Some strong laws of large numbers for arrays of rowwise $\rho^*$-mixing random variables are obtained. The result obtained not only generalizes the result of Hu and Taylor (1997) to $\rho^*$-mixing random variables, but also improves it.

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1. Introduction

Let $\{X, X_n, n \geq 1\}$ be a sequence of independent identically distributed (i.i.d.) random variables. The Marcinkiewicz-Zygmund strong law of large numbers (SLLN) provides that

\[
\frac{1}{n^{1/\alpha}} \sum_{i=1}^{n} (X_i - EX_i) \longrightarrow 0 \quad \text{a.s. for } 1 \leq \alpha < 2,
\]

\[
\frac{1}{n^{1/\alpha}} \sum_{i=1}^{n} X_i \longrightarrow 0 \quad \text{a.s. for } 0 < \alpha < 1
\]

if and only if $E|X|^{\alpha} < \infty$. The case $\alpha = 1$ is due to Kolmogorov. In the case of independence (but not necessarily identically distributed), Hu and Taylor [1] proved the following strong law of large numbers.

Theorem 1.1. Let $\{X_{ni}; 1 \leq i \leq n, n \geq 1\}$ be a triangular array of rowwise independent random variables. Let $\{a_n, n \geq 1\}$ be a sequence of positive real numbers such that $0 < a_n \uparrow \infty$. Let $\psi(t)$ be a positive, even function such that $\psi(|t|)/|t|^p$ is an increasing function of $|t|$ and $\psi(|t|)/|t|^{p+1}$ is a decreasing function of $|t|$, respectively, that is,

\[
\frac{\psi(|t|)}{|t|^p} \uparrow, \quad \frac{\psi(|t|)}{|t|^{p+1}} \downarrow, \quad \text{as } |t| \uparrow
\]
for some nonnegative integer \( p \). If \( p \geq 2 \) and

\[
E X_{ni} = 0, \\
\sum_{n=1}^{\infty} \sum_{i=1}^{n} E \frac{\psi(\left|X_{ni}\right|)}{\psi(a_n)} < \infty, \\
\sum_{n=1}^{\infty} \left( \sum_{i=1}^{n} E \left(\frac{X_{ni}}{a_n}\right)^2 \right)^{2k} < \infty,
\]

where \( k \) is a positive integer, then

\[
\frac{1}{a_n} \sum_{i=1}^{n} X_{ni} \rightarrow 0 \quad a.s.
\]

Let nonempty sets \( S, T \subset \mathcal{N} \), and define \( \mathcal{F}_S = \sigma(X_k, k \in S) \), and the maximal correlation coefficient \( \rho^*_n = \sup \text{corr}(f, g) \) where the supremum is taken over all \( (S, T) \) with \( \text{dist}(S, T) \geq n \) and all \( f \in L_2(\mathcal{F}_S), g \in L_2(\mathcal{F}_T) \), and where \( \text{dist}(S, T) = \inf_{x \in S, y \in T} |x - y| \).

A sequence of random variables \( \{X_n, n \geq 1\} \) on a probability space \( \{\Omega, \mathcal{F}, P\} \) is called \( \rho^* \)-mixing if

\[
\lim_{n \to \infty} \rho^*_n < 1.
\]

An array of random variables \( \{X_{ni}; i \geq 1, n \geq 1\} \) is called rowwise \( \rho^* \)-mixing random variables if for every \( n \geq 1 \), \( \{X_{ni}; i \geq 1\} \) is a \( \rho^* \)-mixing sequence of random variables.


The main purpose of this paper is to establish a strong law of large numbers for arrays of rowwise \( \rho^* \)-mixing random variables. The result obtained not only generalizes the result of Hu and Taylor [1] to \( \rho^* \)-mixing random variables, but also improves it.

### 2. Main results

Throughout this paper, \( C \) will represent a positive constant though its value may change from one appearance to the next, and \( a_n = O(b_n) \) will mean \( a_n \leq Cb_n \).

Let \( \{X, X_n, n \geq 1\} \) be a sequence of independent identically distributed (i.i.d.) random variables and denote \( S_n = \sum_{i=1}^{n} X_i \). The Hsu-Robbins-Erdős law of large numbers (see Hsu and Robbins [7], Erdős [8]) states that

\[
\forall \varepsilon > 0, \quad \sum_{n=1}^{\infty} P(\left|S_n\right| > \varepsilon n) < \infty
\]

is equivalent to \( EX = 0, EX^2 < \infty \).
This is a fundamental theorem in probability theory and has been intensively investigated by many authors in the past decades. One of the most important results is Baum-Katz [9] law of large numbers, which states that for $p < 2$ and $r \geq p$,

$$\forall \varepsilon > 0, \sum_{n=1}^{\infty} n^{r/p-2} P( | S_n | > \varepsilon n^{1/p} ) < \infty$$

(2.2)

if and only if $E|X|^r < \infty$, $r \geq 1$, and $EX = 0$.

There are many extensions in various directions. Some of them can be found by Chow and Lai in [10, 11], where the authors propose a two-sided estimate, and by Petrov in [12].

In order to prove our main result, we need the following lemma.

**Lemma 2.1** (see Utev and Peligrad [6]). Let $\{X_i, i \geq 1\}$ be a $\rho^*$-mixing sequence of random variables, $EX_i = 0$, $E|X_i|^p < \infty$ for some $p \geq 2$ and for every $i \geq 1$. Then there exists $C = C(p)$, such that

$$E \max_{1 \leq k \leq n} \left| \sum_{i=1}^{k} X_i \right|^p \leq C \left\{ \sum_{i=1}^{n} E|X_i|^p + \left( \sum_{i=1}^{n} EX_i^2 \right)^{p/2} \right\}. \quad (2.3)$$

**Theorem 2.2.** Let $\{X_{ni}, i \geq 1, n \geq 1\}$ be an array of rowwise $\rho^*$-mixing random variables. Let $\{a_n, n \geq 1\}$ be a sequence of positive real numbers such that $0 < a_n \uparrow \infty$. Let $\psi(t)$ be a positive, even function such that $\psi(|t|/|t|)$ is an increasing function of $|t|$ and $\psi(|t|/|t|^p)$ is a decreasing function of $|t|$, respectively, that is,

$$\frac{\psi(|t|)}{|t|} \uparrow, \quad \frac{\psi(|t|)}{|t|^p} \downarrow, \quad \text{as} \ |t| \uparrow \quad (2.4)$$

for some nonnegative integer $p$. If $p \geq 2$ and

$$EX_{ni} = 0,$$

$$\sum_{n=1}^{\infty} \sum_{i=1}^{n} E \frac{\psi(|X_{ni}|)}{\psi(a_n)} < \infty, \quad (2.5)$$

$$\sum_{n=1}^{\infty} \left( \sum_{i=1}^{n} E \left( \frac{X_{ni}}{a_n} \right)^2 \right)^{v/2} < \infty,$$

where $v$ is a positive integer, $v \geq p$, then

$$\forall \varepsilon > 0, \sum_{n=1}^{\infty} P \left( \max_{1 \leq k \leq n} \left| \frac{1}{a_n} \sum_{i=1}^{k} X_{ni} \right| > \varepsilon \right) < \infty. \quad (2.6)$$
Proof of Theorem 2.2. For all $i \geq 1$, define $X_i^{(n)} = X_{ni}I(|X_{ni}| \leq a_n)$, $T_j^{(n)} = (1/a_n) \sum_{i=1}^{j} (X_i^{(n)} - EX_i^{(n)})$, then for all $\varepsilon > 0$,

$$P\left( \max_{1 \leq k \leq n} \left| \frac{1}{a_n} \sum_{i=1}^{k} X_{ni} \right| > \varepsilon \right) \leq P\left( \max_{1 \leq j \leq n} |X_{nj}| > a_n \right) + P\left( \max_{1 \leq j \leq n} |T_j^{(n)}| > \varepsilon - \max_{1 \leq j \leq n} \left| \frac{1}{a_n} \sum_{i=1}^{j} EX_i^{(n)} \right| \right).$$

(2.7)

First, we show that

$$\max_{1 \leq j \leq n} \left| \frac{1}{a_n} \sum_{i=1}^{j} EX_i^{(n)} \right| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \quad (2.8)$$

In fact, by $EX_{ni} = 0$, $\psi(|t|)/|t| \uparrow$ as $|t| \uparrow$ and $\sum_{n=1}^{\infty} \sum_{i=1}^{n} E(\psi(|X_{ni}|)/\psi(a_n)) < \infty$, then

$$\max_{1 \leq j \leq n} \left| \frac{1}{a_n} \sum_{i=1}^{j} EX_i^{(n)} \right| = \max_{1 \leq j \leq n} \left| \frac{1}{a_n} \sum_{i=1}^{j} EX_{ni}I(|X_{ni}| \leq a_n) \right|$$

$$= \max_{1 \leq j \leq n} \left| \frac{1}{a_n} \sum_{i=1}^{j} EX_{ni}I(|X_{ni}| > a_n) \right|$$

$$\leq \sum_{i=1}^{n} E|X_{ni}|I(|X_{ni}| > a_n)$$

$$\leq \sum_{i=1}^{n} \frac{E\psi(|X_{ni}|)I(|X_{ni}| > a_n)}{\psi(a_n)}$$

$$\leq \sum_{i=1}^{n} \frac{E\psi(|X_{ni}|)}{\psi(a_n)} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \quad (2.9)$$

From (2.7) and (2.8), it follows that for $n$ large enough,

$$P\left( \max_{1 \leq k \leq n} \left| \frac{1}{a_n} \sum_{i=1}^{k} X_{ni} \right| > \varepsilon \right) \leq \sum_{j=1}^{n} P(|X_{nj}| > a_n) + P\left( \max_{1 \leq j \leq n} |T_j^{(n)}| > \frac{\varepsilon}{2} \right).$$

(2.10)

Hence, we need only to prove that

$$I = \sum_{n=1}^{\infty} \sum_{j=1}^{n} P(|X_{nj}| > a_n) < \infty,$$

$$II = \sum_{n=1}^{\infty} P\left( \max_{1 \leq j \leq n} |T_j^{(n)}| > \frac{\varepsilon}{2} \right) < \infty. \quad (2.11)$$
From the fact that \( \sum_{n=1}^{\infty} \sum_{i=1}^{n} E(\psi(|X_{ni}|)/\psi(a_n)) < \infty \), it follows easily that

\[
I = \sum_{n=1}^{\infty} \sum_{j=1}^{n} P(\ |X_{nj}| > a_n) \leq \sum_{n=1}^{\infty} \sum_{j=1}^{n} \frac{\psi\left(\psi\left(a_n\right)\right)}{\psi(a_n)} < \infty, \tag{2.12}
\]

By \( \nu \geq p \) and \( \psi(|t|)/|t|^p \downarrow \) as \( |t| \uparrow \), then \( \psi(|t|)/|t|^\nu \downarrow \) as \( |t| \uparrow \).

By Markov inequality, Lemma 2.1, and \( \sum_{n=1}^{\infty} (\sum_{j=1}^{n} E(\psi(|X_{nj}|)/\psi(a_n))^{v/2}) < \infty \), we have

\[
II = \sum_{n=1}^{\infty} P\left( \max_{1 \leq j \leq n} \ |T^{(n)}_j| > \frac{\epsilon}{2} \right) \leq \sum_{n=1}^{\infty} \left( \frac{\epsilon}{2} \right)^{-v} E \max_{1 \leq j \leq n} \ |T^{(n)}_j|^{\nu/2} \leq C \sum_{n=1}^{\infty} \left( \frac{\epsilon}{2} \right)^{-\nu} \frac{1}{a_n^{\nu}} \left[ \left( \sum_{j=1}^{n} E|X_j^{(n)}|^2 \right)^{\nu/2} + \sum_{j=1}^{n} E|X_j^{(n)}|^\nu \right] \leq C \sum_{n=1}^{\infty} \frac{1}{a_n^{\nu}} \sum_{j=1}^{n} E|X_j^{(n)}|^\nu + C \sum_{n=1}^{\infty} \frac{1}{a_n^{\nu}} \left( \sum_{j=1}^{n} E|X_j^{(n)}|^2 \right)^{\nu/2} \leq C \sum_{n=1}^{\infty} \frac{1}{a_n^{\nu}} \sum_{j=1}^{n} E|X_{nj}|^\nu I(|X_{nj}| \leq a_n) + C \sum_{n=1}^{\infty} \frac{1}{a_n^{\nu}} \left( \sum_{j=1}^{n} E|X_j^{(n)}|^2 \right)^{\nu/2} \leq C \sum_{n=1}^{\infty} \sum_{i=1}^{n} E\left( \frac{|X_{ni}|}{\psi(a_n)} \right)^\nu + C \sum_{n=1}^{\infty} \frac{1}{a_n^{\nu}} \left[ \sum_{j=1}^{n} E\left( \frac{|X_j^{(n)}|}{\psi(a_n)} \right)^2 \right]^{\nu/2} \leq C \sum_{n=1}^{\infty} \sum_{i=1}^{n} E\left( \frac{|X_{ni}|}{\psi(a_n)} \right)^\nu + C \sum_{n=1}^{\infty} \left( \sum_{i=1}^{n} \left( \frac{X_{ni}}{a_n} \right)^2 \right)^{\nu/2} \leq C \sum_{n=1}^{\infty} \sum_{i=1}^{n} E\left( \frac{|X_{ni}|}{\psi(a_n)} \right)^\nu + C \sum_{n=1}^{\infty} \left( \sum_{i=1}^{n} \left( \frac{X_{ni}}{a_n} \right)^2 \right)^{\nu/2} < \infty. \tag{2.13}
\]

Now we complete the proof of Theorem 2.2. \(\square\)

**Corollary 2.3.** Under the conditions of Theorem 2.2, then

\[
\frac{1}{a_n} \sum_{i=1}^{n} X_{ni} \rightarrow 0 \text{ a.s.} \tag{2.14}
\]

**Proof of Corollary 2.3.** By Theorem 2.2, the Proof of Corollary 2.3 is obvious. \(\square\)

**Remark 2.4.** Corollary 2.3 not only generalizes the result of Hu and Taylor [1] to \(\rho^*\)-mixing random variables, but also improves it.

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