Research Article

Solution Estimates for Semilinear Difference-Delay Equations with Continuous Time

Michael Gil' and Sui Sun Cheng

Received 5 July 2007; Accepted 11 August 2007

We consider semilinear difference-delay equations with continuous time in a Euclidean space. Estimates are found for the solutions. Such estimates are then applied to obtain the stability and boundedness criteria for solutions.

Copyright © 2007 M. Gil' and S. S. Cheng. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

1. Introduction and notation

To motivate our study, let us consider the amount of money $x(t)$ saved in a bank at time t . At $t = 0$, an initial amount of money x_0 is deposited and it remains the same until interest is added at time $t = 1$. The principle plus interest remains the same after $t = 1$ until new interest is added at time $t = 2$. In general, $x = x(t)$ is a function defined on $[0, \infty)$ and satisfies

$$x(t+1) = (1+r)x(t) \quad (t > 0, x(0) = x_0, r = \text{const}). \quad (1.1)$$

In the case when money is invested in instruments other than bank savings and for $0 \leq t < 1$, $x(t)$ can vary due to speculation or other factors. We may then face mathematical models described by

$$x(t+1) = f(x(t)) \quad (t > 0), \quad (1.2)$$

where we may allow $x = x(t)$ to be vector-valued functions. Clearly, the properties of their solutions are worthy of study. Moreover, the importance of continuous time difference equations in applications is very well explained in several books [1–3]. They appear in economics [1], gas dynamics, and propagation models [3]. They appear also as the

internal dynamics of nonlinear differential delay systems, when these systems are input-output linearized and controlled (cf. [4]). Results concerning the stability of the origin, in the linear case, can be found in [1] (see Chapter 9 on the equations of neutral type). In [1, Section 9.6], the stability of linear, time-invariant, continuous time difference equations is studied, taking into account small variations that may occur in the delays. In [1, Chapter 9], necessary and sufficient conditions for the asymptotic stability of linear scalar continuous time difference equations, with arbitrary delays, are given. In [5], asymptotic properties of the solutions of scalar difference equations with continuous argument are treated.

To the best of our knowledge, however, stability of general continuous time difference equations has not been extensively investigated in the literature.

For this reason, we will consider the following difference-delay equation with continuous time in a Euclidean space $\mathbb{C}^n$:

$$x(t) = \sum_{k=1}^m A_k x(t - h_k) + F(t, x(t - h_1), \dots, x(t - h_m)) + f(t), \quad t > 0, \quad (1.3)$$

where f is a given vector-valued function, a mapping $F : [0, \infty) \times \mathbb{C}^n \rightarrow \mathbb{C}^n$ satisfies the conditions pointed below, A_k are n -by- n matrices, and $h_k, k = 1, \dots, m$, are nonnegative numbers. We derive solution estimates to (1.3). These estimates give us explicit stability and boundedness conditions.

2. Preliminaries

Denote by $\|\cdot\|_n$ the Euclidean norm in $\mathbb{C}^n$. As usually, $L^p(\omega)$ ($1 \leq p < \infty$) is the space of functions defined on a set ω with the finite norm

$$\|w\|_{L^p(\omega)} = \left[\int_{\omega} \|w(t)\|_n^p dt \right]^{1/p}, \quad (2.1)$$

and $M(\omega)$ is the space of bounded measurable functions defined on ω with the finite sup-norm

$$\|w\|_{C(\omega)} = \sup_{t \in \omega} \|w(t)\|_n. \quad (2.2)$$

In addition, put $R_+ = [0, \infty)$ and consider in $\mathbb{C}^n$ the linear equation

$$v(t) = \sum_{k=1}^m A_k v(t - h_k) \quad (t > 0), \quad (2.3)$$

where A_k are n -by- n matrices again and

$$0 \leq h_k \leq 1 \quad (k = 1, \dots, m). \quad (2.4)$$

Take the initial condition

$$v(t) = \phi(t), \quad t \in [-1, 0], \quad (2.5)$$

where ϕ is a given continuous vector-valued function. Let us apply the Laplace transform to problems (2.3), (2.5). Then

$$\tilde{v}(z) = \sum_{k=1}^m A_k e^{-h_k z} \tilde{v}(z) + \tilde{\psi}_0(z), \quad (2.6)$$

where $\tilde{v}(z)$ is the Laplace transform to $v(t)$, z is the dual variable, and

$$\tilde{\psi}_0(z) = \sum_{k=1}^m A_k e^{-z h_k} \int_{-h_k}^0 e^{-zt} \phi(t) dt. \quad (2.7)$$

Hence,

$$\tilde{v}(z) = K^{-1}(e^{-z}) \tilde{\psi}_0(z), \quad (2.8)$$

where

$$K(w) := I - \sum_{k=1}^m A_k w^{h_k} \quad (w \in \mathbb{C}) \quad (2.9)$$

is the characteristic matrix-valued function.

We will call $K(w)$ a stable function if all its characteristic values (i.e., the zeros of $\det K(w)$) lie in the disc $|w| < 1$. In the case of the equation $x(t) = Ax(t-1)$ with a matrix A , the stability means that the spectral radius $r_s(A)$ of A is less than one. For some necessary and sufficient conditions for stability of operators, one may consult the papers [6, 7] and the book [8]. In the sequel, K is a stable function. Let

$$G(t) = \frac{1}{2\pi i} \int_{\operatorname{Re} z > b_0} e^{zt} K^{-1}(e^{-z}) dz = \frac{1}{2\pi i} \int_C w^{t-1} K^{-1}(w) dw \quad (2.10)$$

be the Green function to (2.3). Here b_0 is a real constant and C is the Jourdan contour surrounding all the characteristic values of $K(\cdot)$. By the inverse Laplace transform, a solution of (2.3) can be represented as

$$v(t) = \frac{1}{2\pi i} \int_{\operatorname{Re} z > b_0} e^{zt} K^{-1}(e^{-z}) \psi_0(e^{-z}) dz. \quad (2.11)$$

Clearly, if K is stable, then the maximum r_0 of the absolute values of the roots of $\det K$ is less than one and for any $r \in (r_0, 1)$,

$$\sup_{|z|=r} \|K^{-1}(z)\|_n < \infty, \quad (2.12)$$

and thus taking $C = \{z \in \mathbb{C} : |z| = r < 1\}$, we get

$$\|G(t)\|_n \leq c_r r^t \quad (t \geq 0; c_r = \text{const}). \quad (2.13)$$

4 Discrete Dynamics in Nature and Society

Next, consider the nonhomogeneous equation

$$u(t) = \sum_{k=1}^m A_k u(t - h_k) + \zeta(t) \quad (t > 0) \quad (2.14)$$

with a given $\zeta \in L^2(R_+)$ and the zero initial condition

$$u(t) = 0, \quad t \in [-1, 0]. \quad (2.15)$$

Again applying the Laplace transform, we arrive at the equality

$$\tilde{u}(z) = K^{-1}(e^{-z})\tilde{\zeta}(z), \quad (2.16)$$

where $\tilde{u}(z)$ is the Laplace transform to $u(t)$ and $\tilde{\zeta}(z)$ is the Laplace transform to $\zeta(t)$. Hence,

$$y(t) = \int_0^t G(t-s)\zeta(s)ds. \quad (2.17)$$

From (2.14) and the Parseval equality, it easily follows that

$$\|u\|_{L^2(R_+)} \leq \theta(K) \|\zeta\|_{L^2(R_+)}, \quad (2.18)$$

where

$$\theta(K) := \sup_{|z|=1} \|K^{-1}(z)\|_n. \quad (2.19)$$

3. The main result

Consider (1.3) under the initial condition

$$x(t) = \phi(t), \quad t \in [-1, 0]. \quad (3.1)$$

Put $\Omega(\rho) := \{h \in \mathbb{C}^n : \|h\|_n \leq \rho\}$ for a positive number $\rho \leq \infty$. It is assumed that $f \in L^2(R_+)$ and F continuously map $R_+ \times \Omega^m(\rho)$ into $\mathbb{C}^n$ and satisfy the condition

$$\|F(t, v_1, \dots, v_m)\|_n \leq \sum_{k=1}^m \nu_k \|v_k\|_n \quad (v_k \in \Omega(\rho)). \quad (3.2)$$

Here $\nu_k < \infty$ are nonnegative constants. Put

$$\nu := \nu_1 + \dots + \nu_m. \quad (3.3)$$

Now we are in a position to formulate our main result.

THEOREM 3.1. *Under conditions (3.2) and (2.4), let $f \in L^2(R_+)$ and*

$$\nu\theta(K) < 1. \quad (3.4)$$

Then there is a positive constant $c_0 < \infty$, such that the condition

$$c_0(|f|_{L^2(R_+)} + |\phi|_{C(-1,0)}) \leq \rho \quad (3.5)$$

implies that a solution x of problems (1.3), (3.1) is in $L^2(R_+)$. Moreover,

$$|x|_{C(R_+)} \leq c_0(|\phi|_{C(-1,0)} + |f|_{L^2(R_+)}). \quad (3.6)$$

Proof. First let $\rho = \infty$. From (1.3) and (2.17), it follows that

$$x(t) = v(t) + \int_0^t G(t-s)[F(s, x(s-h_1), \dots, x(s-h_m)) + f(s)]ds, \quad t \geq 0, \quad (3.7)$$

where v is a solution of the linear problems (2.3), (2.5). By (2.18),

$$\begin{aligned} & \left| \int_0^t G(t-s)[F(s, x(s-h_1), \dots, x(s-h_m)) + f(s)] \right|_{L^2(R_+)} \\ & \leq \theta(K) |F(s, x(s-h_1), \dots, x(s-h_m)) + f(s)|_{L^2(R_+)}. \end{aligned} \quad (3.8)$$

But by (3.2),

$$\begin{aligned} |F(s, x(s-h_1), \dots, x(s-h_m))|_{L^2(R_+)} & \leq \sum_{k=1}^m \gamma_k |x(s-h_k)|_{L^2(R_+)} \\ & \leq \sum_{k=1}^m \gamma_k (|x|_{L^2(R_+)} + |x|_{L^2(-1,0)}) = \nu(|x|_{L^2(R_+)} + |\phi|_{L^2(-1,0)}). \end{aligned} \quad (3.9)$$

Now from (3.7), it follows that

$$|x|_{L^2(R_+)} \leq b_1 + \theta(K)\nu|x|_{L^2(R_+)}, \quad (3.10)$$

where

$$b_1 = |\nu|_{L^2(R_+)} + \nu\theta(K)|\phi|_{L^2(-1,0)} + \theta(K)|f|_{L^2(R_+)}. \quad (3.11)$$

Since K is stable, (2.11) yields us the inequalities

$$|\nu|_{C(R_+)} \leq \text{const}|\phi|_{C(-1,0)}, \quad |\nu|_{L^2(R_+)} \leq \text{const}|\phi|_{C(-1,0)}. \quad (3.12)$$

Thus due to (3.4),

$$|x|_{L^2(R_+)} \leq (1 - \nu\theta(K))^{-1}b_1 \leq \text{const}(|f|_{L^2(R_+)} + |\phi|_{C(-1,0)}). \quad (3.13)$$

Furthermore, from (3.7) and the Schwarz inequality now it follows that

$$|x|_{C(R_+)} \leq |\nu|_{C(R_+)} + |G|_{L^2(R_+)}(|F(s, x(s-h_1), \dots, x(s-h_m))|_{L^2(R_+)} + |f|_{L^2(R_+)}). \quad (3.14)$$

But according to (3.2) and (3.13),

$$\begin{aligned} & \|F(s, x(s-h_1), \dots, x(s-h_m))\|_{L^2(R_+)} \\ & \leq \nu(\|x\|_{L^2(R_+)} + \|\phi\|_{L^2(-1,0)}) \leq \text{const}(\|f\|_{L^2(R_+)} + \|\phi\|_{C(-1,0)}). \end{aligned} \quad (3.15)$$

Hence, the required inequality (3.6) follows, provided $\rho = \infty$.

Next, let $\rho < \infty$. On $\mathbb{C}^n$, let us define the function

$$\mu(y) = \begin{cases} 1, & \|y\|_n \leq \rho, \\ 0, & \|y\|_n > \rho \end{cases} \quad (3.16)$$

for all $y \in \mathbb{C}^n$. Such a function always exists due to the Urysohn theorem [9, page 15]. Consider the equation

$$\tilde{x}(t) = \sum_{k=1}^m A_k \tilde{x}(t-h_k) + \mu(x)F(t, \tilde{x}(t-h_1), \dots, \tilde{x}(t-h_m)) + f(t), \quad t \geq 0. \quad (3.17)$$

Since

$$\mu(w(t))\|F(t, w(t-h_1), \dots, w(t-h_m))\|_n \leq \sum_{k=1}^m \nu_k \|w(t-h_k)\|_n \quad (3.18)$$

for any function $w \in M(0, T)$ with an arbitrary positive $T < \infty$, we can apply our above results to (3.17). Namely, for a solution $\tilde{x}$ of (3.17), estimate (3.6) holds according to the above arguments and (3.18). But $\mu(w(t))F(t, w(t-h_1), \dots, w(t-h_m))$ and $F(t, w(t-h_1), \dots, w(t-h_m))$ coincide for any function w with values in $\Omega(\rho)$. According to condition (3.5), $\tilde{x}(t) \in \Omega(\rho)$, $t \geq 0$, and $x(t) = \tilde{x}(t)$ for $t \geq 0$. This proves the theorem. $\square$

4. Stability of solutions and bounds for $\theta(K)$

We first recall some usual definitions related to the asymptotic properties of solutions.

(i) The zero solution of the equation

$$x(t) = \sum_{k=1}^m A_k x(t-h_k) + F(t, x(t-h_1), \dots, x(t-h_m)), \quad t > 0, \quad (4.1)$$

is said to be stable if, for any $\varepsilon > 0$, there exists $\delta > 0$, such that the inequality $\|\phi\|_{C(-1,0)} \leq \delta$ implies $\|x(t)\| \leq \varepsilon$, $t > 0$, for a solution $x(t)$ of problems (4.1), (3.1).

(ii) The zero solution of (4.1) is said to be attractive if there exists $\delta > 0$, such that the inequality $\|\phi\|_{C(-1,0)} \leq \delta$ implies $x(t) \rightarrow 0$ as $t \rightarrow \infty$ for a solution $x(t)$ of problems (3.1), (4.1).

(iii) The zero solution of (4.1) is said to be asymptotically stable if it is both stable and attractive. It is globally asymptotically stable if one can take $\delta = \infty$.

Theorem 3.1 implies the following result.

COROLLARY 4.1. *Under the hypothesis of Theorem 1.1, the zero solution of (4.1) is asymptotically stable. Moreover, it is globally asymptotically stable if (3.2) holds with $\rho = \infty$.*

Let us say that (4.1) is a *quasilinear equation* if

$$\lim_{v_1, \dots, v_m \rightarrow 0} \frac{\|F(t, v_1, \dots, v_m)\|_n}{\sum_{k=1}^m \|v_k\|_n} = 0 \quad (4.2)$$

uniformly in $t \geq 0$.

THEOREM 4.2 (stability in the linear approximation). *Let (4.1) be a quasilinear equation with a stable characteristic function K . Then the zero solution to (4.1) is asymptotically stable.*

Proof. In view of (4.2), condition (3.2) holds for all sufficiently small $\rho > 0$, and $v \rightarrow 0$ as $\rho \rightarrow 0$. So we can take ρ sufficiently small in such a way that condition (3.4) holds. By Corollary 4.1, we may then obtain the desired stability. $\square$

Let us derive estimates for $\theta(K)$. Let $N(A)$ be the Frobenius (Hilbert-Schmidt) norm of an $n \times n$ matrix A , that is,

$$N^2(A) = \text{Trace } AA^*. \quad (4.3)$$

Here A^* is the adjoint. As it was proved in [10, Chapter 2],

$$\|A^{-1}\|_n \leq \frac{N^{n-1}(A)}{|\det A| (n-1)^{(n-1)/2}}, \quad (4.4)$$

provided A is invertible. Hence,

$$\|K^{-1}(z)\|_n \leq \frac{N^{n-1}(K(z))}{|\det K(z)| (n-1)^{(n-1)/2}}. \quad (4.5)$$

But

$$N^{n-1}(K(z)) = N\left(I - \sum_{k=1}^m A_k z^{h_k}\right) \leq \sqrt{n} + \sum_{k=1}^m N(A_k) \quad (|z| = 1). \quad (4.6)$$

Thus

$$\theta(K) \leq \tilde{\theta}(K) := \frac{\left(\sqrt{n} + \sum_{k=1}^m N(A_k)\right)^{n-1}}{\inf_{|z|=1} |\det K(z)|}. \quad (4.7)$$

So in Theorems 3.1 and 4.2, one can replace $\theta(K)$ by $\tilde{\theta}(K)$. Note also that any eigenvalue $\lambda_j(K(z))$ of $K(z)$ with fixed K is equal to $1 - \lambda_j(B(z))$, where $\lambda_j(B(z))$ is the eigenvalue of

$$B(z) := \sum_{k=1}^m A_k z^{h_k}. \quad (4.8)$$

So if in some norm $\|\cdot\|_a$ the inequality

$$\gamma_0 := \sum_{k=1}^m \|A_k\|_a < 1 \quad (4.9)$$

holds, then $\inf_{|z|=1} |\det K(z)| \geq (1 - \gamma_0)^n$. About localization of the characteristic values of matrix-valued functions, see, for instance, [11, 12] and references therein. For lower estimates for determinants, consult [13].

References

- [1] J. K. Hale and S. M. Verduyn Lunel, *Introduction to Functional-Differential Equations*, vol. 99 of *Applied Mathematical Sciences*, Springer, New York, NY, USA, 1993.
- [2] V. Kolmanovskii and A. Myshkis, *Applied Theory of Functional-Differential Equations*, vol. 85 of *Mathematics and Its Applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1992.
- [3] S.-I. Niculescu, *Delay Effects on Stability: A Robust Control Approach*, vol. 269 of *Lecture Notes in Control and Information Sciences*, Springer, London, UK, 2001.
- [4] P. Pepe, “The Liapunov’s second method for continuous time difference equations,” *International Journal of Robust and Nonlinear Control*, vol. 13, no. 15, pp. 1389–1405, 2003.
- [5] A. N. Sharkovsky, Yu. L. Maistrenko, and E. Yu. Romanenko, *Difference Equations and Their Applications*, vol. 250 of *Mathematics and Its Applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1993.
- [6] S. S. Cheng and Y.-Z. Lin, “Exact region of stability for an investment plan with three parameters,” *Applied Mathematics E-Notes*, vol. 5, pp. 194–201, 2005.
- [7] S. S. Cheng and S. S. Chiou, “Exact stability regions for quartic polynomials,” *Bulletin of the Brazilian Mathematical Society*, vol. 38, no. 1, pp. 21–38, 2007.
- [8] M. I. Gil, *Difference Equations in Normed Spaces: Stability and Oscillations*, vol. 206 of *North-Holland, Mathematics Studies*, Elsevier, Amsterdam, The Netherlands, 2007.
- [9] N. Dunford and J. T. Schwartz, *Linear Operators. Part I: General Theory*, Interscience, New York, NY, USA, 1966.
- [10] M. I. Gil, *Operator Functions and Localization of Spectra*, vol. 1830 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 2003.
- [11] M. I. Gil, “Bounds for characteristic values of entire matrix pencils,” *Linear Algebra and Its Applications*, vol. 390, no. 1, pp. 311–320, 2004.
- [12] M. I. Gil, “On variations of characteristic values of entire matrix pencils,” *Journal of Approximation Theory*, vol. 136, no. 1, pp. 115–128, 2005.
- [13] T.-Z. Huang and X.-P. Liu, “Estimations for certain determinants,” *Computers & Mathematics with Applications*, vol. 50, no. 10–12, pp. 1677–1684, 2005.

Michael Gil: Department of Mathematics, Ben-Gurion University, P.O. Box 653,
Beer Sheva 84105, Israel
Email address: gilmi@bezeqint.net

Sui Sun Cheng: Department of Mathematics, Tsing Hua University, Hsinchu, Taiwan 30043, Taiwan
Email address: sscheng@math.nthu.edu.tw