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Research Article

Stability of the Positive Point of Equilibrium of Nicholson's Blowflies Equation with Stochastic Perturbations: Numerical Analysis

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Known Nicholson's blowflies equation (which is one of the most important models in ecology) with stochastic perturbations is considered. Stability of the positive (nontrivial) point of equilibrium of this equation and also a capability of its discrete analogue to preserve stability properties of the original differential equation are studied. For this purpose, the considered equation is centered around the positive equilibrium and linearized. Asymptotic mean square stability of the linear part of the considered equation is used to verify stability in probability of nonlinear origin equation. From known previous results connected with B. Kolmanovskii and L. Shaikhet, general method of Lyapunov functionals construction, necessary and sufficient condition of stability in the mean square sense in the continuous case and necessary and sufficient conditions for the discrete case are deduced. Stability conditions for the discrete analogue allow to determinate an admissible step of discretization for numerical simulation of solution trajectories. The trajectories of stable and unstable solutions of considered equations are simulated numerically in the deterministic and the stochastic cases for different values of the parameters and of the initial data. Numerous graphical illustrations of stability regions and solution trajectories are plotted.

1. Introduction

Consider the nonlinear differential equation with exponential nonlinearity

$$\dot{x}(t) = ax(t-h)e^{-bx(t-h)} - cx(t), \tag{1.1}$$

which is one of the most important ecological models. It describes the population dynamics of Nicholson's blowflies. Here $x(t)$ is the size of the population at time t, a is the maximum per capita daily egg production rate, $1/b$ is the size at which the population reproduces at the maximum rate, c is the per capita daily adult death rate, and h is the generation time.

Equation (1.1) is enough popular with researches [1–11]. The majority of the results on (1.1) deal with the global attractiveness of the positive point of equilibrium and oscillatory behaviors of solutions [2, 4, 6–9, 12–14]. In connection with numerical simulation, a special interest has an investigation of discrete analogues of (1.1) [1, 4, 7, 10].

In this paper, we consider stability in probability of the positive point of equilibrium of (1.1) by stochastic perturbations and also of one discrete analogue of this equation. A capability of a discrete analogue to preserve stability properties of the original differential equation is studied. Sufficient stability conditions for discrete analogue obtained here are much more better than similar conditions known earlier in deterministic case [4, 10].

The following method for stability investigation is used here. The considered nonlinear equation is exposed to stochastic perturbations and is linearized in the neighborhood of the positive point of equilibrium. Conditions for asymptotic mean square stability of the trivial solution of the constructed linear equation are obtained. In the case if the order of nonlinearity is more than 1, these conditions are sufficient ones (both for continuous and discrete time [15–19]) for stability in probability of the initial nonlinear equation by stochastic perturbations.

This method was used already for stability investigation of other biological systems with delays: SIR epidemic model [15] predator-prey model [19]. Conditions for asymptotic mean square stability that are used here were obtained via the general method of Lyapunov functionals construction for stability investigation of stochastic differential and difference equations [22–28].

2. Some definitions and auxiliary statements

Let $\{\Omega,\sigma,\mathbf{P}\}$ be a probability space and let $\mathbf{E}$ be the expectation. Consider stochastic differential equation [20, 21, 29]

$$\begin{gathered}\dot{z}(t)=a(t,z_t)+\sigma(t,z_t)\dot{w}(t),\quad t\geq 0,\\ z(s)=\phi_0(s),\quad s\leq 0.\end{gathered}\tag{2.1}$$

Here $z(t)$ is a value of the process z at time t, z_t is a trajectory of the process z to the time t, w is a standard Wiener process, a, σ are functionals on $T\times H$, where $T=\{t:t\geq 0\}$, H is a space of trajectories ϕ of (2.1). It is supposed also that $a(t,0)=\sigma(t,0)=0$. In this case, (2.1) has the trivial solution.

Two definitions for stability of the trivial solution of (2.1) are used here.

Definition 2.1. The trivial solution of (2.1) is called stable in probability if for any $\epsilon_1>0$ and $\epsilon_2>0$, there exists $\delta>0$ such that the solution $z(t)=z(t,\phi_0)$ satisfies the condition $\mathbf{P}\{\sup_{t\geq 0}|z(t,\phi_0)|>\epsilon_1\}<\epsilon_2$ for any initial function $\phi_0\in H$ such that $\mathbf{P}\{\sup_{s\leq 0}|\phi_0(s)|\leq\delta\}=1$.

Definition 2.2. The trivial solution of (2.1) is called mean square stable if for any $\epsilon > 0$, there exists $\delta > 0$ such that the solution $z(t) = z(t, \phi_0)$ satisfies the condition $\mathbf{E}|z(t, \phi_0)|^2 < \epsilon$ for any initial function $\phi_0 \in H$ such that $\sup_{s \le 0} \mathbf{E}|\phi_0(s)|^2 < \delta$. If besides $\lim_{t\to\infty} \mathbf{E}|z(t,\phi_0)|^2 = 0$ for any initial function $\phi_0 \in H$, then the trivial solution of (2.1) is called asymptotically mean square stable.

Consider scalar linear stochastic differential equation with delay

$$\dot{z}(t) = -az(t) - bz(t-h) + \sigma z(t)\dot{w}(t). \tag{2.2}$$

Here a, b, σ are arbitrary constants and $h > 0$. Put

$$p = \frac{1}{2}\sigma^2. \tag{2.3}$$

Lemma 2.3 [30]. *Necessary and sufficient condition for asymptotic mean square stability of the trivial solution of (2.2) is*

$$a + b > 0, \qquad pG < 1, \tag{2.4}$$

where

$$G = \begin{cases} \dfrac{1 + bq^{-1}\sin(qh)}{a + b\cos(qh)}, & b > |a|,\ q = \sqrt{b^2 - a^2}, \\ \dfrac{1 + ah}{2a}, & b = a > 0, \\ \dfrac{1 + bq^{-1}\sinh(qh)}{a + b\cosh(qh)}, & a > |b|,\ q = \sqrt{a^2 - b^2}. \end{cases} \tag{2.5}$$

In particular, if $p > 0$, $h = 0$ then stability condition (2.4), (2.5) takes the form $a + b > p$; if $p = 0$, $h > 0$ then the region of stability is bounded by the lines $a + b = 0$ and $a + b\cos(qh) = 0$.

Consider the scalar stochastic difference equation

$$z_{i+1} = az_i + bz_{i-k} + \sigma z_i \xi_{i+1}, \quad i = 0, 1, 2, \dots. \tag{2.6}$$

Here ξ_i, $i = 1, 2, \dots$, are mutually independent random variables such that $\mathbf{E}\xi_i = 0$, $\mathbf{E}\xi_i^2 = 1$, a, b, σ are arbitrary constants and $k > 0$ is an integer.

Two sufficient conditions for asymptotic mean square stability of the trivial solution of (2.6) are given by the following lemma.

Lemma 2.4 [24, 26]. *If at least one of the following inequalities holds*

$$\left(|a| + |b|\right)^2 + \sigma^2 < 1, \tag{2.7}$$

$$(a + b)^2 + 2k\left|b(a + b - 1)\right| + \sigma^2 < 1, \tag{2.8}$$

then the trivial solution of (2.6) is asymptotically mean square stable.

Consider also the necessary and sufficient condition for asymptotic mean square stability of the trivial solution of (2.6).

Let U and A be two square matrices of dimension $k+1$ such that $U = \|u_{ij}\|$ has all zero elements except for $u_{k+1,k+1} = 1$ and

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ b & 0 & 0 & \dots & 0 & a \end{pmatrix}. \tag{2.9}$$

Lemma 2.5 [31]. *Let the matrix equation*

$$A'DA - D = -U \tag{2.10}$$

have a positively semidefinite solution D with $d_{k+1,k+1} > 0$. Then the inequality

$$\sigma^2 d_{k+1,k+1} < 1 \tag{2.11}$$

is the necessary and sufficient condition for asymptotic mean square stability of the trivial solution of (2.6).

Remark 2.6. For $k = 1$. Lemma 2.5 gives the following necessary and sufficient condition for asymptotic mean square stability of the trivial solution of (2.6):

$$0 < d_{22} = \frac{1-b}{(1+b)\left((1-b)^2 - a^2\right)} < \sigma^{-2}. \tag{2.12}$$

In particular, if $\sigma = 0$, this condition has the form $|b| < 1$, $|a| < 1 - b$.

For $k = 2$ from (2.10), (2.11) follows the necessary and sufficient condition for asymptotic mean square stability of the trivial solution of (2.6) in the form

$$0 < d_{33} = \left(1 - a^2 - b^2 - \frac{2a^2 b(a+b)}{1 - b(a+b)}\right)^{-1} < \sigma^{-2}. \tag{2.13}$$

3. Positive point of equilibrium, stochastic perturbations, centering, and linearization

Let us suppose that in (1.1) $a > c > 0$, $b > 0$. By these conditions, (1.1) has a positive point x^* of equilibrium. This point is obtained from the condition $\dot{x}(t) = 0$ and is defined as follows:

$$x^* = \frac{1}{b} \ln \frac{a}{c}. \tag{3.1}$$

As it was proposed in [15, 19] and used later in [32, 33] let us assume that (1.1) is exposed to stochastic perturbation, which is of white noise type, is directly proportional to the deviation of $x(t)$ from the point of equilibrium x^*, and influences on $\dot{x}(t)$ immediately. In this way, (1.1) is transformed to the form

$$\dot{x}(t) = a x(t-h) e^{-bx(t-h)} - c x(t) + \sigma\left(x(t) - x^*\right)\dot{w}(t). \tag{3.2}$$

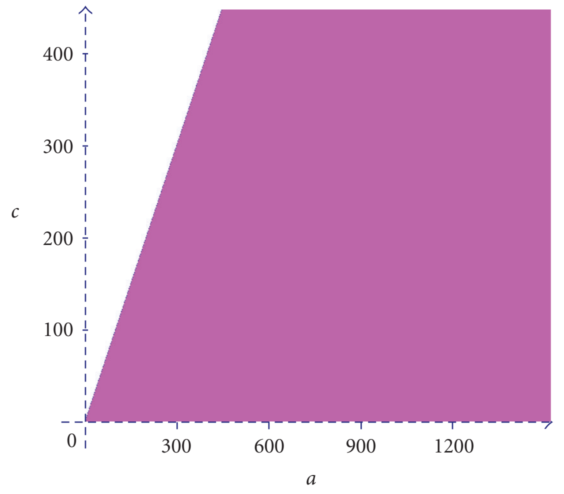

Figure 4.1. Region of necessary and sufficient stability condition for (3.4): $p = 0$, $h = 0$.

Let us center (3.2) on the positive point of equilibrium using the new variable $y(t) = x(t) - x^*$. By this way, via (3.1) we obtain

$$\dot{y}(t) = -cy(t) + cy(t-h)e^{-by(t-h)} + \frac{c}{b}\ln\frac{a}{c}\left(e^{-by(t-h)} - 1\right) + \sigma y(t)\dot{w}(t). \tag{3.3}$$

It is clear that stability of (3.2) equilibrium x^* is equivalent to stability of the trivial solution of (3.3).

Along with (3.3), we will consider the linear part of this equation. Using the representation $e^y = 1 + y + o(y)$ (where $o(y)$ means that $\lim_{y\to 0}(o(y)/y) = 0$) and neglecting $o(y)$, we obtain the linear part (process $z(t)$) of (3.3) in the form

$$\dot{z}(t) = -cz(t) - c\left(\ln\frac{a}{c} - 1\right)z(t-h) + \sigma z(t)\dot{w}(t). \tag{3.4}$$

As it is shown in [16–18], if the order of nonlinearity of the equation under consideration is more than 1, then a sufficient condition for asymptotic mean square stability of the linear part of the initial nonlinear equation is also a sufficient condition for stability in probability of the initial equation. So, we will investigate sufficient conditions for asymptotic mean square stability of the linear part (3.4) of nonlinear (3.3).

4. Stability condition in the case of continue time

Note that condition (2.4), (2.5) for (3.4) takes the form

$$pG < 1, \tag{4.1}$$

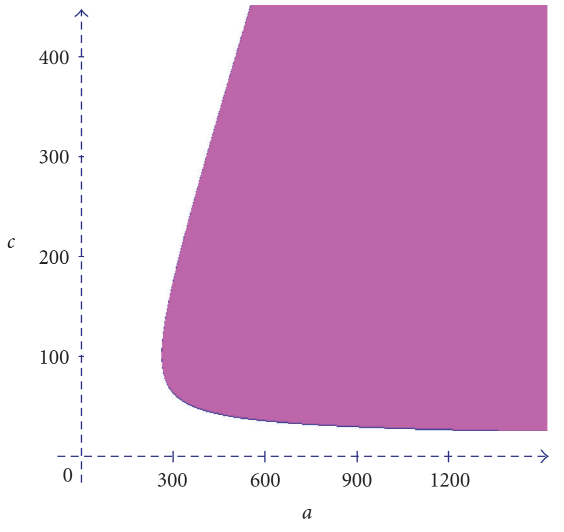

FIGURE 4.2. Region of necessary and sufficient stability condition for (3.4): $p = 100$, $h = 0$.

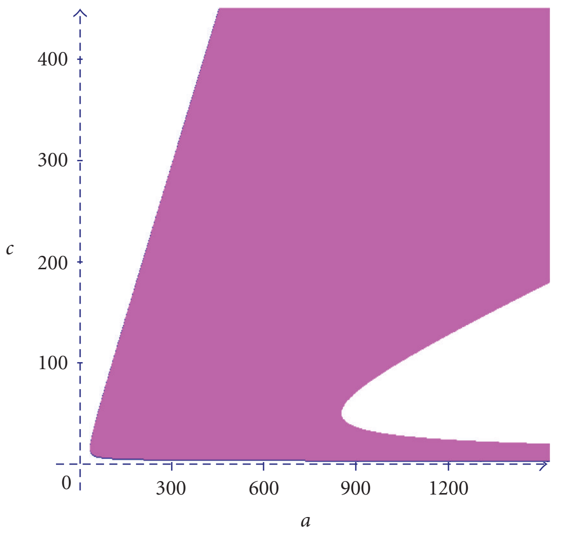

FIGURE 4.3. Region of necessary and sufficient stability condition for (3.4): $p = 12$, $h = 0.024$.

where

$$G = \begin{cases} \dfrac{1 + (c/q)\left(\ln(a/c) - 1\right)\sin(qh)}{c\left[1 + \left(\ln(a/c) - 1\right)\cos(qh)\right]}, & a > ce^2,\ q = c\sqrt{\ln\dfrac{a}{c}\left(\ln\dfrac{a}{c} - 2\right)}, \\ \dfrac{1 + ch}{2c}, & a = ce^2, \\ \dfrac{1 + (c/q)\left(\ln(a/c) - 1\right)\sinh(qh)}{c\left[1 + \left(\ln(a/c) - 1\right)\cosh(qh)\right]}, & c < a < ce^2,\ q = c\sqrt{\ln\dfrac{a}{c}\left(2 - \ln\dfrac{a}{c}\right)}. \end{cases} \tag{4.2}$$

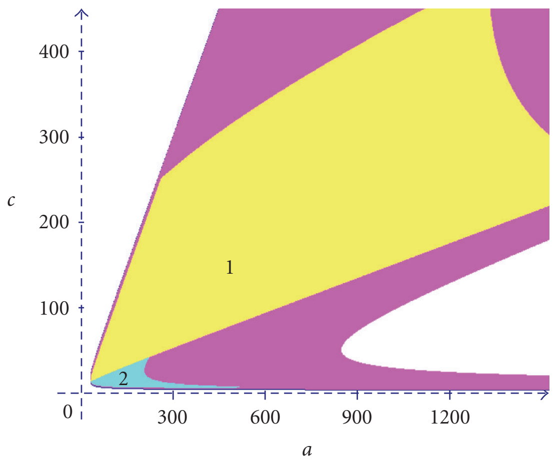

FIGURE 5.1. Region of sufficient stability condition for (5.2): $p = 12$, $h = 0.024$, $\Delta = 0.004$.

In particular, if $p > 0$, $h = 0$, then stability condition takes the form $c\ln(a/c) > p$; if $p = 0$, $h > 0$ then the region of stability is bounded by the lines $c = 0$, $c = a$ and $1 + (\ln(a/c) - 1)\cos(qh) = 0$ for $a > ce^2$.

Condition (4.1), (4.2) gives us regions (in the space of the parameters (a, c)) for asymptotic mean square stability of the trivial solution of (3.4) (and at the same time regions for stability in probability of the positive point of equilibrium x^* of (3.2)). In Figure 4.1, the region of stability given by condition (4.1), (4.2) is shown for $p = 0$, $h = 0$. Similar regions of stability are shown also for $p = 100$, $h = 0$ (Figure 4.2) and for $p = 12$, $h = 0.024$ (Figure 4.3).

Remark 4.1. Note that stability condition (4.1), (4.2) has the following property: if the point (a, c) belongs to the stability region with some p and h then for arbitrary positive α, the point $(a_0, c_0)\ \ (\alpha a, \alpha c)$ belongs to the stability region with $p_0 = \alpha p$ and $h_0 = \alpha^{-1}h$.

5. Stability of the discrete analogue

Consider a difference analogue of nonlinear (3.3) using the Euler scheme

$$y_{i+1} = (1 - c\Delta)y_i + c\Delta y_{i-k}e^{-by_{i-k}} + \frac{c}{b}\ln\frac{a}{c}\Delta\left(e^{-by_{i-k}} - 1\right) + \sigma\sqrt{\Delta}y_i\xi_{i+1}. \tag{5.1}$$

Here k is an integer, $\Delta = h/k$ is the step of discretization, $y_i = y(t_i)$, $t_i = i\Delta$, $\xi_{i+1} = (1/\sqrt{\Delta})(w(t_{i+1}) - w(t_i))$, $i = 0, 1, \ldots$.

In compliance with (3.4), the linear part of (5.1) is

$$z_{i+1} = (1 - c\Delta)z_i + c\Delta\left(1 - \ln\frac{a}{c}\right)z_{i-k} + \sigma\sqrt{\Delta}z_i\xi_{i+1}. \tag{5.2}$$

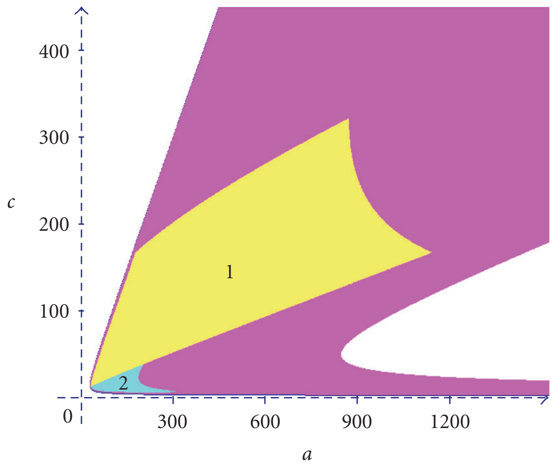

FIGURE 5.2. Region of sufficient stability condition for (5.2): $p = 12$, $h = 0.024$, $\Delta = 0.006$.

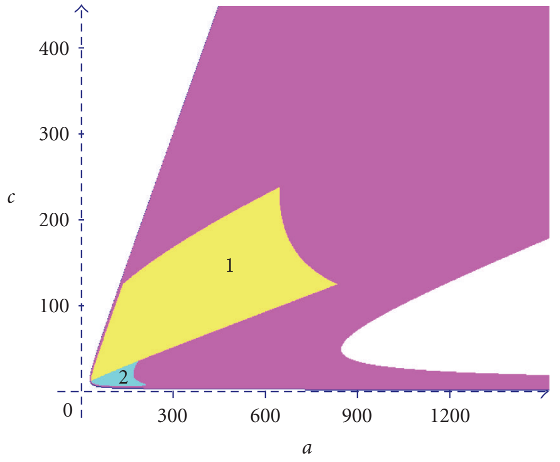

FIGURE 5.3. Region of sufficient stability condition for (5.2): $p = 12$, $h = 0.024$, $\Delta = 0.008$.

Via Lemma 2.4, we obtain two sufficient conditions for asymptotic mean square stability of the trivial solution of (5.2):

$$\frac{p}{c} + \left|1 - \ln\frac{a}{c}\right| |1 - c\Delta| + \frac{1}{2}c\Delta\left(1 + \left|1 - \ln\frac{a}{c}\right|^2\right) < 1, \tag{5.3}$$

$$\frac{p}{c} + \frac{1}{2}c\Delta\ln^2\frac{a}{c} < \left(1 - ch\ln\frac{a}{c}\left|1 - \ln\frac{a}{c}\right|\right)\ln\frac{a}{c}. \tag{5.4}$$

Regions for asymptotic mean square stability of the trivial solution of (5.2) (and at the same time regions for stability in probability of the trivial solution of (5.1)), obtained

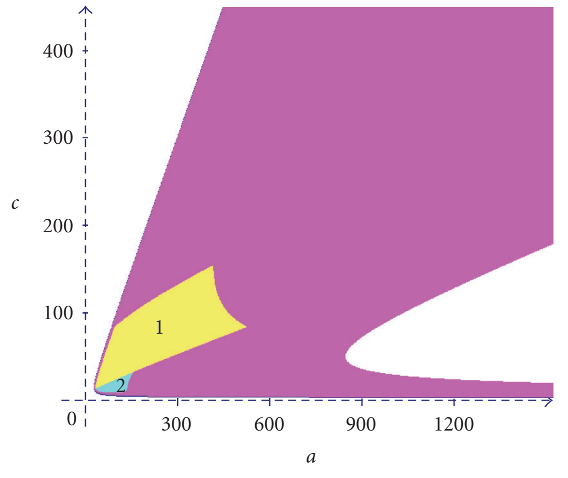

Figure 5.4. Region of sufficient stability condition for (5.2): $p = 12$, $h = 0.024$, $\Delta = 0.012$.

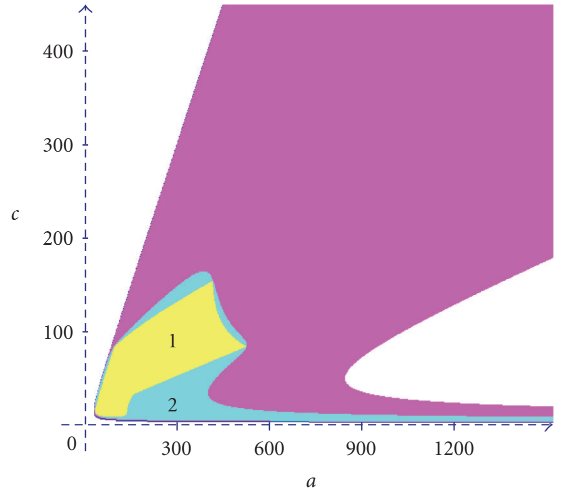

Figure 5.5. Regions of sufficient stability condition and necessary and sufficient stability condition for (5.2): $p = 12$, $h = 0.024$, $\Delta = 0.012$.

by conditions (5.3), (5.4), are shown in the space of the parameters (a,c) for $p = 12$, $h = 0.024$, and $\Delta = 0.004$ (Figure 5.1), $\Delta = 0.006$ (Figure 5.2), $\Delta = 0.008$ (Figure 5.3), $\Delta = 0.012$ (Figure 5.4). The main part (with number 1) of the stability region is obtained via condition (5.3), the additional part (with number 2) is obtained via condition (5.4).

Let us show how much sufficient conditions (5.3), (5.4) are close to the necessary and sufficient condition. Consider the case $p = 12$, $h = 0.024$, $\Delta = 0.012$. Since here $k = h/\Delta = 2$, we can use necessary and sufficient condition (2.13). For (5.2) it can be represented in

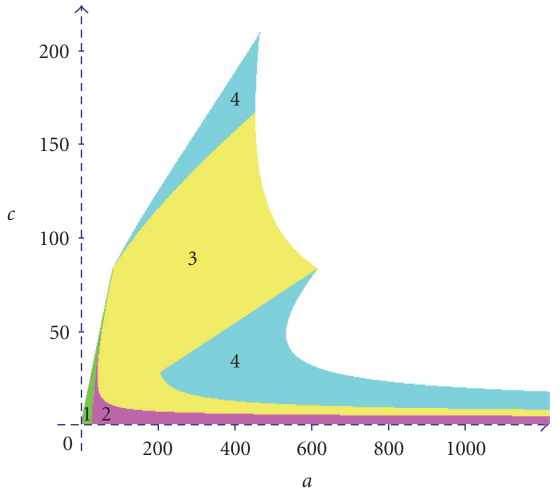

FIGURE 5.6. Regions of sufficient stability condition and necessary and sufficient stability condition for (5.2): $p = 0$, $h = 0.024$, $\Delta = 0.012$.

the form

$$\frac{p}{c} + \frac{1}{2}c\Delta\left[1 + \left(1 - \ln\frac{a}{c}\right)^2\right] + \frac{(1 - c\Delta)^2(1 - \ln(a/c))(1 - c\Delta\ln(a/c))}{1 - c\Delta(1 - \ln(a/c))(1 - c\Delta\ln(a/c))} < 1. \tag{5.5}$$

In Figure 5.5, the stability region, obtained via sufficient conditions (5.3), (5.4) (number 1), is shown inside the stability region, obtained via necessary and sufficient condition (5.5) (number 2).

Remark 5.1. Conditions (5.3), (5.4), (5.5) for arbitrary values of the parameters of (3.4) allow to choose the admissible step of discretization Δ by numerical simulation of the stable solution of this equation. For example, from Figures 5.1, 5.2, we can see that for simulation of (3.4) solution with $a = 900$, $c = 200$, we can use $\Delta = 0.004$ or $\Delta = 0.006$; but taking in account Figures 5.3, 5.4, we cannot be sure that it is possible to use $\Delta = 0.008$ or $\Delta = 0.012$.

Remark 5.2. Note that stability conditions (5.3), (5.4) have the following property: if the point (a, c) belongs to the stability region with some p, h, and Δ, then for arbitrary positive α, the point $(a_0, c_0) = (\alpha a, \alpha c)$ belongs to the stability region with $p_0 = \alpha p$, $h_0 = \alpha^{-1}h$, and $\Delta_0 = \alpha^{-1}\Delta$.

Remark 5.3. In [4, 10], the discrete analogue of (1.1) was considered in the form (in our notations)

$$x_{i+1} = (1 - c\Delta)x_i + a\Delta x_{i-k}e^{-bx_{i-k}}. \tag{5.6}$$

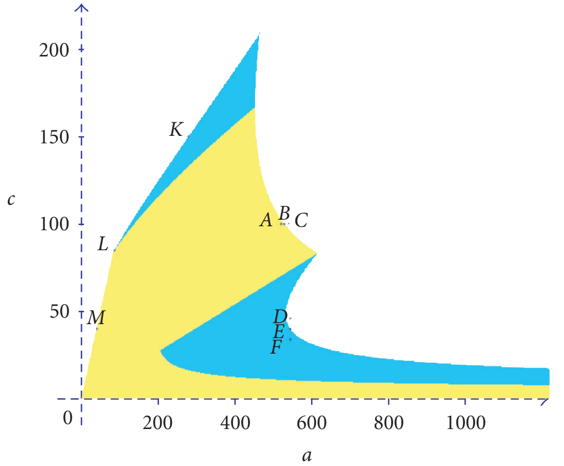

FIGURE 6.1. Regions of sufficient stability condition and necessary and sufficient stability condition for (5.2): $p = 0$, $h = 0.024$, $\Delta = 0.012$.

By the assumption $0 < c\Delta < 1$, the sufficient condition for asymptotic stability of the positive equilibrium (3.1) was obtained in [4]

$$\left[(1-c\Delta)^{-(k+1)}-1\right]\left(\frac{a}{c}-1\right)<1 \tag{5.7}$$

and improved in [10]

$$\left[(1-c\Delta)^{-(k+1)}-1\right]\ln\frac{a}{c}\leq 1. \tag{5.8}$$

Conditions (5.3), (5.4), and (5.5) give much better results. In Figure 5.6, one can see stability regions for $h = 0.024$ and $\Delta = 0.012$ given by condition (5.7) (number 1), given by condition (5.8) (numbers 1 and 2), given by conditions (5.3), (5.4) (numbers 1, 2, and 3), and given by condition (5.5) (numbers 1, 2, 3, and 4).

6. Numerical analysis in the deterministic case

Consider (3.4) at first in the deterministic case ($p = 0$) with delay $h = 0.024$. We will simulate solutions of this equation via its discrete analogue (5.2) with $\Delta = 0.012$. Corresponding stability region is shown in Figure 6.1. Note that for $p = 0$ stability region slightly differs from the similar region for $p = 12$ (Figure 5.5). The initial function is $z(s) = a_0\cos(s)$, $s \in [-h, 0]$, where a_0 has different values in different points.

In Figure 6.1, one can see the points $A(520, 100)$, $B(529.45, 100)$, $C(540, 100)$, $D(544.5, 46)$, $E(544.5, 40)$, $F(544.5, 34)$, $K(279.9, 150)$, $L(87.5, 85)$, $M(40, 40)$. Trajectories of (5.2) solutions in these points are shown accordingly in Figure 6.2 (A, $a_0 = 5$), Figure 6.3 (B, $a_0 = 5$), Figure 6.4 (C, $a_0 = 0.1$), Figure 6.5 (D, $a_0 = 0.4$), Figure 6.6 (E, $a_0 = 4$), Figure 6.7 (F, $a_0 = 5$), Figure 6.8 (K, $a_0 = 6$), Figure 6.9 (L, $a_0 = 5$), and Figure 6.10 (M, $a_0 = 3$).

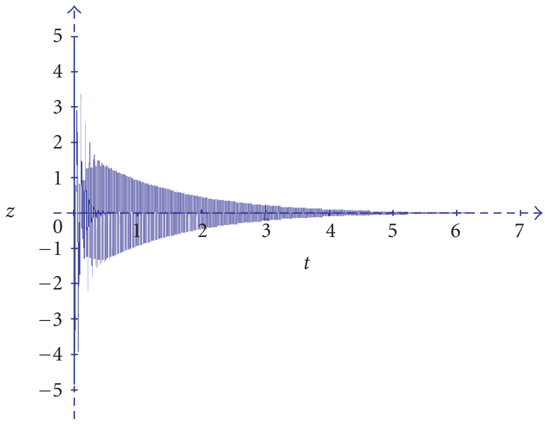

Figure 6.2. Stable solution of (5.2) in the point $A(520, 100)$, $a_0 = 5$.

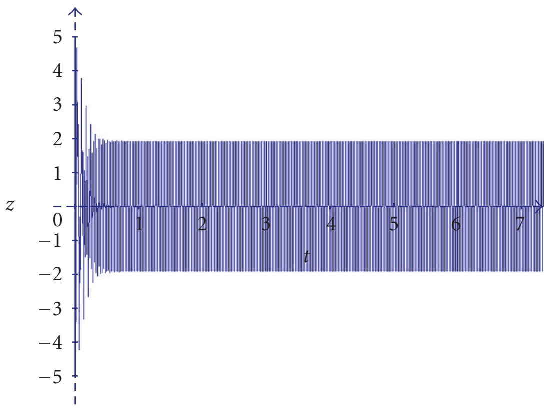

Figure 6.3. Bounded solution of (5.2) in the point $B(529.45, 100)$, $a_0 = 5$.

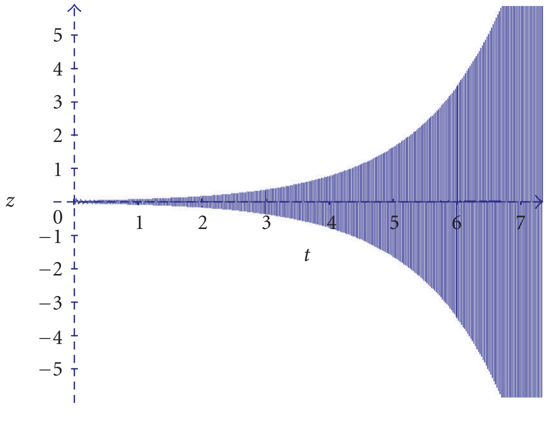

Figure 6.4. Unstable solution of (5.2) in the point $C(540, 100)$, $a_0 = 0.1$.

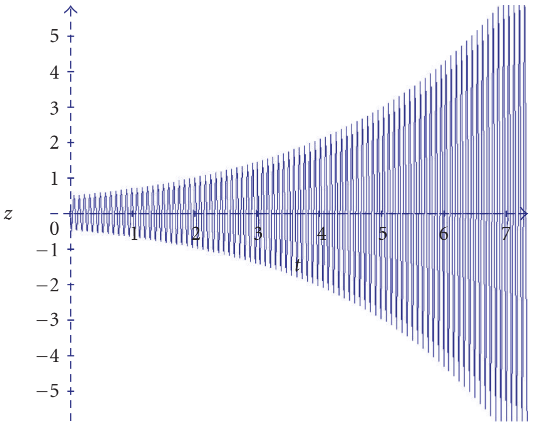

FIGURE 6.5. Unstable solution of (5.2) in the point $D(544.5, 46)$, $a_0 = 0.4$.

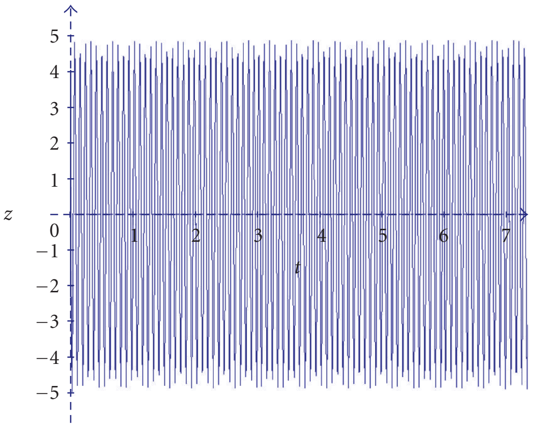

FIGURE 6.6. Bounded solution of (5.2) in the point $E(544.5, 40)$, $a_0 = 4$.

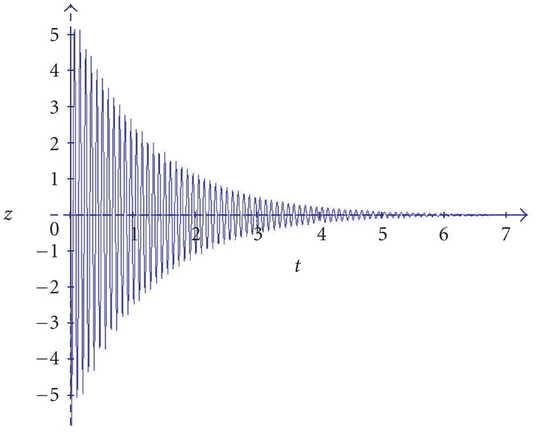

FIGURE 6.7. Stable solution of (5.2) in the point $F(544.5, 34)$, $a_0 = 5$.

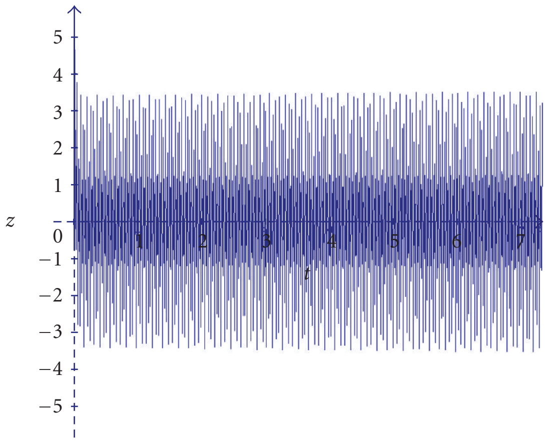

FIGURE 6.8. Bounded solution of (5.2) in the point $K(279.9, 150)$, $a_0 = 6$.

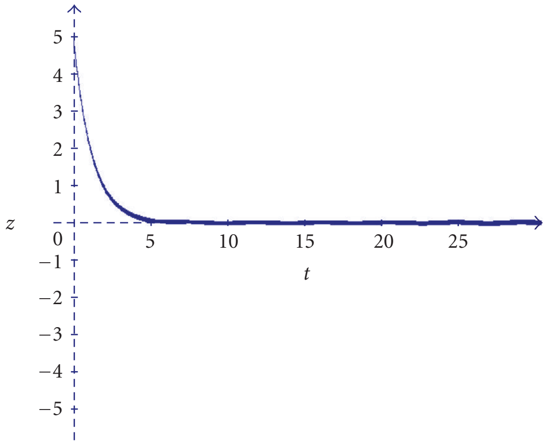

FIGURE 6.9. Bounded solution of (5.2) in the point $L(87.5, 85)$, $a_0 = 5$.

The points A and F belong to stability region, the solutions of (5.2) in these points are stable. The points C and D do not belong to stability region, the solutions of (5.2) in these points are unstable. The points B, E, K, L, and M are placed on the bound of the stability region, the solutions of (5.2) in these points do not converge to zero but converge to bounded functions, in particular, to a constant as in the point M. Note however that in the point M (i.e., in the case $b > 0$, $a = c > 0$) initial (1.1) has only zero equilibrium.

Comparing the solutions of (5.2) in the points A, B, C and in the points D, E, F, one can see also that a bit removal outside of stability region gives an unstable solution and a bit removal inside of stability region gives a stable solution. Similar result one can obtain comparing the solution of (5.2) in the point $L(87.5, 85)$ (Figure 6.9) with the solutions in the points $L_1(88, 85)$ (Figure 6.11) and $L_2(87, 85)$ (Figure 6.12). This fact can be used to construct the exact bound of stability region in the case when we

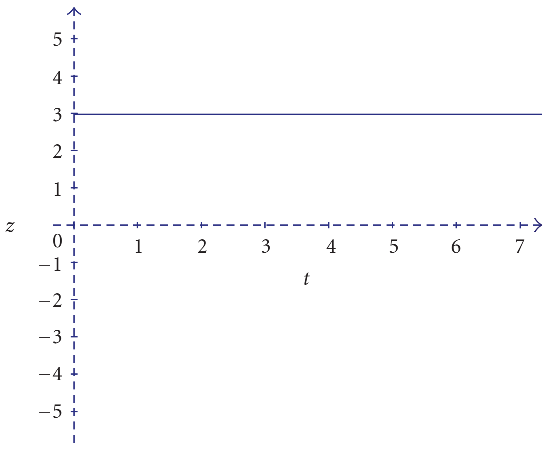

FIGURE 6.10. Bounded solution of (5.2) in the point $M(40,40)$, $a_0 = 3$.

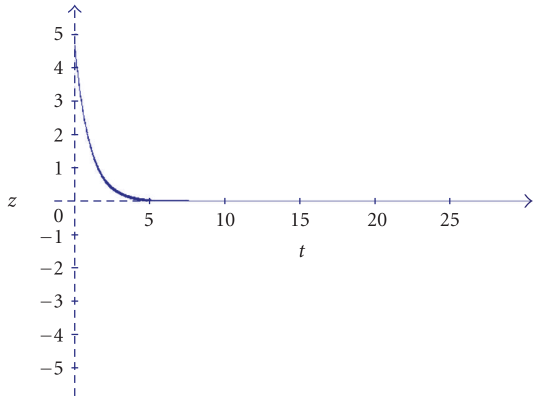

FIGURE 6.11. Stable solution of (5.2) in the point $L_1(88,85)$, $a_0 = 5$.

have sufficient stability condition only. For example, in the case $p = 0$, $h = 0.024$, $\Delta = 0.008$, the points $P(50,50)$, $Q(288.65,170)$, $R(680,250.079)$, $S(810,170)$, $T(923.63,125)$, $U(652.6,50)$, $V(1000,24.16)$ (Figure 6.13) belong to the bound of the stability region since in all these points the solutions of (5.2) do not converge to zero but are bounded functions (Figure 6.14 (P, $a_0 = 3$), Figure 6.15 (Q, $a_0 = 5$), Figure 6.16 (R, $a_0 = 5$), Figure 6.17 (S, $a_0 = 5$), Figure 6.18 (T, $a_0 = 5$), Figure 6.19 (U, $a_0 = 4$), and Figure 6.20 (V, $a_0 = 4$)). Note that in the point P similar to the point M initial (1.1) has only zero equilibrium.

To illustrate Remark 5.2, consider the point $A_0(5.2,1)$ which corresponds to the point $A(520,100)$ with $\alpha = 0.01$. The solution of (5.2) in the point A_0 is stable with $h = 2.4$, $\Delta = 1.2$ (Figure 6.21, $a_0 = 5$). Note that in spite of the fact that (5.2) has the same coefficients in both these cases, the graphic of the solution in the point A_0 differs from the graphic of the solution in the point A (Figure 6.2, $a_0 = 5$) since the initial functions in both cases are

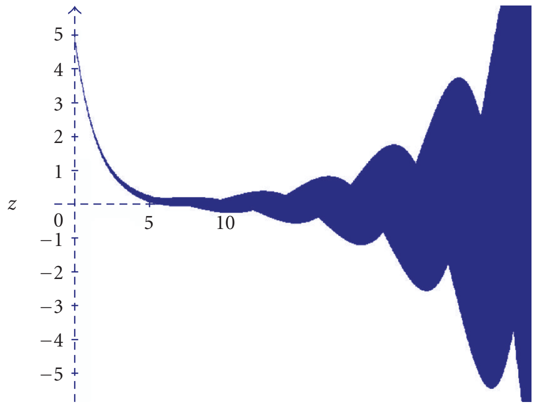

Figure 6.12. Unstable solution of (5.2) in the point $L_2(87, 85)$, $a_0 = 5$.

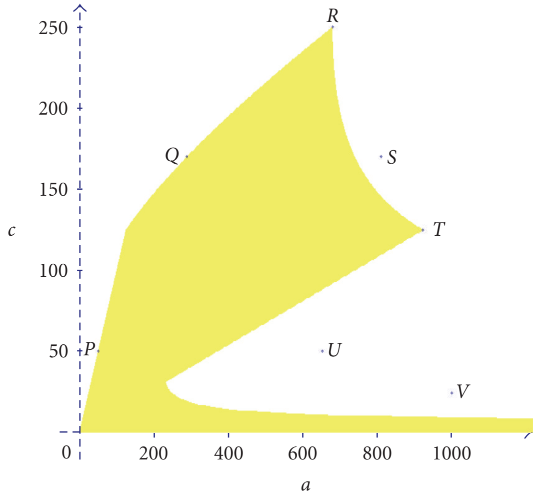

Figure 6.13. Region of sufficient stability condition for (5.2): $p = 0$, $h = 0.024$, $\Delta = 0.008$.

different: in the point A, it is $z_{-2} = 5\cos(0.024)$, $z_{-1} = 5\cos(0.012)$, $z_0 = 5$; in the point A_0, it is $z_{-2} = 5\cos(2.4)$, $z_{-1} = 5\cos(1.2)$, $z_0 = 5$.

Consider now the behavior of the solution of nonlinear (3.3) in the case $p = 0$. We will simulate solutions of this equation via its discrete analogue (5.1) with $\Delta = 0.012$. If in the point (a, c) the trivial solution of (5.2) is asymptotically stable (it means that for arbitrary initial function the solution of (5.2) goes to zero), then the trivial solution of (5.1) is stable in the first approximation (it means that for each enough small initial function the solution of (5.1) goes to zero). On the other hand, if the trivial solution of (5.2) is not asymptotically stable, then for arbitrary indefinitely small initial function the solution of (5.1) does not go to zero.

These facts are illustrated by the following examples. In the point $A(520, 100)$, the trivial solution of (5.2) is asymptotically stable (Figure 6.2, $a_0 = 5$) so, in this point the

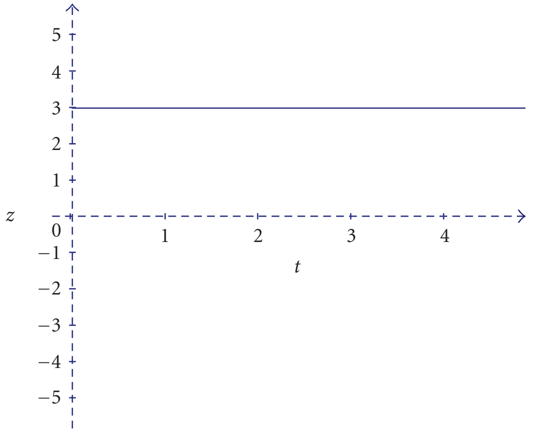

FIGURE 6.14. Bounded solution of (5.2) in the point $P(50,50)$, $a_0 = 3$.

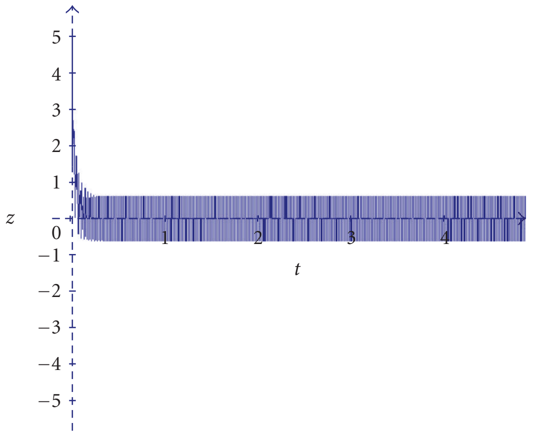

FIGURE 6.15. Bounded solution of (5.2) in the point $Q(288.65,170)$, $a_0 = 5$.

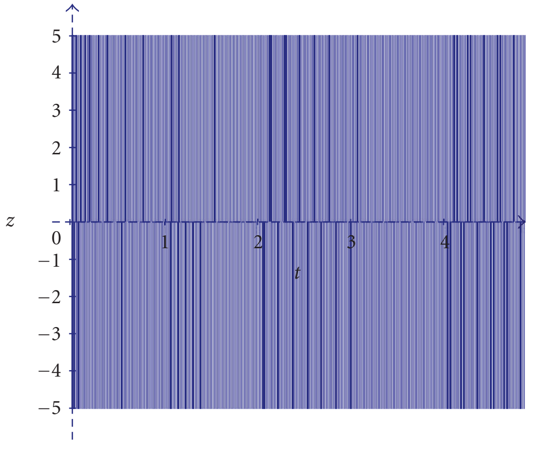

FIGURE 6.16. Bounded solution of (5.2) in the point $R(680,250.079)$, $a_0 = 5$.

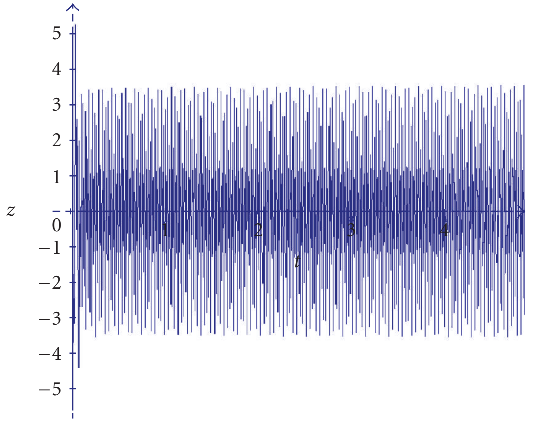

FIGURE 6.17. Bounded solution of (5.2) in the point $S(810, 170)$, $a_0 = 5$.

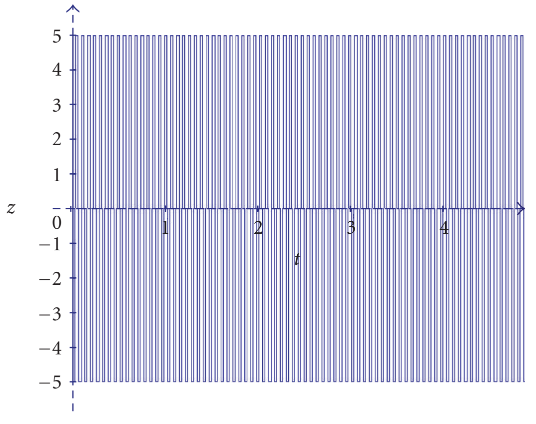

FIGURE 6.18. Bounded solution of (5.2) in the point $T(923.63, 125)$, $a_0 = 5$.

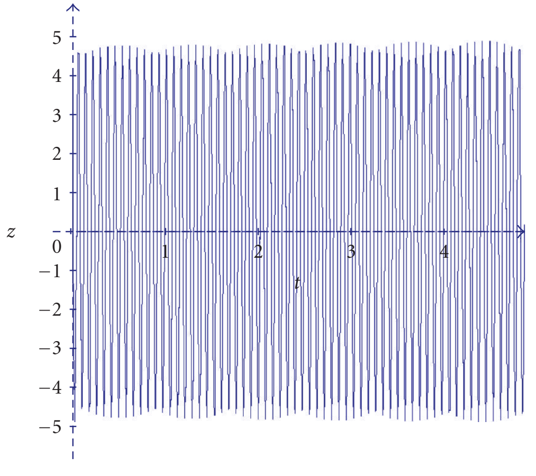

FIGURE 6.19. Bounded solution of (5.2) in the point $U(652.6, 50)$, $a_0 = 4$.

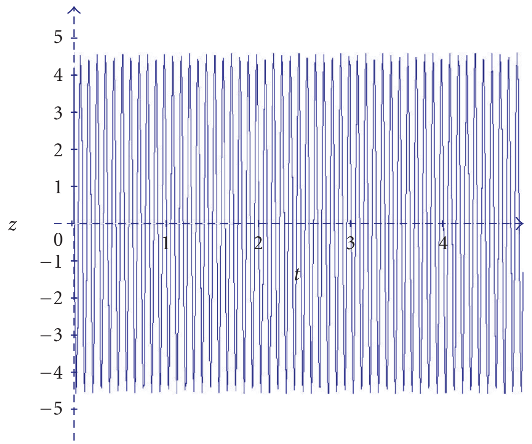

FIGURE 6.20. Bounded solution of (5.2) in the point $V(1000, 24.16)$, $a_0 = 4$.

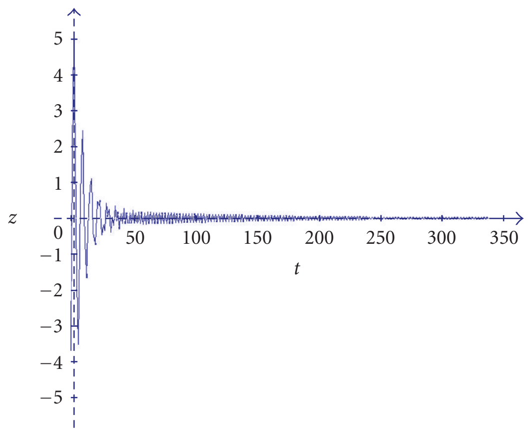

FIGURE 6.21. Stable solution of (5.2) in the point $A_0(5.2, 1)$, $p = 0$, $h = 2.4$, $\Delta = 1.2$, $a_0 = 5$.

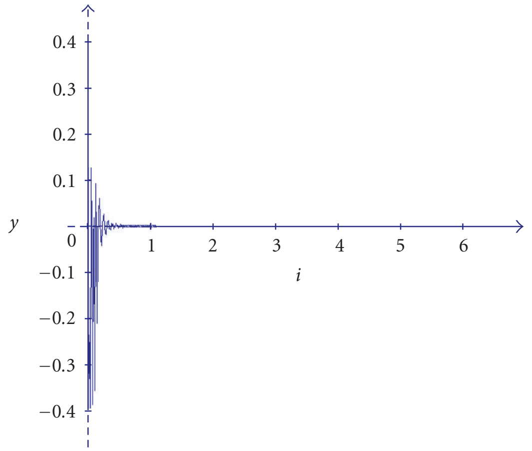

FIGURE 6.22. Stable solution of (5.1) in the point $A(520, 100)$, $a_0 = 0.437$.

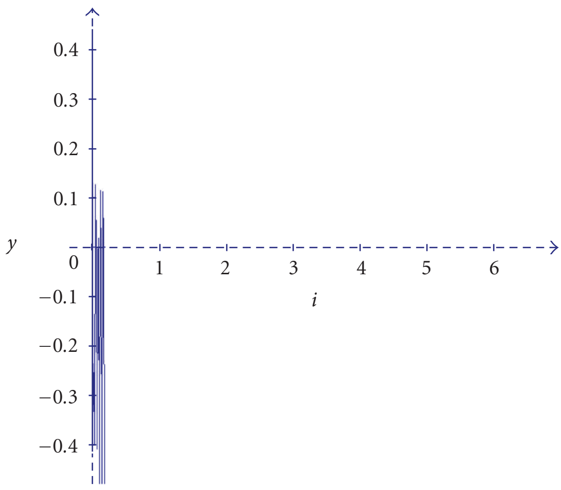

FIGURE 6.23. Unstable solution of (5.1) in the point $A(520, 100)$, $a_0 = 0.438$.

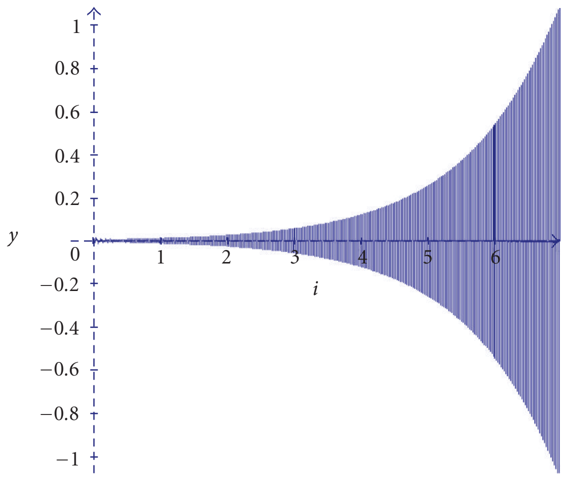

FIGURE 6.24. Unstable solution of (5.1) in the point $C(540, 100)$, $a_0 = 0.001$.

solution of (5.1) ($b = 4$) goes to zero for enough small initial function (Figure 6.22, $a_0 = 0.437$) and enough quickly goes to infinity for a little larger initial function (Figure 6.23, $a_0 = 0.438$). In the point $C(540, 100)$, the trivial solution of (5.2) is not asymptotically stable (Figure 6.4, $a_0 = 0.1$) and the solution of (5.1) ($b = 1$) goes to infinity for indefinitely small initial function (Figure 6.24, $a_0 = 0.001$).

7. Numerical analysis in the stochastic case

Consider at last a behavior of the solution of (3.4) in the stochastic case with $p = 12$, delay $h = 0.024$, and the initial function $z(s) = a_0 \cos(s)$, $s \in [-h, 0]$. A solution of this

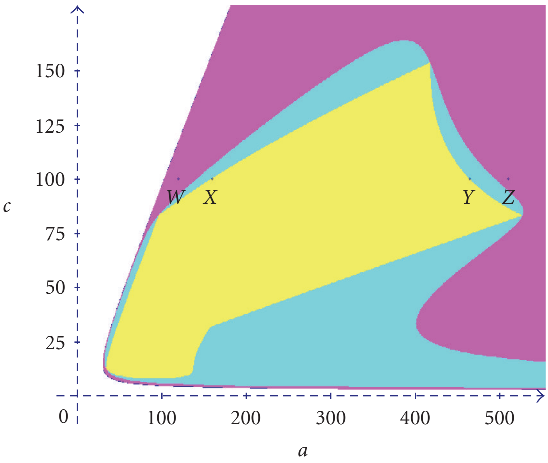

FIGURE 7.1. Regions of sufficient stability condition and necessary and sufficient stability condition for (5.2): $p = 12$, $h = 0.024$, $\Delta = 0.012$.

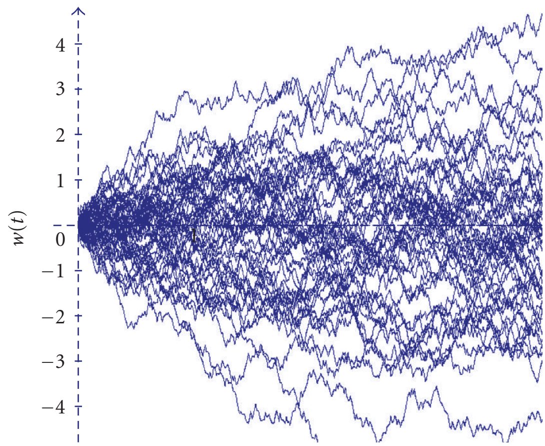

FIGURE 7.2. Trajectories of Wiener process.

equation is simulated here via its discrete analogue (5.2) with $\Delta = 0.012$. Corresponding stability region is shown in Figure 7.1 that is the increasing copy of Figure 5.5 with the additional points $X(160, 100)$, $Y(465, 100)$ that belong to stability region of (5.2) and the points $W(120, 100)$, $Z(510, 100)$ that do not belong to stability region of (5.2).

For numerical simulation of the solution of (5.2) some special algorithm of numerical simulation of Wiener process trajectories [34] is used. In Figure 7.2 50 trajectories of Wiener process obtained via this algorithm are shown.

In Figure 7.3, ten trajectories of the solution of (5.2) are shown in the point W with $a_0 = 0.1$. The point W belongs to stability region of stochastic differential (3.4) but does not belong to stability region of its difference analogue (5.2). One can see that each trajectory of the solution of (5.2) in the point W oscillates and go to infinity. A similar

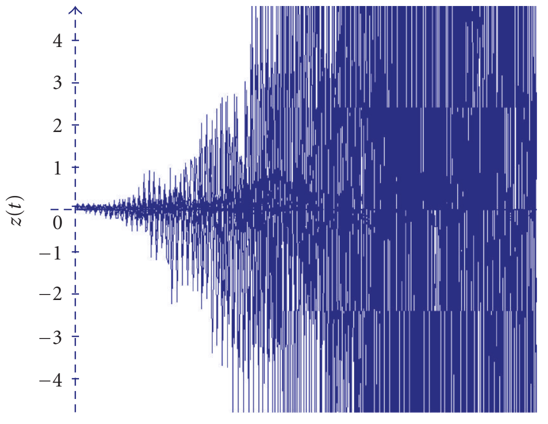

FIGURE 7.3. Unstable solution of (5.2) in the point $W(120, 100)$, $a_0 = 0.1$.

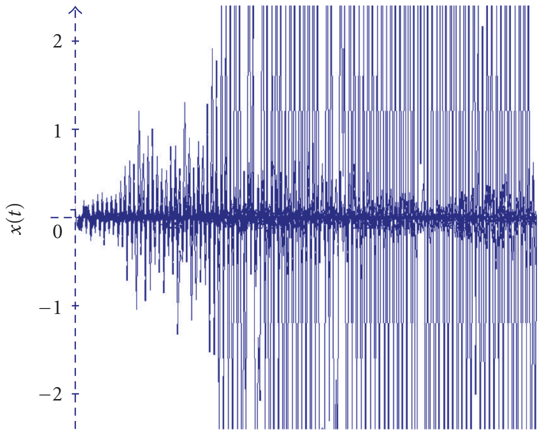

FIGURE 7.4. Unstable solution of (5.2) in the point $Z(510, 100)$, $a_0 = 0.1$.

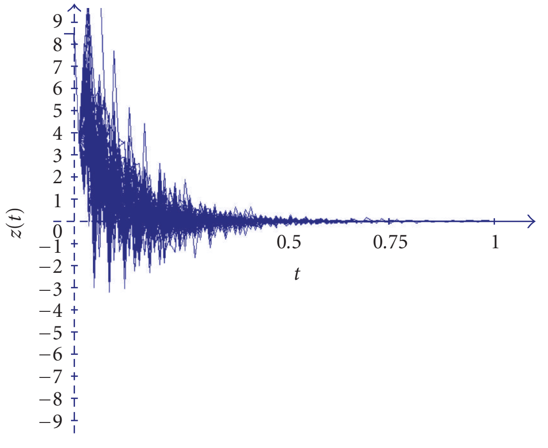

FIGURE 7.5. Stable solution of (5.2) in the point $X(160, 100)$, $a_0 = 8.5$.

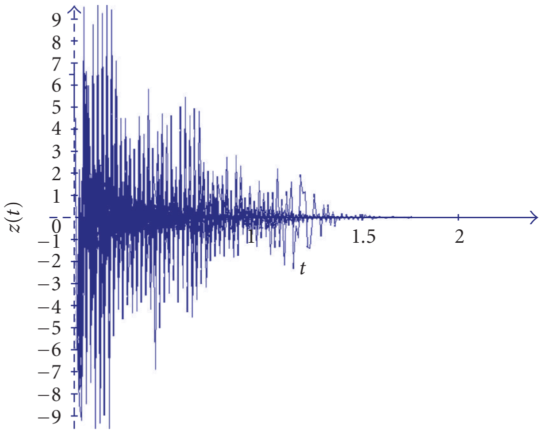

FIGURE 7.6. Stable solution of (5.2) in the point $Y(465,100)$, $a_0 = 6.5$.

picture one can see in Figure 7.4 where one hundred trajectories of the solution of (5.2) are shown in the point Z with $a_0 = 0.1$.

In Figure 7.5, one hundred trajectories of the solution of (5.2) are shown in the point X with $a_0 = 8.5$. The point X belongs to stability region of (5.2) and all trajectories go to zero. One hundred trajectories of the stable solution of (5.2) are shown also in Figure 7.6 in the point Y with $a_0 = 6.5$.

8. Conclusion

In the paper, it is shown that investigating of stability of some differential equation together with its difference analogue via general method of Lyapunov functionals construction, one can obtain a size of the step of discretization for which a difference analogue preserves the stability properties of the original differential equation. All theoretical results are verified by numerical simulation. Besides, it is shown that numerical simulation of the solution of difference analogue allows to define more exactly a bound of stability region obtained by sufficient stability condition.

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