Research Article

A Global Convergence Result for a Higher Order Difference Equation

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Let $f(z_1, \dots, z_k) \in C(I^k, I)$ be a given function, where I is (bounded or unbounded) subinterval of $\mathbb{R}$, and $k \in \mathbb{N}$. Assume that $f(y_1, y_2, \dots, y_k) \geq f(y_2, \dots, y_k, y_1)$ if $y_1 \geq \max\{y_2, \dots, y_k\}$, $f(y_1, y_2, \dots, y_k) \leq f(y_2, \dots, y_k, y_1)$ if $y_1 \leq \min\{y_2, \dots, y_k\}$, and f is non-decreasing in the last variable z_k . We then prove that every bounded solution of an autonomous difference equation of order k , namely, $x_n = f(x_{n-1}, \dots, x_{n-k})$, $n = 0, 1, 2, \dots$, with initial values $x_{-k}, \dots, x_{-1} \in I$, is convergent, and every unbounded solution tends either to $+\infty$ or to $-\infty$.

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1. Introduction and the main results

Let us consider an autonomous difference equation of order $k \in \mathbb{N}$:

$$x_n = f(x_{n-1}, \dots, x_{n-k}), \quad n \in \mathbb{N}_0 := \{0, 1, 2, \dots\}, \quad (1.1)$$

where $f: I^k \rightarrow I$, interval I is bounded or unbounded interval of the real line $\mathbb{R}$, and f is a function that satisfies the condition $f(x, \dots, x) \leq x$ for all $x \in I$. The difference equation (1.1) was investigated by many authors (see, e.g., [1–11] and the references cited therein). In most of these results the monotonicity of the function $f(z_1, \dots, z_k)$ in its variables $z_1, \dots, z_k$ plays the main role.

For example, in [7], Stević proved the following theorem.

THEOREM 1.1. *Assume that f is a continuous real-valued function defined on $\mathbb{R}^k$ satisfying the following conditions:*

- (a) *f is nondecreasing in each of its arguments;*
- (b) *$f(z_1, \dots, z_k)$ is strictly increasing in z_1 ;*

(c) for every $x \in \mathbb{R}$ it holds

$$f(x, x, \dots, x) \leq x. \quad (1.2)$$

Then every bounded solution of the difference inequality

$$x_n \leq f(x_{n-1}, \dots, x_{n-k}), \quad n \in \mathbb{N}_0, \quad (1.3)$$

converges.

This result generalizes the main results in [1, 3]. For closely related results see also [2, 6, 9, 10, 12–18] and the references cited therein. The asymptotic behavior of the solutions of (1.1) was investigated in [11].

The following result was proved (also by Stević) in [8] (for a particular case of the result in which condition (c) below is replaced with the fact that every real number $\bar{x}$ is an equilibrium point of $x_n = f(x_{n-1}, \dots, x_{n-k})$, see [4, Theorem 2.2]).

THEOREM 1.2. Assume that f is a continuous real-valued function on $\mathbb{R}^k$, where f satisfies conditions (a) and (c) from Theorem 1.1, and where $f(z_1, \dots, z_k)$ is strictly increasing in at least two of its arguments z_i and z_j , where i and j are relatively prime integers. Then any bounded solution of difference inequality (1.3) converges.

Remark 1.3. The fact that the last condition in Theorem 1.2 is sharp, in the sense that primeness cannot be omitted, was shown in [9].

Here we will improve these results in some sense. We present a global convergence result regarding the solutions of (1.1) in which the function f need not be monotonous in each variable, and the condition $f(x, \dots, x) \leq x$ can be omitted as an irrelevant one.

THEOREM 1.4. Let $f(z_1, \dots, z_k) \in C(I^k, I)$ be a given function, where I is bounded or unbounded interval of $\mathbb{R}$, which satisfies the following conditions:

(a)

$$f(y_1, y_2, \dots, y_k) \geq f(y_2, \dots, y_k, y_1) \quad (1.4)$$

if $y_1 \geq \max\{y_2, \dots, y_k\}$;

(b)

$$f(y_1, y_2, \dots, y_k) \leq f(y_2, \dots, y_k, y_1) \quad (1.5)$$

if $y_1 \leq \min\{y_2, \dots, y_k\}$;

(c) f is nondecreasing in the last variable z_k .

Then every bounded solution of (1.1) with initial values $x_{-k}, \dots, x_{-1} \in I$ converges, and every unbounded solution of (1.1) tends either to $+\infty$ or to $-\infty$.

2. Proof of the main result

In this section, we prove the main result of this paper. Before this, we formulate and prove an auxiliary result.

LEMMA 2.1. Assume that (x_n) is a solution of (1.1) for which there is an $n_0 \in \mathbb{N}$ such that

$$x_{n_0-1} \leq \min \{x_{n_0-2}, \dots, x_{n_0-k-1}\} \quad (2.1)$$

or

$$x_{n_0-1} \geq \max \{x_{n_0-2}, \dots, x_{n_0-k-1}\}. \quad (2.2)$$

Then under the conditions of Theorem 1.4, the solution (x_n) is eventually monotonous.

Proof. Assume that condition (2.1) holds. From (2.1) and (1.5), and by employing conditions (b) and (c) from Theorem 1.4, it follows that

$$\begin{aligned} x_{n_0} &= f(x_{n_0-1}, x_{n_0-2}, \dots, x_{n_0-k}) \\ &\leq f(x_{n_0-2}, \dots, x_{n_0-k}, x_{n_0-1}) \\ &\leq f(x_{n_0-2}, \dots, x_{n_0-k}, x_{n_0-k-1}) = x_{n_0-1} \end{aligned} \quad (2.3)$$

and, consequently,

$$x_{n_0} \leq \min \{x_{n_0-1}, \dots, x_{n_0-k}\}. \quad (2.4)$$

From this and by induction, it follows that $x_{n+1} \leq x_n$ for $n \geq n_0 - 1$, which means that the solution (x_n) of (1.1) is eventually nonincreasing, as desired. The proof of the lemma under condition (2.2) is very similar and, therefore, will be omitted. This finishes the proof. $\square$

Proof of Theorem 1.4. By Lemma 2.1, we may suppose that for every $n \in \mathbb{N}$,

$$\min \{x_{n-1}, \dots, x_{n-k}\} < x_n < \max \{x_{n-1}, \dots, x_{n-k}\}. \quad (2.5)$$

Otherwise, the sequence is eventually monotonous and, as such, it converges to a finite limit or to $+\infty$ or $-\infty$.

Since $\min \{x_{n-1}, \dots, x_{n-k}\} \leq \min \{x_{n-1}, \dots, x_{n-k+1}\}$ and from (2.5), we have $\min \{x_{n-1}, \dots, x_{n-k}\} < x_n$. It is easily seen that the sequence

$$m_n = \min \{x_n, \dots, x_{n-k+1}\}, \quad n \in \mathbb{N}, \quad (2.6)$$

is nondecreasing; similarly, the sequence

$$M_n = \max \{x_n, \dots, x_{n-k+1}\}, \quad n \in \mathbb{N}, \quad (2.7)$$

is nonincreasing.

Our aim is to prove that

$$\lim_{n \rightarrow \infty} m_n = \lim_{n \rightarrow \infty} M_n. \quad (2.8)$$

Since sequences (m_n) and (M_n) are bounded, this is equivalent with

$$\lim_{n \rightarrow \infty} (\max \{x_n, \dots, x_{n-k+1}\} - \min \{x_n, \dots, x_{n-k+1}\}) = 0, \quad (2.9)$$

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that motivates us to consider the function $h : I^k \rightarrow \mathbb{R}^+ \cup \{0\}$ given by

$$h(z_1, \dots, z_k) = \max \{z_1, \dots, z_k\} - \min \{z_1, \dots, z_k\}. \quad (2.10)$$

Let us introduce the sequence

$$X_n = (x_n, x_{n-1}, \dots, x_{n-k+1}), \quad n \in \mathbb{N}, \quad (2.11)$$

and $\mathcal{A} = \overline{\{X_n : n \in \mathbb{N}\}}$, that is, the closure of the set $\{X_n : n \in \mathbb{N}\}$. Note that set $\mathcal{A}$ is compact, as a bounded and a closed subset of $\mathbb{R}^k$. Since X_n can be written in the form

$$X_n = (f(x_{n-1}, \dots, x_{n-k}), x_{n-1}, \dots, x_{n-k+1}), \quad n \in \mathbb{N}, \quad (2.12)$$

we also have that set $\mathcal{A}$ is invariant under the vector function

$$F(z_1, \dots, z_k) = (f(z_1, \dots, z_k), z_1, \dots, z_{k-1}). \quad (2.13)$$

Function h , as a continuous one, attains its minimum on $\mathcal{A}$ at some $\mathbf{c}_0 \in \mathcal{A}$. Let us remember that $h(\mathbf{c}_0) \geq 0$. Let (y_n) be the solution of (1.1) with initial values equal to $\mathbf{c}_0 = (y_{-1}, \dots, y_{-k})$. We claim that, for this solution, there is a number $n_0 \in \mathbb{N}$ such that

$$y_{n_0-1} \leq \min \{y_{n_0-2}, \dots, y_{n_0-k-1}\} \quad \text{or} \quad y_{n_0-1} \geq \max \{y_{n_0-2}, \dots, y_{n_0-k-1}\}. \quad (2.14)$$

Since the set $\mathcal{A}$ is invariant under F ,

$$h(y_{k-1}, \dots, y_0) \geq h(y_{-1}, \dots, y_{-k}), \quad (2.15)$$

where the right-hand side of the inequality is the minimum of function h , we have that there is an index $i_0 \in \{0, 1, \dots, k-1\}$ such that

$$y_{i_0} = \max \{y_{k-1}, \dots, y_0\} \geq \max \{y_{-1}, \dots, y_{-k}\} \quad (2.16)$$

or

$$y_{i_0} = \min \{y_{k-1}, \dots, y_0\} \leq \min \{y_{-1}, \dots, y_{-k}\}. \quad (2.17)$$

Indeed, if $\max \{y_{k-1}, \dots, y_0\} \geq \max \{y_{-1}, \dots, y_{-k}\}$, we finish. Otherwise, from $\max \{y_{k-1}, \dots, y_0\} - \min \{y_{k-1}, \dots, y_0\} \geq \max \{y_{-1}, \dots, y_{-k}\} - \min \{y_{-1}, \dots, y_{-k}\} = h(\mathbf{c}_0) \geq 0$, we deduce $\min \{y_{-1}, \dots, y_{-k}\} - \min \{y_{k-1}, \dots, y_0\} \geq \max \{y_{-1}, \dots, y_{-k}\} - \max \{y_{k-1}, \dots, y_0\} > 0$, which leads to $\min \{y_{-1}, \dots, y_{-k}\} \geq \min \{y_{k-1}, \dots, y_0\}$. Note that the indices $-k, -k+1, \dots, -1, 0, 1, \dots, i_0$ are consecutive, with $i_0 \leq k-1$. Hence,

$$y_{i_0} \geq \max \{y_{i_0-1}, \dots, y_{i_0-k}\} \quad (2.18)$$

or

$$y_{i_0} \leq \min \{y_{i_0-1}, \dots, y_{i_0-k}\}, \quad (2.19)$$

as claimed.

By Lemma 2.1, we have that (y_n) is eventually nondecreasing or nonincreasing. Now, note that (y_n) cannot be unbounded, since $\mathbf{c}_0 \in \mathcal{A}$ and $\mathcal{A}$ is invariant under F . Hence, the sequence

$$Y_n = (y_n, y_{n-1}, \dots, y_{n-k+1}), \quad n \in \mathbb{N}, \quad (2.20)$$

converges to a point, which must be of the form $\mathbf{c}_0 = (x^*, x^*, \dots, x^*) \in I^k$, and for which $h(\mathbf{c}_0) = 0$. In particular, we have that

$$\lim_{n \rightarrow \infty} \sum_{j=0}^{k-1} |y_{n-j} - x^*| = 0. \quad (2.21)$$

On the other hand, since $Y_n \in \mathcal{A}$, and set $\mathcal{A}$ is closed, we have that for every $\varepsilon > 0$ and $n \in \mathbb{N}$, there is a vector X_m such that

$$\sum_{j=0}^{k-1} |y_{n-j} - x_{m-j}| < \varepsilon. \quad (2.22)$$

From this and (2.21), it follows that for every $\varepsilon > 0$, there is an index $m \in \mathbb{N}$ such one that

$$\sum_{j=0}^{k-1} |x_{m-j} - x^*| < 2\varepsilon. \quad (2.23)$$

Using (2.5) and (2.23), we obtain

$$x^* - 2\varepsilon < \min \{x_m, \dots, x_{m-k+1}\} < x_{m+1} < \max \{x_m, \dots, x_{m-k+1}\} < x^* + 2\varepsilon. \quad (2.24)$$

From this, (2.5), and the fact that (m_n) and (M_n) are nondecreasing and nonincreasing, respectively, we obtain that $|x_n - x^*| < 2\varepsilon$, for $n \geq m+1$. Since ε is an arbitrary positive real number, we have $\lim_{n \rightarrow \infty} x_n = x^*$, finishing the proof of the main result. $\square$

Example 2.2. There are many natural examples for a difference equation (1.1) in which the function f satisfies conditions (1.4) and (1.5) from Theorem 1.4, such as particular maps; for instance, $f(z_1, z_2) = \alpha z_1 + \beta z_2$ with constants $\alpha \geq \beta \geq 0$.

Now, let us consider the function f and suppose that it is a linear combination of any monotonous nondecreasing function g with respect to variables z_i , $i = 1, 2, \dots, k$, that is,

$$f(z_1, \dots, z_k) = a_1 g(z_1) + \dots + a_k g(z_k), \quad a_1 \geq a_2 \geq \dots \geq a_k \geq 0. \quad (2.25)$$

The case when all constants a_i , $i = 1, 2, \dots, k$, are the same, obviously satisfies all conditions. So, suppose that $a_1 > a_k$. Now, assume that $z_1 \geq z_i$ for all $i = 1, 2, \dots, k$. It is well known that for any monotonous nondecreasing function g (which need not necessarily be a differentiable one), inequality $g(z_1) \geq g(z_i)$ holds for all $i = 1, 2, \dots, k$. After the multiplication of all these inequalities by $(a_{i-1} - a_i)/(a_1 - a_k)$, respectively, and the summation

from $i = 2$ to $i = k$, for some $\eta \leq z_1$, we get (notice that $\sum_{i=2}^k ((a_{i-1} - a_i)/(a_1 - a_k)) = 1$)

$$g(z_1) \geq g(\eta) = \frac{1}{a_1 - a_k} \left(\sum_{i=2}^k a_{i-1} g(z_i) - \sum_{i=2}^k a_i g(z_i) \right) \quad (2.26)$$

that is

$$\sum_{i=1}^k a_i g(z_i) \geq \sum_{i=2}^k a_{i-1} g(z_i) + a_k g(z_1), \quad (2.27)$$

which constitutes the condition (1.4) from Theorem 1.4. A similar algebraic manipulation can connect $y_1 \leq \min\{y_2, \dots, y_k\}$ (i.e., the condition (b)) from Theorem 1.4 and inequality (1.5) for such functions g . This implies that any function f from the previously described class of functions (2.25) satisfies the conditions from Theorem 1.4. Hence, this theorem is applicable to the respective difference equation (1.1).

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