in Section 7. A brief account of results regarding existence of nontrivial solutions of the so-called Putnam-type difference equations as well as a conjecture are found in the last section.

We also leave some other open problems and conjectures which are of some interest to the experts in this research field.

2. Berg's inclusion theorem and its application

Consider a general real nonlinear difference equation of order $m \in \mathbb{N}$ of the form

$$F(x_n, \ldots, x_{n+m}) = 0, \tag{2.1}$$

where $F : \mathbb{R}^{m+1} \to \mathbb{R}$, $n \in \mathbb{N}_0$. Also, let φ_n and ψ_n be two sequences such that $\psi_n > 0$ and $\psi_n = o(\varphi_n)$ as $n \to \infty$. Then (under certain additional conditions) for arbitrary $\varepsilon > 0$, there exist a solution x_n of (2.1) and an $n_0(\varepsilon) \in \mathbb{N}$, such that

$$\varphi_n - \varepsilon\psi_n \le x_n \le \varphi_n + \varepsilon\psi_n, \tag{2.2}$$

for $n \ge n_0(\varepsilon)$. The set of all sequences x_n satisfying (2.2) is called an asymptotic stripe $X(\varepsilon)$, that is, $y_n \in X(\varepsilon)$ implies the existence of a real sequence C_n with $y_n = \varphi_n + C_n\psi_n$ and $|C_n| \le \varepsilon$ for $n \ge n_0(\varepsilon)$. Hints for the construction of the pairs φ_n, ψ_n can be found in [5–7, 9].

The next theorem is the main result in [7]. (See also [9] for a correction of the proof.)

Theorem 2.1 (see [7, Theorem 2.1]). *Let $F(w_0, w_1, \ldots, w_m)$ be continuously differentiable when $w_i = y_{n+i}$, for $i = 0, 1, \ldots, m$, and $y_n \in X(1)$. Let the partial derivatives of F satisfy*

$$F_{w_i}(y_n, \ldots, y_{n+m}) \sim F_{w_i}(\varphi_n, \ldots, \varphi_{n+m}) \tag{2.3}$$

as $n \to \infty$ uniformly in C_j for $|C_j| \le 1$, $n \le j \le n+m$, so far as $F_{w_i} \not\equiv 0$. Assume that there exist a sequence $f_n > 0$ and constants $A_0, A_1, \ldots, A_m$ such that

$$F(\varphi_n, \ldots, \varphi_{n+m}) = o(f_n), \tag{2.4}$$

$$\psi_{n+i} F_{w_i}(\varphi_n, \ldots, \varphi_{n+m}) \sim A_i f_n, \tag{2.5}$$

for $i = 0, 1, \ldots, m$ as $n \to \infty$, and suppose there exists an integer k, with $0 \le k \le m$, such that

$$|A_0| + \cdots + |A_{k-1}| + |A_{k+1}| + \cdots + |A_m| < |A_k|. \tag{2.6}$$

Then, for sufficiently large n, there exists a solution (x_n) of (2.1) satisfying (2.2).

In the following example, which is motivated by [15, Open Problem 6.2.1], we demonstrate how Theorem 2.1 can be applied in showing the existence of solutions of difference equations converging to zero.

Example 2.2. Consider the equation

$$x_{n+1} = \frac{bx_n^r}{1+x_{n-1}^r}, \quad n \in \mathbb{N}_0, \tag{2.7}$$

with $b > 0$ and $r > 1$. Equation (2.7) can be written in the following form:

$$F(x_{n-1}, x_n, x_{n+1}) := x_{n+1}(1+x_{n-1}^r) - bx_n^r = 0. \tag{2.8}$$

We look for a solution of (2.7) such that $x_n = o(1)$ as $n \to \infty$. Therefore, as $n \to \infty$ we have that

$$x_{n+1} \sim bx_n^r. \tag{2.9}$$

If we consider the equation

$$x_{n+1} = bx_n^r, \tag{2.10}$$

instead of the asymptotic relation in (2.9), then we find that the solution of (2.10) is

$$x_n = \frac{1}{b^{1/(r-1)}} e^{-r^n} \tag{2.11}$$

up to a constant factor nearby r^n (the general solution of (2.10) is $x_n = b^{-1/(r-1)} e^{r^n \ln(b^{1/(r-1)} x_0)}$). In order to prove that (2.7) has a solution with the asymptotic behavior as in (2.11), we make the following ansatz:

$$\varphi_n = \frac{1}{b^{1/(r-1)}} \left(e^{-r^n} + ce^{-ar^n}\right), \tag{2.12}$$

where $a > 1$ and $c > 0$ are two parameters which will be determined below. If we put (2.12) into (2.8), use the well-known asymptotic formula for the function $(1+x)^\alpha$, and by some simple calculation, we have that

$$\begin{aligned}
&b^{1/(r-1)} F(\varphi_{n-1}, \varphi_n, \varphi_{n+1}) \\
&\quad = \left(e^{-r^{n+1}} + ce^{-ar^{n+1}}\right)\left(1 + \frac{1}{b^{r/(r-1)}} e^{-r^n} \left(1 + ce^{-r^{n-1}(a-1)}\right)^r\right) \\
&\qquad - e^{-r^{n+1}} \left(1 + ce^{-r^n(a-1)}\right)^r \\
&\quad = e^{-r^{n+1}} + ce^{-ar^{n+1}} + \frac{1}{b^{r/(r-1)}} e^{-r^n(r+1)} + \frac{rc}{b^{r/(r-1)}} e^{-r^{n-1}(r^2+r+a-1)} \\
&\qquad + o\left(e^{-r^{n-1}(r^2+r+a-1)}\right) + \frac{c}{b^{r/(r-1)}} e^{-r^{n-1}(r(ar+1))} + o\left(e^{-r^{n-1}(r(ar+1))}\right) \\
&\qquad - e^{-r^{n+1}} - rce^{-r^n(r+a-1)} - \frac{r(r-1)}{2} c^2 e^{-r^n(r+2(a-1))} + o\left(e^{-r^n(r+2(a-1))}\right).
\end{aligned} \tag{2.13}$$

Now note that the first terms in the first and third rows cancel, and that the factors in the exponents standing by $-r^{n-1}$ are the following:

$$\begin{gathered} ar^2, \qquad r^2+r, \qquad r^2+r+a-1, \qquad ar^2+r, \\ r(r+a-1), \qquad r(r+2(a-1)), \end{gathered} \tag{2.14}$$

also

$$\begin{gathered} \min\{ar^2, r^2+r, r^2+r+a-1, ar^2+r\} = \min\{ar^2, r^2+r\}, \\ \min\{r(r+a-1), r(r+2(a-1))\} = r(r+a-1). \end{gathered} \tag{2.15}$$

Choose a such that

$$\min\{ar^2, r^2+r\} = r(r+a-1). \tag{2.16}$$

Assume that

$$\min\{ar^2, r^2+r\} = r^2+r. \tag{2.17}$$

Note that in this case, $a \geq 1+1/r$. Then we have $r^2+r = r(r+a-1)$, which implies that $a=2$. For such chosen a, we have that $a > 1+1/r$ for every $r>1$, so that (2.17) indeed holds.

Since the corresponding terms in (2.13) must cancel, it follows that $c = 1/(rb^{r/(r-1)})$. The next smallest term is $r^2+r+a-1 = r^2+r+1$ or $ar^2 = 2r^2$, depending on whether r is less or greater than $(1+\sqrt{5})/2$.

Hence

$$F(\varphi_{n-1}, \varphi_n, \varphi_{n+1}) = O\left(e^{-r^{n-1}(r^2+r+1)}\right) \quad \text{if } r > \frac{1+\sqrt{5}}{2}, \tag{2.18}$$

or

$$F(\varphi_{n-1}, \varphi_n, \varphi_{n+1}) = O\left(e^{-r^{n-1}(2r^2)}\right) \quad \text{if } 1 < r \leq \frac{1+\sqrt{5}}{2}. \tag{2.19}$$

Let φ_n be the sequence in (2.12), $\psi_n = e^{-2r^n}$, and $y_n \in X(1)$, that is, $y_n = \varphi_n + C_n\psi_n$ where $|C_n| \leq 1$. Then, we have that

$$\begin{gathered} F_{w_1}(y_{n-1}, y_n, y_{n+1}) \sim F_{w_1}(\varphi_{n-1}, \varphi_n, \varphi_{n+1}) = 1+\varphi_{n-1}^r \sim 1, \\ F_{w_0}(y_{n-1}, y_n, y_{n+1}) \sim F_{w_0}(\varphi_{n-1}, \varphi_n, \varphi_{n+1}) = -rb\varphi_n^{r-1} \sim -re^{-r^n(r-1)}, \\ F_{w_{-1}}(y_{n-1}, y_n, y_{n+1}) \sim F_{w_{-1}}(\varphi_{n-1}, \varphi_n, \varphi_{n+1}) = r\varphi_{n+1}\varphi_{n-1}^{r-1} \sim \frac{r}{b^{r/(r-1)}} e^{-r^{n-1}(r^2+r-1)}, \end{gathered} \tag{2.20}$$

from which it follows that condition (2.3) in Theorem 2.1 is satisfied. Further, from (2.20) it follows that

$$\begin{gathered} \psi_{n+1}F_{w_1}(\varphi_{n-1}, \varphi_n, \varphi_{n+1}) \sim e^{-2r^{n+1}}, \\ \psi_n F_{w_0}(\varphi_{n-1}, \varphi_n, \varphi_{n+1}) \sim -re^{-r^n(r+1)}, \\ \psi_{n-1}F_{w_{-1}}(\varphi_{n-1}, \varphi_n, \varphi_{n+1}) \sim rb^{-/r(r-1)}e^{-r^{n-1}(r^2+r+1)}. \end{gathered} \tag{2.21}$$

Now we choose $f_n = e^{-r^n(r+1)}$, for which condition (2.4) in Theorem 2.1 is satisfied, as well as condition (2.5) with $A_1 = A_{-1} = 0$ and $A_0 = -r$ (i.e., $k = 1$ in the notations of the theorem).

Remark 2.3. Ansatz (2.12) can also be obtained in the following fashion. Solving (2.7) with respect to x_n, we have that

$$x_n = \sqrt[r]{\frac{1}{b} x_{n+1}\left(1 + x_{n-1}^r\right)}. \tag{2.22}$$

Putting (2.11) into the right-hand side of the last equation, it follows that

$$x_n = \frac{1}{b^{1/(r-1)}} \sqrt[r]{e^{-r^{n+1}}\left(1 + \frac{1}{b^{r/(r-1)}} e^{-r^n}\right)} = \frac{1}{b^{1/(r-1)}} e^{-r^n}\left(1 + \frac{1}{rb^{r/(r-1)}} e^{-r^n}\right) \tag{2.23}$$

up to the smaller terms. Therefore, we obtain (2.12) with the foregoing a and c.

3. Asymptotics of some solutions of the difference equation $x_{n+1} = x_{n-1}/(p + x_{n-1} + x_n)$

In [27], we proved that every positive solution of the difference equation

$$x_{n+1} = \frac{x_{n-1}}{p + x_{n-1} + x_n}, \quad n \in \mathbb{N}_0, \tag{3.1}$$

where $p \in [0, \infty)$ and the initial conditions x_{-1}, x_0 are positive real numbers, converges to a, not necessarily prime, periodic-two solution. Moreover, we showed that positive solutions of (3.1) converge to the corresponding periodic-two solutions geometrically. One of the conjectures we posed in [27] is the following.

Conjecture 3.1. *Let $p \in (0,1)$ and let (x_n) be a nonnegative solution of (3.1) such that $(x_{2n-1}, x_{2n}) \to (\rho, 1 - \rho - p)$, as $n \to \infty$. Then*

(a) $x_{2n-1} = \rho + \rho c t^n + \mathbb{O}(t^{2n})$;

(b) $x_{2n} = 1 - \rho - p + (1 - \rho)(1 - \rho - p) c t^n + \mathbb{O}(t^{2n})$;

where $t = (1 - \rho)(\rho + p)$ and the constant c depends on initial values x_{-1} and x_0.

It is easy to see that the linearized equation about the positive equilibrium $\overline{x} = (1 - p)/2$, associated to (3.1), is

$$y_{n+1} + \overline{x} y_n + (\overline{x} - 1) y_{n-1} = 0. \tag{3.2}$$

The characteristic roots of the characteristic polynomial $P(\lambda)$ of (3.2) are $\lambda_1 = -1$ and $\lambda_2 = 1 - \overline{x} = (1 + p)/2 := s$, that is, $P(\lambda) = (\lambda + 1)(\lambda - s)$.

In this section, we confirm the conjecture for the case $\rho = (1-p)/2$, by proving the following theorem.

Theorem 3.2. *Assume that $p \in (0,1)$. Then (3.1) has solutions with the following asymptotics:*

$$x_n = \overline{x} + \sum_{k=1}^{\infty} q_k c^k s^{nk}, \tag{3.3}$$

where c is a positive parameter.

Proof. Since $p \in (0,1)$, then $s \in (1/2,1)$. Replacing (3.3) into (3.1) and equating the coefficients nearby c^k, it follows that

$$q_1 s^{n-1}\left(s^2 + \overline{x}s + \overline{x} - 1\right) = 0. \tag{3.4}$$

Since $P(s) = 0$, it follows that q_1 is an arbitrary number. Without loss of generality, we may take $q_1 = \overline{x}$.

For $k = 2$, we have that

$$q_2 s^{2(n+1)} + q_1^2 s^{n+1}\left(s^{n-1} + s^n\right) + q_2\overline{x}\left(s^{2(n-1)} + s^{2n}\right) = q_2 s^{2(n-1)}, \tag{3.5}$$

which implies that

$$\left|q_2\right| = \frac{q_1^2 s^2(1+s)}{\left|P(s^2)\right|} \le \left|q_1\right| \frac{s(1+s)}{\left(s^2+1\right)} < \left|q_1\right|, \tag{3.6}$$

where we have used the fact that $\overline{x} = q_1 = (1-p)/2 = 1-s$.

For $k \ge 3$, we have that

$$q_k s^{k(n+1)} + \sum_{i=0}^{k-2} q_{k-1-i} q_{i+1} s^{(k-1-i)(n+1)}\left(s^{(i+1)(n-1)} + s^{(i+1)n}\right) + \overline{x} q_k\left(s^{k(n-1)} + s^{kn}\right) = q_k s^{k(n-1)}, \tag{3.7}$$

from which it follows that

$$q_k = -\frac{1}{P\left(s^k\right)} \sum_{i=0}^{k-2} q_{k-1-i} q_{i+1} s^{2(k-1-i)}\left(1 + s^{i+1}\right), \tag{3.8}$$

and consequently

$$\left|q_k\right| = \frac{1}{\left|P\left(s^k\right)\right|} \max_{i=\overline{0,k-2}} q_{k-1-i} q_{i+1} \sum_{i=0}^{k-2} s^{2(k-1-i)}\left(1 + s^{i+1}\right). \tag{3.9}$$

Now we prove by the induction that $0 < q_k < q_1$ for every $k \ge 2$. Since we have already proved that $0 < q_2 < q_1$, the statement holds true for $k = 2$. Assume that the statement

holds true for $i \le k-1$. Then by (3.9) and the induction hypothesis, it follows that

$$\begin{aligned} q_k &\le q_1 \frac{q_1}{|P(s^k)|}\left(s^2+\cdots+s^{2(k-1)}+s^{k+1}+\cdots+s^{2k-1}\right) \\ &= q_1 \frac{s^2+\cdots+s^{2(k-1)}+s^{k+1}+\cdots+s^{2k-1}}{s(s^k+1)(1+s+\cdots+s^{k-2})} < q_1, \end{aligned} \tag{3.10}$$

finishing the inductive proof. □

From this, it follows that the series in (3.3) converges if $|cs^n| < 1$. Since for each $c \in \mathbb{R}$ there is an $n_0 \in \mathbb{N}$ such that $|cs^{n_0}| < 1$, we have that the series are not only asymptotic ones as $n \to \infty$, but they even converge for every $c \in \mathbb{R}$ and sufficiently large n, as desired.

4. Periodic character of positive solutions of a generalization of (3.1)

In this section, we investigate the periodic character of positive solutions of the following extension of (3.1):

$$x_{n+1} = \frac{x_{n-k}}{p+x_{n-k}+\cdots+x_n}, \quad n \in \mathbb{N}_0, \tag{4.1}$$

where $p \in [0,\infty)$ and the initial conditions $x_{-k},\dots,x_{-1},x_0$ are positive real numbers. The case $p \in (0,1)$ is more interesting, since when $p \ge 1$ it is easy to see that the zero equilibrium of (4.1) is global attractor of all positive solutions. Our results extend those ones in [27].

Theorem 4.1. *Consider (4.1) where $p \in (0,1)$. Then every positive solution of (4.1) converges to a, not necessarily prime, period-$(k+1)$ solution $\rho_1,\dots,\rho_{k+1}$, such that $p+\rho_1+\cdots+\rho_{k+1}=1$. If $p+x_0+x_{-1}+\cdots+x_{-k}>1$ the sequences $x_{(k+1)n+i}$, $(i=0,1,2,\dots,k)$ are decreasing, if $p+x_0+x_{-1}+\cdots+x_{-k}<1$, the sequences $x_{(k+1)n+i}$ $(i=0,1,2,\dots,k)$ are increasing, and if $p+x_0+x_{-1}+\cdots+x_{-k}=1$ the sequence x_n is a period-$(k+1)$ solution of (4.1).*

Proof. Using the change of variable $x_n = 1/z_n$, we obtain

$$z_{n+1} = \frac{p\prod_{i=0}^{k} z_{n-i} + \sum_{j=0}^{k}\prod_{i=0,i\ne j}^{k} z_{n-i}}{\prod_{i=0}^{k-1} z_{n-i}}, \tag{4.2}$$

and consequently

$$u_{n+1} = z_{n+1}z_n\cdots z_{n-k+1} = p\prod_{i=0}^{k} z_{n-i} + \sum_{j=0}^{k}\prod_{i=0,i\ne j}^{k} z_{n-i}. \tag{4.3}$$

Using (4.3) and calculating $u_{n+1}-u_n$, it follows that

$$z_{n+1}-z_{n-k} = \left(z_n - z_{n-(k+1)}\right)\frac{\left(p\prod_{i=1}^{k} z_{n-i} + \sum_{j=1}^{k}\prod_{i=1,i\ne j}^{k} z_{n-i}\right)}{\prod_{i=0}^{k-1} z_{n-i}}, \tag{4.4}$$

which implies that

$$z_{n+1} - z_{n-k} = (z_1 - z_{-k}) \prod_{l=1}^{n} \frac{\left(p \prod_{i=1}^{k} z_{l-i} + \sum_{j=1}^{k} \prod_{i=1, i\neq j}^{k} z_{l-i}\right)}{\prod_{i=0}^{k-1} z_{l-i}}. \tag{4.5}$$

From (4.5), we obtain that the signum of $z_{n+1} - z_{n-k}$ remains invariant for $n \in \mathbb{N}_0$ and that the sequences $(z_{(k+1)n+i})$, $i = 0, 1, \ldots, k$, are nondecreasing or nonincreasing at the same time which implies that the sequences $(x_{(k+1)n+i})$, $i = 0, 1, \ldots, k$, are nonincreasing or nondecreasing at the same time.

Since

$$z_1 - z_{-k} = \frac{p + x_0 + x_{-1} + \cdots + x_{-k} - 1}{x_{-k}}, \tag{4.6}$$

we see from (4.5) that if $p + x_0 + x_{-1} + \cdots + x_{-k} < 1$, then the sequences $(x_{(k+1)n+i})$, $i = 0, 1, \ldots, k$, are increasing, if $p + x_0 + x_{-1} + \cdots + x_{-k} > 1$, the sequences $(x_{(k+1)n+i})$, $i = 0, 1, \ldots, k$, are decreasing, and if $p + x_0 + x_{-1} + \cdots + x_{-k} = 1$, then $(x_{-k}, \ldots, x_{-1}, x_0, x_{-k}, \ldots, x_{-1}, x_0, \ldots)$ is a period-$(k+1)$ solution of (4.1).

Assume that the sequences $(x_{(k+1)n+i})$, $i = 0, 1, \ldots, k$, are decreasing, that is, $p + x_0 + x_{-1} + \cdots + x_{-k} > 1$. Then there are finite limits

$$\lim_{n\to\infty} x_{(k+1)n+i} = \rho_i, \quad i = 0, 1, \ldots, k. \tag{4.7}$$

It is clear that

$$(\rho_0, \rho_1, \ldots, \rho_k, \rho_0, \rho_1, \ldots, \rho_k, \ldots) \tag{4.8}$$

is a period-$(k+1)$ solution of (4.1). Suppose that all ρ_i are equal to zero. Since $(x_{(k+1)n+i})$, $i = 0, 1, \ldots, k$, are decreasing from (4.1), we obtain

$$p + x_{n-k} + \cdots + x_n > 1, \quad n \in \mathbb{N}_0. \tag{4.9}$$

Letting $n \to \infty$ in (4.9), we obtain $p \geq 1$, which is a contradiction. Hence $(\rho_0, \rho_1, \ldots, \rho_k) \neq (0, \ldots, 0)$, that is, (4.8) is a nontrivial period-$(k+1)$ solution of (4.1).

Without loss of generality, we may assume that $\rho_0 \neq 0$. Then letting $n \to \infty$ in the equation

$$x_{(k+1)n} = \frac{x_{(k+1)(n-1)}}{p + x_{(k+1)(n-1)} + \cdots + x_{(k+1)n-1}}, \tag{4.10}$$

we obtain the equality $p + \rho_0 + \rho_1 + \cdots + \rho_k = 1$.

Now suppose that the sequences $(x_{(k+1)n+i})$, $i = 0, 1, \ldots, k$, are increasing, that is, $p + x_0 + x_{-1} + \cdots + x_{-k} < 1$. Then there are finite or infinite limits

$$\lim_{n\to\infty} x_{(k+1)n+i} = \rho_i, \quad i = 0, 1, \ldots, k. \tag{4.11}$$

From (4.1), we see that $0 < x_{n+1} < 1$, that is, all solutions of (4.1) are bounded. Hence $\rho_i < \infty$, $i = 0,1,\dots,k$. On the other hand, since $(x_{(k+1)n+i})$, $i = 0,1,\dots,k$, are increasing, $\rho_i > x_i > 0$, $i = -k,\dots,-1,0$. Similarly as above, we obtain that (4.8) is a period-$(k+1)$ solution of (4.1) and $p+\rho_0+\rho_1+\cdots+\rho_k = 1$. □

Remark 4.2. For the initial conditions $x_{-k} = \cdots = x_{-1} = x_0 = (1-p)/(k+1)$, we have $x_n = (1-p)/(k+1)$, when $n \geq -k$, which shows that there is a solution which converges to a not prime period-$(k+1)$ solution.

Remark 4.3. From (4.5), we see that condition (4.9) for $n = 0$, that is, $p + x_0 + x_{-1} + \cdots + x_{-k} > 1$, implies (4.9) for all greater n, that is, for $n \geq 1$, moreover the sequence $u_n = p + x_{n-k} + \cdots + x_n$ is also decreasing.

Also, the condition $p + x_0 + x_{-1} + \cdots + x_{-k} < 1$ implies that the sequence $u_n = p + x_{n-k} + \cdots + x_n$ is increasing and

$$p + x_{n-k} + \cdots + x_n < 1, \quad n \in \mathbb{N}_0. \tag{4.12}$$

From this and by Theorem 4.1, it follows that the distance from the point $(x_{n-k},\dots,x_n)$ to the limit hyperplane $p + y_1 + y_2 + \cdots + y_{k+1} = 1$, that is,

$$d_n = \frac{p + x_n + \cdots + x_{n-k} - 1}{\sqrt{k+1}}, \tag{4.13}$$

also converges monotonously to zero.

Note also that from this and the fact that the sequences $(x_{(k+1)n+i})$, $i = 0,1,\dots,k$, are nonincreasing and positive or nondecreasing and bounded from above by 1, we obtain another proof of Theorem 3.2.

Remark 4.4. The fact that there is no solutions of (4.1) converging to zero follows also from the main result in [26].

Remark 4.5. For $k = 0$, (4.1) is a Riccati equation with the general solution

$$x_n^{(1)} = \frac{1-p}{1+cp^n}. \tag{4.14}$$

Hence, (4.1) with arbitrary k has the particular solution $x_{(k+1)n} = x_n^{(1)}$, and $x_i = 0$ if $i \neq 0 \pmod{k+1}$.

The following open problem will be interesting to the experts in the field.

Open problem 4.6. Let

$$\dots,\rho_0,\rho_1,\dots,\rho_k,\rho_0,\rho_1,\dots,\rho_k,\dots \tag{4.15}$$

be a positive period-$(k+1)$ solution of (4.1). Find the basin of attraction of this solution.

Now we give an estimation of the convergence rate of the positive solutions of (4.1).

Theorem 4.7. *Every positive solution of (4.1) converges to the corresponding period-$(k+1)$ solution $(\rho_0,\rho_1,\ldots,\rho_k)$ geometrically, that is, there are an $M>0$ and $q\in(0,1)$ such that*

$$|x_{(k+1)n}-\rho_0|+\cdots+|x_{(k+1)n+k}-\rho_k|\le Mq^{(k+1)n},\quad n\ge 0. \tag{4.16}$$

Proof. If we turn back to the sequence x_n in (4.4), we have that

$$\frac{p+x_n+x_{n-1}+\cdots+x_{n-k}-1}{x_{n-k}}=\frac{p+x_{n-k}+\cdots+x_{n-1}}{x_{n-k}}x_n\frac{p+x_{n-1}+x_{n-2}+\cdots+x_{n-k-1}-1}{x_{n-k-1}}, \tag{4.17}$$

that is,

$$d_n=\frac{p+x_{n-k}+\cdots+x_{n-1}}{x_{n-k-1}}x_n d_{n-1}=(1-x_n)d_{n-1}, \tag{4.18}$$

where $d_n=p+x_{n-k}+\cdots+x_n-1$, and consequently

$$d_n=\left(\prod_{i=0}^{k}(1-x_{n-i})\right)d_{n-(k+1)}. \tag{4.19}$$

On the other hand, by Theorem 4.1, we have that

$$\lim_{n\to\infty}\prod_{i=0}^{k}(1-x_{n-i})=\prod_{i=0}^{k}(1-\rho_i). \tag{4.20}$$

In the proof of Theorem 4.1, we have seen that all $\rho_i\in[0,1)$, $i\in\{0,1,\ldots,k\}$, cannot be equal to zero, hence

$$L=\prod_{i=0}^{k}(1-\rho_i)<1. \tag{4.21}$$

From (4.18)–(4.21), we have that for every $\varepsilon\in(0,1-L)$, there is an $n_0\in\mathbb{N}$ such that

$$\prod_{i=0}^{k}(1-x_{n-i})<L+\varepsilon, \tag{4.22}$$

for all $n\ge n_0$.

From this and (4.19), it follows that

$$|d_n|\le(L+\varepsilon)|d_{n-k-1}|, \tag{4.23}$$

for every $n\ge n_0$.

Now from (4.23) and the equality

$$\begin{aligned}|d_{(k+1)n+k}|&=|x_{(k+1)n+k}-\rho_k+\cdots+x_{(k+1)n}-\rho_0|\\&=|x_{(k+1)n+k}-\rho_k|+\cdots+|x_{(k+1)n}-\rho_0|,\end{aligned} \tag{4.24}$$

which follows by monotonicity of the sequences $(x_{(k+1)n+i})$, $i = 0,1,\dots,k$, we see that the result follows for

$$q = \sqrt[k+1]{L+\varepsilon}. \tag{4.25}$$

□

Corollary 4.8. *The distance d_n from the point $(x_{n-k},\dots,x_n)$ to the limit hyperplane $p + y_1 + y_2 + \cdots + y_{k+1} = 1$ converges to zero monotonously and moreover geometrically.*

5. Existence of nonoscillatory solutions of equation $y_n = A + (y_{n-k}/\sum_{j=1}^{m}\beta_j y_{n-q_j})^p$

This section studies the asymptotics of some positive solutions of the recursive equation

$$y_n = A + \left(\frac{y_{n-k}}{\sum_{j=1}^{m}\beta_j y_{n-q_j}}\right)^p, \quad n \in \mathbb{N}_0, \tag{5.1}$$

where $p, A \in (0,\infty)$, $k, m \in \mathbb{N}$, q_j, $j \in \{1,\dots,m\}$, are natural numbers such that $q_1 < q_2 < \cdots < q_m$, $\beta_j \in (0,+\infty)$, $j \in \{1,\dots,m\}$, $\sum_{j=1}^{m}\beta_j = 1$, and $y_{-s}, y_{-s+1},\dots,y_{-1} \in (0,\infty)$, where $s = \max\{k, q_m\}$.

In [28], the present author posed the problem of investigating the existence of nonequilibrium nonoscillatory solutions of (5.1) for the case $p > 0$, $k = 2$, $\beta_j = 0$, $j \in \{0,1,\dots,m-1\}$, and $q_m = m$. The case $p = 1$, $k \in \mathbb{N}$ and $\beta_j = 0$, $j \in \{0,1,\dots,m-1\}$, was considered independently in [11] by DeVault et al., where among other results it was proven that all nonoscillatory solutions of (5.1) for the case converge to the positive equilibrium $\overline{y} = A + 1$, and where they posed the above problem for the case $p = 1$, $k \in \mathbb{N}$, and $m = q_m = 1$. The problem was solved by the present author in [31]. Here we extend one of the results from [31], for the case of (5.1). For some other results on (5.1) and some closely related equations, see, for example, [1–3, 12, 14, 16, 26, 29, 37, 38, 40] and the references therein.

Note that the linearized equation for (5.1) about the positive equilibrium $\overline{y} = A + 1$ is

$$(A+1)z_n + p\sum_{j=1}^{m}\beta_j z_{n-q_j} - pz_{n-k} = 0. \tag{5.2}$$

The characteristic polynomial associated with (5.2) depends on the sign of $k - q_m$. Thus, we have two different cases, namely, if $k > q_m$, then

$$P_1(t) = (A+1)t^k + p\sum_{j=1}^{m}\beta_j t^{k-q_j} - p = 0, \tag{5.3}$$

and if $k < q_m$, then

$$P_2(t) = (A+1)t^{q_m} - pt^{q_m-k} + p\sum_{j=1}^{m}\beta_j t^{q_m-q_j} = 0. \tag{5.4}$$

We will consider only the first case. Since $P_1(0) = -p < 0$, $P_1(1) = A + 1$, and

$$P_1'(t) = (A+1)kt^{k-1} + p\sum_{j=1}^{m}\beta_j(k-q_j)t^{k-q_j-1} > 0, \tag{5.5}$$

when $t \in (0,1]$, it follows that for each $A > -1$, there is a unique positive root t_0 of the polynomial P_1 belonging to the interval $(0,1)$.

These facts motivated us to believe that in the case $k > q_m$, there are solutions of (5.1) which have the following asymptotics:

$$y_n = A + 1 + at_0^n + o(t_0^n), \tag{5.6}$$

where $a \in \mathbb{R}$ and t_0 is the above-mentioned root of the polynomial in (5.3).

Now we show that a nonoscillatory solution of (5.1) for the case $k > q_m$ exists by developing Berg's ideas in [6] which are based on asymptotics. Asymptotics for solutions of difference equations have been investigated by Berg and Stević, see, for example, [4–10, 22, 24, 30–32] and the reference therein. The problem is solved by constructing two appropriate sequences y_n and z_n with

$$y_n \le x_n \le z_n \tag{5.7}$$

for sufficiently large n. As we have already mentioned, some methods for the construction of these bounds can be found in [5, 6], also see [7, 9].

From (5.6) and results in Berg's paper [6], we expect that for $k \ge 2$, such solutions have the first three members in their asymptotics in the following form:

$$\varphi_n = A + 1 + at^n + bt^{2n}. \tag{5.8}$$

In order to solve the open problem for the case $k > q_m$, we use an inclusion theorem which can be proved as in [32].

Theorem 5.1. *Let $f : I^{k+2} \to I$ be a continuous and nondecreasing function in each argument on the interval $I \subset \mathbb{R}$, and let (y_n) and (z_n) be sequences with $y_n < z_n$ for $n \ge n_0$ and such that*

$$y_{n-k} \le f(n, y_{n-k+1}, \dots, y_{n+1}), \qquad f(n, z_{n-k+1}, \dots, z_{n+1}) \le z_{n-k}, \tag{5.9}$$

for $n > n_0 + k - 1$.

Then there is a solution of the following difference equation:

$$x_{n-k} = f(n, x_{n-k+1}, \dots, x_{n+1}), \tag{5.10}$$

with property (5.7) for $n \ge n_0$.

The following is the main result of this section.

Theorem 5.2. *Suppose that $k > q_m$, $A > -1$, and $p > 0$. Then, there is a nonoscillatory solution of (5.1) converging to the positive equilibrium $\overline{y} = A + 1$, as $n \to \infty$, moreover this solution has the asymptotics*

$$y_n = A + 1 + at^n + bt^{2n} + o(t^{2n}), \tag{5.11}$$

for some $t \in (0,1)$ and some $a, b \neq 0$.

Proof. Equation (5.1) can be written in the following equivalent form:

$$F(y_{n-k}, y_{n-q_m}, \ldots, y_{n-q_1}, y_n) = (y_n - A)^{1/p}\left(\sum_{j=1}^{m} \beta_j y_{n-q_j}\right) - y_{n-k} = 0. \tag{5.12}$$

We expect that solutions of (5.1) have the asymptotic approximation (5.8). Thus, we calculate

$$F(\varphi_{n-k}, \varphi_{n-q_m}, \ldots, \varphi_{n-q_1}, \varphi_n). \tag{5.13}$$

We have

$$\begin{aligned} F &= (1 + at^n + bt^{2n})^{1/p}\left(A + 1 + \sum_{j=1}^{m} \beta_j (at^{n-q_j} + bt^{2(n-q_j)})\right) - (A + 1 + at^{n-k} + bt^{2(n-k)}) \\ &= \frac{a}{p} t^n \left((A+1) + p\sum_{j=1}^{m} \frac{\beta_j}{t^{q_j}} - \frac{p}{t^k}\right) \\ &\quad + \frac{1}{p} t^{2n}\left(b(A+1) + bp\sum_{j=1}^{m} \frac{\beta_j}{t^{2q_j}} - \frac{bp}{t^{2k}} + a^2 \sum_{j=1}^{m} \frac{\beta_j}{t^{q_j}} + \frac{(1-p)a^2(A+1)}{2p}\right) + o(t^{2n}). \end{aligned} \tag{5.14}$$

Let

$$D(t) = (A+1) + p\sum_{j=1}^{m} \beta_j t^{-q_j} - pt^{-k}. \tag{5.15}$$

Now, choose $t \in (0,1)$ such that $D(t) = 0$, and $a, b \in \mathbb{R}$, $a \neq 0$, such that the coefficients in (5.14) are equal to zero. Since $D(t) = 0$ implies that $t = t_0$ (see above discussion), by (5.14) we have that

$$b = -\frac{a^2 \sum_{j=1}^{m} \beta_j t_0^{-q_j} + (1-p)a^2(A+1)/2p}{(A+1) + p\sum_{j=1}^{m} \beta_j t^{-2q_j} - pt_0^{-2k}} = -\frac{a^2 \sum_{j=1}^{m} \beta_j t_0^{-q_j} + (1-p)a^2(A+1)/2p}{D(t_0^2)}. \tag{5.16}$$

Observe that since $k > q_m \geq m$, we have that

$$D'(t) = \frac{kp}{t^{k+1}} - p\sum_{j=1}^{m} \frac{q_j \beta_j}{t^{q_j+1}} > \frac{p}{t^{k+1}}\left(k - \sum_{j=1}^{m} q_j \beta_j\right) > 0 \tag{5.17}$$

when $t \in (0,1]$. Thus we have that $D(t_0^2) < D(t_0) = 0$.

If

$$\widehat{\varphi}_n = A + 1 + at_0^n + qt_0^{2n}, \tag{5.18}$$

we obtain

$$F(\widehat{\varphi}_{n-k},\dots,\widehat{\varphi}_n) \sim \frac{1}{p}\left(qD(t_0^2) + a^2\sum_{j=1}^{m}\beta_j t_0^{-q_j} + \frac{(1-p)a^2(A+1)}{2p}\right)t_0^{2n}. \tag{5.19}$$

Let

$$H_{t_0}(q) = qD(t_0^2) + a^2\sum_{j=1}^{m}\beta_j t_0^{-q_j} + \frac{(1-p)a^2(A+1)}{2p}. \tag{5.20}$$

Since $H'_{t_0}(q) = D(t_0^2) < 0$, we obtain that there are $q_1 < b$ and $q_2 > b$ such that

$$H_{t_0}(q_1) > 0, \qquad H_{t_0}(q_2) < 0. \tag{5.21}$$

With the notations

$$u_n = A + 1 + at_0^n + q_1 t_0^{2n}, \qquad z_n = A + 1 + at_0^n + q_2 t_0^{2n}, \tag{5.22}$$

we get

$$\begin{aligned} F(u_{n-k},\dots,u_n) &\sim \frac{1}{p}\left(q_1 D(t_0^2) + a^2\sum_{j=1}^{m}\beta_j t_0^{-q_j} + \frac{(1-p)a^2(A+1)}{2p}\right)t_0^{2n} > 0, \\ F(z_{n-k},\dots,z_n) &\sim \frac{1}{p}\left(q_2 D(t_0^2) + a^2\sum_{j=1}^{m}\beta_j t_0^{-q_j} + \frac{(1-p)a^2(A+1)}{2p}\right)t_0^{2n} < 0. \end{aligned} \tag{5.23}$$

These relations show that inequalities in (5.9), where instead of y_n appears u_n, are satisfied for sufficiently large n, and where $f = F + y_{n-k}$ and F is given by (5.12). Since for sufficiently large n, $y_n > A$, we can apply Theorem 5.1 with $I = (A, \infty)$, and see that there are an $n_0 \geq 0$ and a solution of (5.1) with the asymptotics $y_n = \widehat{\varphi}_n + o(t_0^{2n})$, for $n \geq n_0$, where $q = b$ in $\widehat{\varphi}_n$; in particular, the solution converges monotonically to the positive equilibrium $\overline{y} = A + 1$ of (5.1), for $n \geq n_0$. Hence, the solution y_{n+n_0+k} is also such a solution when $n \geq -k$. □

Remark 5.3. Since $a \in \mathbb{R} \setminus \{0\}$ is an arbitrary parameter, by Theorem 5.2 we find a set of nonoscillatory solutions of (5.1) converging to the positive equilibrium. A natural question is in what extent these one-parameter family of solutions covers solutions of (5.1).

Remark 5.4. From the proof of Theorem 5.2, we see that the parameter A can be replaced by a nonincreasing sequence with the asymptotics $A_n = A + o(t_0^{2n})$, and that in this case there is a positive solution of the corresponding equation which is eventually nonoscillatory. This means that the method is quite powerful since it proves the existence of eventually nonoscillatory solutions also for some nonautonomous nonlinear difference equations.

6. A new inclusion theorem

Motivated by Theorem 5.1, here we present a new inclusion theorem which could be useful in the investigation of the nonlinear difference equations and finding the asymptotics of some of their solutions.

Theorem 6.1. *Let the function $f(u_1,\dots,u_k)$ be continuous, nondecreasing in variables u_i, $i \in I \subset \{1,\dots,k\}$, and nonincreasing in variables u_j, $j \in \{1,\dots,k\} \setminus I$, and let (y_n) and (z_n) be sequences with $y_n < z_n$ for $n \geq n_0$ as well as*

$$y_{n+1} \leq f(v_n,\dots,v_{n-k+1}), \qquad f(w_n,\dots,w_{n-k+1}) \leq z_{n+1}, \tag{6.1}$$

where

$$\begin{aligned} v_l &= \begin{cases} z_l & \text{if } v_l \text{ appears at a variable which index belongs to } \{1,\dots,k\} \setminus I, \\ y_l & \text{if } v_l \text{ appears at a variable which index belongs to } I, \end{cases} \\ w_l &= \begin{cases} y_l & \text{if } v_l \text{ appears at a variable which index belongs to } \{1,\dots,k\} \setminus I, \\ z_l & \text{if } v_l \text{ appears at a variable which index belongs to } I. \end{cases} \end{aligned} \tag{6.2}$$

Then there exists a solution of the difference equation

$$x_{n+1} = f(x_n,\dots,x_{n-k+1}) \tag{6.3}$$

such that

$$y_n \leq x_n \leq z_n. \tag{6.4}$$

Proof. We will prove the theorem for the case $k = 2$, and where $f(u_1,u_2)$ is nonincreasing in u_1 and nondecreasing in u_2. The proof in general case is only technically complicated.

Assume that (6.4) holds when $n \in \{n_0, n_0+1\}$. Then, from (6.1) and the monotonicity of the function f in each variable, we have that

$$y_{n_0+2} \leq f(z_{n_0+1}, y_{n_0}) \leq f(x_{n_0+1}, x_{n_0}) \leq f(y_{n_0+1}, z_{n_0}) \leq z_{n_0+2}. \tag{6.5}$$

Hence,

$$y_{n_0+2} \leq x_{n_0+2} \leq z_{n_0+2}, \tag{6.6}$$

that is, inequality (6.4) holds when $n = n_0 + 2$. By induction, the result easily follows. □

7. On the difference equation $x_{n+1} = 1 + x_n^{1+r}/x_{n-1}^r$

In [34], we investigated behavior of positive solutions of the difference equation

$$x_{n+1} = 1 + \frac{x_n^p}{x_{n-1}^r}, \tag{7.1}$$

where $p, q \in (0,\infty)$.

Among other results, we showed that in the case $p = r+1$, $r \in (0,1)$, (7.1), that is, the equation

$$x_{n+1} = 1 + \frac{x_n^{1+r}}{x_{n-1}^r}, \tag{7.2}$$

has unbounded solutions.

The following open problem regarding (7.2) is of some interest.

Open problem 7.1. Assume that $r \in (0,1)$. Find asymptotics of some unbounded solutions of (7.2).

Our idea is to find some solutions of (7.2) with the following asymptotics:

$$\varphi_n = an + b\ln n + c + d\frac{\ln n}{n} + \frac{e}{n}, \tag{7.3}$$

for some positive constants a, b, c, d, and e.

Since Open problem 7.1 cannot be solved by Theorems 2.1 or 5.1, we need another theorem which could be appropriate for treating the problem. This was the starting point for getting Theorem 6.1.

Although we are not able to solve the problem also by Theorem 6.1, we present here an interesting property of natural candidate (7.3) for the asymptotics for some solutions of (7.2). In order to avoid too much complicated calculations, we confine ourself to the case $r = 1/2$.

Hence, assume that $r = 1/2$ and write (7.2) in the following form:

$$F_n = F = \sqrt{(y_n)^3/z_{n-1}} - y_{n+1} + 1. \tag{7.4}$$

Set

$$y_n = 2n + b\ln n + c + d\frac{\ln n}{n} + \frac{e}{n}, \qquad z_n = 2n + p\ln n + q + r\frac{\ln n}{n} + \frac{s}{n}. \tag{7.5}$$

We have

$$\begin{aligned}
F_n &= \left[\frac{(2n + b\ln n + c + d(\ln n/n) + e/n)^3}{2n + p\ln(n-1) + q - 2 + r(\ln(n-1)/(n-1)) + s/(n-1)}\right]^{1/2} \\
&\quad + 1 - 2(n+1) - b\ln(n+1) - c - d\frac{\ln(n+1)}{n+1} - \frac{e}{n+1} \\
&= 2n\left(1 + b\frac{\ln n}{2n} + \frac{c}{2n} + d\frac{\ln n}{2n^2} + \frac{e}{2n^2}\right)^{3/2} \\
&\quad \times \left(1 + p\frac{\ln(n-1)}{2n} + \frac{q-2}{2n} + r\frac{\ln(n-1)}{2n(n-1)} + \frac{s}{2n(n-1)}\right)^{-1/2} \\
&\quad - 2n - b\ln(n+1) - c - 1 - d\frac{\ln(n+1)}{n+1} - \frac{e}{n+1}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{b-p}{2}\ln n + \frac{c-q}{2} + \left(\frac{3(b^2+p^2-2bp)}{16}\right)\frac{\ln^2 n}{n} \\
&\quad + \left(\frac{d-r}{2} + \frac{3}{8}(bc+p(q-2)-b(q-2)-cp)\right)\frac{\ln n}{n} \\
&\quad \times\left(\frac{e-s}{2} + \frac{3}{16}(c^2+(q-2)^2-2c(q-2)) + \frac{p}{2} - b\right)\frac{1}{n} + o\left(\frac{1}{n}\right).
\end{aligned}
\tag{7.6}
$$

From this and since it must be $b \le p$, we must choose $b = p$, which implies that $b^2 + p^2 - 2bp = 0$. Similarly, since it must be $c \le q$, we must choose $c = q$. Hence

$$
\begin{aligned}
\frac{d-r}{2} + \frac{3}{8}(bc+p(q-2)-b(q-2)-cp) &= \frac{d-r}{2}, \\
\frac{e-s}{2} + \frac{3}{16}(c^2+(q-2)^2-2c(q-2)) + \frac{p}{2} - b &= \frac{e-s-b}{2} + \frac{3}{4}.
\end{aligned}
\tag{7.7}
$$

Since $d \le r$, we take that $d = r$. It is possible to find b, e, and s such that $b + s - e < 3/2$, so that the first inequality in (6.1) holds. On the other hand, by symmetry, we obtain that for the above chosen parameters, we have that

$$
\sqrt{\frac{(z_n)^3}{y_{n-1}}} - z_{n+1} + 1 \sim \left(\frac{s-e-b}{2} + \frac{3}{4}\right)\frac{1}{n}. \tag{7.8}
$$

Since it must be $3/2 < b - (s - e)$, we arrive at a contradiction with $b + s - e < 3/2$ and $s \ge e$. Hence, it is natural to choose $b = 3/2$ and try with more members in the asymptotics of y_n and z_n.

This fact motivated us to choose

$$
y_n = 2n + \frac{3}{2}\ln n + a_k\frac{\ln^\alpha n}{n^\beta}, \qquad z_n = 2n + \frac{3}{2}\ln n + b_k\frac{\ln^\alpha n}{n^\beta}, \tag{7.9}
$$

$\alpha,\beta \in \mathbb{N}$, with $a_k \le b_k$. Interestingly, by similar calculations as above, we obtain that it must be

$$
F(y_{n-1}, y_{n+1}, z_n) \sim (a_k - b_k)\frac{\ln^{\alpha_1} n}{n^{\beta_1}} \ge 0 \tag{7.10}
$$

for some natural α_1 and β_1, which implies that $a_k = b_k$.

Remark 7.2. Although we have not managed to prove that (7.3) is a solution of (7.2), the asymptotic relation in (7.6) (or (7.10)) means that (7.3) is an asymptotic solution of (7.2) in sense of [5].

8. Nontrivial solutions of Putnam-type difference equations

By using the part metric technique and a theorem from [17], Yang [39] recently proved that every positive solution of the generalization of the so-called Putnam difference equation [20]

$$x_{n+1} = \frac{x_n + x_{n-1} + \cdots + x_{n-(k-1)} + x_{n-k}x_{n-(k+1)}}{x_n x_{n-1} + x_{n-2} + \cdots + x_{n-(k+1)}}, \quad n \in \mathbb{N}_0, \tag{8.1}$$

where $k \in \mathbb{N}$, converges to the positive equilibrium $\overline{x} = 1$, confirming a conjecture posed by Kruse and Nesemann in [17]. The asymptotic stability of (8.1), for the case $k = 2$, was studied previously in [17].

A somewhat challenging problem is to show that there is a positive solution of (8.1) which is not eventually equal to unity (for the case $k = 2$, the problem was posed in [19] and was solved in [33]).

The problem for the case $k \geq 3$ has been recently solved by the present author in [35] by proving the following theorem, where in the proof we used Theorem 2.1.

Theorem 8.1. *For every $k \in \mathbb{N}$ and $\varepsilon > 0$, (8.1) has a solution with the following asymptotics:*

$$x_n = 1 + (k+1)\varphi_n + (k+1)\varepsilon_n \psi_n, \tag{8.2}$$

where

$$\varphi_n = e^{-\lambda^n} + e^{-c\lambda^n}, \qquad \psi_n = e^{-c\lambda^n}, \tag{8.3}$$

for some $c > 1$ depending on k, λ is the positive root of the polynomial $P(\lambda) = \lambda^{k+2} - \lambda - 1$ belonging to the interval $(1,2)$, and $|\varepsilon_n| \leq \varepsilon$.

It was proved in [39] that every positive solution of the generalization of Putnam-type difference equation

$$x_{n+1} = \frac{\sum_{i \in \mathbb{Z}_{k+1} \setminus \{j,l\}} x_{n-i} + x_{n-j}x_{n-l} + A}{\sum_{i \in \mathbb{Z}_{k+1} \setminus \{s,t\}} x_{n-i} + x_{n-s}x_{n-t} + A}, \quad n \in \mathbb{N}_0, \tag{8.4}$$

where $k \in \mathbb{N}_0$, $j,l,s,t \in \mathbb{Z}_{k+2} = \{0,1,\ldots,k+1\}$ with $0 \leq j < l \leq k+1$, $0 \leq s < t \leq k+1$, $s+t < j+l$, and $A \in [0,\infty)$, converges to the positive equilibrium $\overline{x} = 1$.

It is expected that (8.4) also has nontrivial solutions but the calculations connected with the application of Theorem 2.1 are in some cases more involved, hence we leave the following conjecture to the interested readers.

Conjecture 8.2. *Equation (8.4) has a solution which is not eventually equal to unity.*

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References

[1] K. S. Berenhaut, J. D. Foley, and S. Stević, "The global attractivity of the rational difference equation $y_n = 1 + y_{n-k}/y_{n-m}$," *Proceedings of the American Mathematical Society*, vol. 135, no. 4, pp. 1133–1140, 2007.

[2] K. S. Berenhaut and S. Stević, "A note on the difference equation $x_{n+1} = 1/(x_n x_{n-1}) + 1/(x_{n-3}x_{n-4})$," *Journal of Difference Equations and Applications*, vol. 11, no. 14, pp. 1225–1228, 2005.

[3] K. S. Berenhaut and S. Stević, "The behaviour of the positive solutions of the difference equation $x_n = A + (x_{n-2}/x_{n-1})^p$," *Journal of Difference Equations and Applications*, vol. 12, no. 9, pp. 909–918, 2006.

[4] K. S. Berenhaut and S. Stević, "The difference equation $x_{n+1} = \alpha + (x_{n-k}/(\sum_{i=0}^{k-1} c_i x_{n-i}))$ has solutions converging to zero," *Journal of Mathematical Analysis and Applications*, vol. 326, no. 2, pp. 1466–1471, 2007.

[5] L. Berg, *Asymptotische Darstellungen und Entwicklungen*, Hochschulbücher für Mathematik, Band 66, VEB Deutscher Verlag der Wissenschaften, Berlin, Germany, 1968.

[6] L. Berg, "On the asymptotics of nonlinear difference equations," *Zeitschrift für Analysis und ihre Anwendungen*, vol. 21, no. 4, pp. 1061–1074, 2002.

[7] L. Berg, "Inclusion theorems for non-linear difference equations with applications," *Journal of Difference Equations and Applications*, vol. 10, no. 4, pp. 399–408, 2004.

[8] L. Berg, "Oscillating solutions of rational difference equations," *Rostocker Mathematisches Kolloquium*, no. 58, pp. 31–35, 2004.

[9] L. Berg, "Corrections to: "Inclusion theorems for non-linear difference equations with applications"," *Journal of Difference Equations and Applications*, vol. 11, no. 2, pp. 181–182, 2005.

[10] L. Berg and L. von Wolfersdorf, "On a class of generalized autoconvolution equations of the third kind," *Zeitschrift für Analysis und ihre Anwendungen*, vol. 24, no. 2, pp. 217–250, 2005.

[11] R. DeVault, C. Kent, and W. Kosmala, "On the recursive sequence $x_{n+1} = p + (x_{n-k}/x_n)$," *Journal of Difference Equations and Applications*, vol. 9, no. 8, pp. 721–730, 2003.

[12] H. M. El-Owaidy, A. M. Ahmed, and M. S. Mousa, "On asymptotic behaviour of the difference equation $x_{n+1} = \alpha + (x_{n-1}^p/x_n^p)$," *Journal of Applied Mathematics & Computing*, vol. 12, no. 1-2, pp. 31–37, 2003.

[13] G. Karakostas, "Convergence of a difference equation via the full limiting sequences method," *Differential Equations and Dynamical Systems*, vol. 1, no. 4, pp. 289–294, 1993.

[14] G. Karakostas, "Asymptotic 2-periodic difference equations with diagonally self-invertible responses," *Journal of Difference Equations and Applications*, vol. 6, no. 3, pp. 329–335, 2000.

[15] V. L. Kocić and G. Ladas, *Global Behavior of Nonlinear Difference Equations of Higher Order with Applications*, vol. 256 of *Mathematics and Its Applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1993.

[16] W. Kosmala and C. Teixeira, "More on the difference equation $y_{n+1} = (p + y_n)/(qy_n + y_{n-1})$," *Applicable Analysis*, vol. 81, no. 1, pp. 143–151, 2002.

[17] N. Kruse and T. Nesemann, "Global asymptotic stability in some discrete dynamical systems," *Journal of Mathematical Analysis and Applications*, vol. 235, no. 1, pp. 151–158, 1999.

[18] H. Levy and F. Lessman, *Finite Difference Equations*, Dover, New York, NY, USA, 1992.

[19] G. Ladas, "Open problems and conjectures," *Journal of Difference Equations and Applications*, vol. 4, pp. 497–499, 1998.

[20] "Putnam exam," *American Mathematical Monthly*, pp. 734–736, 1965.

[21] S. Stević, "On stability results for a new approximating fixed points iteration process," *Demonstratio Mathematica*, vol. 34, no. 4, pp. 873–880, 2001.

[22] S. Stević, "Asymptotic behavior of a sequence defined by iteration with applications," *Colloquium Mathematicum*, vol. 93, no. 2, pp. 267–276, 2002.

[23] S. Stević, "A global convergence results with applications to periodic solutions," *Indian Journal of Pure and Applied Mathematics*, vol. 33, no. 1, pp. 45–53, 2002.

[24] S. Stević, "Asymptotic behavior of a nonlinear difference equation," *Indian Journal of Pure and Applied Mathematics*, vol. 34, no. 12, pp. 1681–1687, 2003.

[25] S. Stević, "On the recursive sequence $x_{n+1} = A/(\prod_{i=0}^{k} x_{n-i}) + 1/(\prod_{j=k+2}^{2(k+1)} x_{n-j})$," *Taiwanese Journal of Mathematics*, vol. 7, no. 2, pp. 249–259, 2003.

[26] S. Stević, "A note on periodic character of a difference equation," *Journal of Difference Equations and Applications*, vol. 10, no. 10, pp. 929–932, 2004.

[27] S. Stević, "Periodic character of a difference equation," *Rostocker Mathematisches Kolloquium*, no. 59, pp. 3–10, 2005.

[28] S. Stević, "On the recursive sequence $x_{n+1} = \alpha + (x_{n-1}^p/x_n^p)$," *Journal of Applied Mathematics & Computing*, vol. 18, no. 1-2, pp. 229–234, 2005.

[29] S. Stević, "On the recursive sequence $x_{n+1} = (\alpha + \beta x_{n-k}/f(x_n,\ldots,x_{n-k+1}))$," *Taiwanese Journal of Mathematics*, vol. 9, no. 4, pp. 583–593, 2005.

[30] S. Stević, "Global stability and asymptotics of some classes of rational difference equations," *Journal of Mathematical Analysis and Applications*, vol. 316, no. 1, pp. 60–68, 2006.

[31] S. Stević, "On monotone solutions of some classes of difference equations," *Discrete Dynamics in Nature and Society*, vol. 2006, Article ID 53890, 9 pages, 2006.

[32] S. Stević, "On positive solutions of a $(k+1)$th order difference equation," *Applied Mathematics Letters*, vol. 19, no. 5, pp. 427–431, 2006.

[33] S. Stević, "Existence of nontrivial solutions of a rational difference equation," *Applied Mathematics Letters*, vol. 20, no. 1, pp. 28–31, 2007.

[34] S. Stević, "On the recursive sequence $x_{n+1} = A + (x_n^p/x_{n-1}^p)$," *Discrete Dynamics in Nature and Society*, vol. 2007, Article ID 34517, 9 pages, 2007.

[35] S. Stević, "Nontrivial solutions of a higher-order rational difference equation," submitted.

[36] T. Sun and H. Xi, "Global asymptotic stability of a family of difference equations," *Journal of Mathematical Analysis and Applications*, vol. 309, no. 2, pp. 724–728, 2005.

[37] T. Sun, H. Xi, and H. Wu, "On boundedness of the solutions of the difference equation $x_{n+1} = x_{n-1}/(p + x_n)$," *Discrete Dynamics in Nature and Society*, vol. 2006, Article ID 20652, 7 pages, 2006.

[38] S.-E. Takahasi, Y. Miura, and T. Miura, "On convergence of a recursive sequence $x_{n+1} = f(x_{n-1}, x_n)$," *Taiwanese Journal of Mathematics*, vol. 10, no. 3, pp. 631–638, 2006.

[39] X. Yang, "Global asymptotic stability in a class of generalized Putnam equations," *Journal of Mathematical Analysis and Applications*, vol. 322, no. 2, pp. 693–698, 2006.

[40] X.-X. Yan, W.-T. Li, and Z. Zhao, "On the recursive sequence $x_{n+1} = \alpha - (x_n/x_{n-1})$," *Journal of Applied Mathematics & Computing*, vol. 17, no. 1-2, pp. 269–282, 2005.

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