Research Article

A New Part-Metric-Related Inequality Chain and an Application

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Part-metric-related (PMR) inequality chains are elegant and are useful in the study of difference equations. In this paper, we establish a new PMR inequality chain, which is then applied to show the global asymptotic stability of a class of rational difference equations.

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1. Introduction

A part-metric related (PMR) inequality chain is a chain of inequalities of the form

\[
\min_{1 \leq i \leq n} \left\{ a_i, \frac{1}{a_j} \right\} \leq f(a_1, \ldots, a_n) \leq \max_{1 \leq i \leq n} \left\{ a_i, \frac{1}{a_j} \right\},
\]

which is closely related to the well-known part metric [1] and has important applications in the study of difference equations [2–13]. Below are three previously known PMR inequality chains:

\[
\min_{1 \leq i \leq k} \left\{ a_i, \frac{1}{a_j} \right\} \leq \frac{a_1 + a_2 + a_3a_4}{a_1a_2 + a_3 + a_4} \leq \max_{1 \leq i \leq k} \left\{ a_i, \frac{1}{a_j} \right\}, \quad \text{(see [5])},
\]

\[
\min_{1 \leq i \leq k} \left\{ a_i, \frac{1}{a_j} \right\} \leq \frac{a_1 + \cdots + a_{k-1}a_k}{a_1a_2 + a_3 + \cdots + a_k} \leq \max_{1 \leq i \leq k} \left\{ a_i, \frac{1}{a_j} \right\}, \quad \text{(see [11])},
\]

\[
\min_{1 \leq i \leq 5} \left\{ a_i, \frac{1}{a_j} \right\} \leq \frac{(1 + \omega)a_1a_2a_3 + a_4 + a_5}{a_1a_2 + a_1a_3 + a_2a_3 + \omega a_4a_5} \leq \max_{1 \leq i \leq 5} \left\{ a_i, \frac{1}{a_j} \right\}, \quad 1 \leq \omega \leq 2 \quad \text{(see [13])}.
\]
In this article, we establish the following PMR inequality chain:

\[
\min_{1 \leq i \leq 2^p-1} \left\{ \frac{a_i}{a_i} \right\} \leq h_w(a_1, \ldots, a_{2^p-1}) \leq \max_{1 \leq i \leq 2^p-1} \left\{ \frac{a_i}{a_i} \right\},
\]

(1.5)

where \(h_w\) will be defined in the next section, \(p - 2 \leq w \leq p - 1\). When \(p = 3\), chain (1.5) reduces to chain (1.4). On this basis, we prove that the difference equation

\[
x_n = h_w(x_{n-2^p+1}, \ldots, x_{n-1}), \quad n = 1, 2, \ldots,
\]

(1.6)

with positive initial conditions admits a globally asymptotically stable equilibrium \(c = 1\).

2. Main results

This section establishes the main results of this paper. For a function \(f(a_1, \ldots, a_{2^p-1})\), let

\[
f(a_1, \ldots, a_{2^p-1}) \big|_{a_i = m, 1 \leq i \leq r} = f(a_1, \ldots, a_{2^p-1}) \big|_{a_i = m, 1 \leq i \leq r}.
\]

(2.1)

**Lemma 2.1.** Let \(a_1, \ldots, a_n, b_1, \ldots, b_n > 0\). Then \(\min_{1 \leq i \leq n} \{a_i/b_i\} \leq (a_1 + \cdots + a_n)/(b_1 + \cdots + b_n) \leq \max_{1 \leq i \leq n} \{a_i/b_i\}\). One equality in the chain holds if and only if \(a_1/b_1 = \cdots = a_n/b_n\).

For \(p \geq 3\) and \(w > 0\), define a function \(h_w : (\mathbb{R}_+)^{2^p-1} \to \mathbb{R}_+\) as follows:

\[
h_w(a_1, \ldots, a_{2^p-1}) = \frac{(1 + w) \prod_{i=1}^{p} a_i + \prod_{i=p+1}^{2^p-1} a_i \times \sum_{i=p+1}^{2^p-1} (1/a_i)}{\prod_{i=1}^{p} a_i \times \sum_{j=1}^{p} (1/a_i) + w \prod_{i=p+1}^{2^p-1} a_i}.
\]

(2.2)

Below are two examples of this function:

\[
\begin{align*}
h_w(a_1, \ldots, a_5) &= \frac{(1 + w)a_1a_2a_3 + a_4 + a_5}{a_1a_2 + a_1a_3 + a_2a_3 + wa_4a_5}, \\
h_w(a_1, \ldots, a_7) &= \frac{(1 + w)a_1a_2a_3a_4 + a_5a_6 + a_5a_7 + a_6a_7}{a_1a_2a_3 + a_1a_2a_4 + a_1a_3a_4 + a_2a_3a_4 + wa_5a_6a_7}.
\end{align*}
\]

(2.3)

For brevity, let \(h_w = h_w(a_1, \ldots, a_{2^p-1})\). Note that, for each \(a_r, h_w\) is linear fractional in \(a_r\). As a consequence, \(h_w\) is monotone in \(a_r\). Through simple calculations, we get the following two lemmas.
Lemma 2.2. Let \( p \geq 3, a_1, \ldots, a_{2p-1} > 0, m = \min_{1 \leq i \leq 2p-1} \{ a_i \}, 1 \leq r \leq p \).

(1) If \( h_{p-2} \) is increasing in \( a_r \), then \( h_{p-2} \leq \frac{(p - 1)}{\sum_{i=1, i \neq r}^p (1/a_i)} \). The equality holds if and only if \( h_{p-2} \) is constant in \( a_r \).

(2) If \( h_{p-2} \) is strictly decreasing in \( a_r \), then \( h_{p-2} \leq h_{p-2}|_{a_r=m} \). The equality holds if and only if \( a_r = m \).

Lemma 2.3. Let \( p \geq 3, a_1, \ldots, a_{2p-1} > 0, m = \min_{1 \leq i \leq 2p-1} \{ a_i \}, p + 1 \leq r \leq 2p - 1 \).

(1) If \( h_{p-2} \) is increasing in \( a_r \), then \( h_{p-2} \leq \frac{\sum_{i=p+1}^{2p-1} (1/a_i)}{(p - 2)} \). The equality holds if and only if \( h_{p-2} \) is constant in \( a_r \).

(2) If \( h_{p-2} \) is strictly decreasing in \( a_r \), then \( h_{p-2} \leq h_{p-2}|_{a_r=m} \). The equality holds if and only if \( a_r = m \).

Theorem 2.4. Let \( p \geq 3, a_1, \ldots, a_{2p-1} > 0 \). Then \( \min_{1 \leq i \leq 2p-1} \{ a_i, 1/a_i \} \leq h_{p-2} \leq \max_{1 \leq i \leq 2p-1} \{ a_i, 1/a_i \} \). One of the two equalities holds if and only if \( a_1 = \cdots = a_{2p-1} = 1 \).

Proof. Let \( m = \min_{1 \leq i \leq 2p-1} \{ a_i \}, M = \max_{1 \leq i \leq 2p-1} \{ a_i \} \).

We prove only \( h_{p-2} \leq \max \{ M, 1/m \} \) because \( \min \{ M, 1/m \} \leq h_{p-2} \) can be proved similarly. We proceed by distinguishing two possible cases.

Case 1. There is a permutation \( i_1, \ldots, i_{2p-1} \) of \( 1, 2, \ldots, 2p - 1 \) such that for each \( 1 \leq k \leq 2p - 1 \), either \( a_{i_k} = m \) or \( h_{p-2}|_{i_1, i_2, \ldots, i_{2p-1}} \) is strictly decreasing in \( a_{i_k} \). Then

\[
h_{p-2} \leq h_{p-2}|_{i_1} \leq \cdots \leq h_{p-2}|_{i_{2p-1}} = \frac{1}{2} \left( m + \frac{1}{m} \right) \leq \max \left\{ m, \frac{1}{m} \right\} \leq \max \left\{ M, \frac{1}{m} \right\}. \tag{2.4}
\]

Case 2. There is a partial permutation \( i_1, \ldots, i_r \) of \( 1, 2, \ldots, 2p - 1 \) (\( 1 \leq r \leq 2p - 2 \)) such that (a) for each \( 1 \leq k \leq r \), either \( a_{i_k} = m \) or \( h_{p-2}|_{i_1, i_2, \ldots, i_{2p-1}} \) is strictly decreasing in \( a_{i_k} \), and (b) for each \( t \in \{ 1, \ldots, 2p - 1 \} - \{ i_1, \ldots, i_r \}, a_{i_k} \neq m \) and \( h_{p-2}|_{i_1, i_2, \ldots, i_{2p-1}} \) is increasing in \( a_{i_k} \). Then

\[
m < M, \quad h_{p-2} \leq h_{p-2}|_{i_1} \leq h_{p-2}|_{i_2} \leq \cdots \leq h_{p-2}|_{i_r}. \tag{2.5}
\]

Since \( r \leq 2p - 2 \), there is an \( t \in \{ 1, \ldots, 2p - 1 \} - \{ i_1, \ldots, i_r \} \). If \( t \in \{ 1, \ldots, p \} - \{ i_1, \ldots, i_r \} \), it follows from (2.5) and Lemma 2.2 that

\[
h_{p-2} \leq h_{p-2}|_{i_1} \leq \frac{(p - 1)}{\sum_{i=1, i \neq r}^p (1/a_i)} \leq \min_{1 \leq i \leq 2p-1} \{ a_i \}|_{i_1} \leq M \leq \max \left\{ M, \frac{1}{m} \right\}. \tag{2.6}
\]

Whereas if \( t \in \{ p + 1, \ldots, 2p - 1 \} - \{ i_1, \ldots, i_r \} \), it follows from (2.5) and Lemma 2.3 that

\[
h_{p-2} \leq h_{p-2}|_{i_1} \leq \frac{\sum_{i=p+1}^{2p-1} (1/a_i)}{(p - 2)} \leq \max_{p + 1 \leq i \leq 2p-1} \left\{ \frac{1}{a_i} \right\}|_{i_1} \leq \frac{1}{m} \leq \max \left\{ M, \frac{1}{m} \right\}. \tag{2.7}
\]

Hence, \( h_{p-2} \leq \max \{ M, 1/m \} \) is proven.

Second, we prove that \( a_1 = \cdots = a_{2p-1} = 1 \) if \( h_{p-2} = \max \{ M, 1/m \} \). The claim of “\( a_1 = \cdots = a_{2p-1} = 1 \) if \( h_{p-2} = \min \{ M, 1/m \} \)” can be treated similarly. To this end, we need to prove the following.

Claim 1. If \( h_{p-2} = \max \{ M, 1/m \} \), then there is a permutation \( i_1, \ldots, i_{2p-1} \) of \( 1, \ldots, 2p - 1 \) such that for each \( 1 \leq k \leq 2p - 1 \), either \( a_{i_k} = m \) or \( h_{p-2}|_{i_1, i_2, \ldots, i_{2p-1}} \) is strictly decreasing in \( a_{i_k} \).
Proof of Claim 1. On the contrary, assume that Claim 1 is not true. Then there is a partial permutation $i_1,\ldots,i_r$ of $1,2,\ldots,2p-1$ $(1 \leq r \leq 2p - 2)$ such that (a) for each $1 \leq k \leq r$, either $a_{i_k} = m$ or $h_{p-2}|_{i_k-i_{k-1}}$ is strictly decreasing in $a_{i_k}$ and (b) for each $t \in \{1,\ldots,2p-1\} - \{i_1,\ldots,i_r\}$, $a_t \neq m$ and $h_{p-2}|_{i_k-t}$ is increasing in $a_t$. One of the following two cases must occur.

Case 1. There is $t \in \{1,\ldots,2p-1\} - \{i_1,\ldots,i_r\}$ such that $h_{p-2}|_{i_k-t}$ is strictly increasing in $a_t$. If $t \in \{1,\ldots,p\} - \{i_1,\ldots,i_r\}$, it follows by (2.5), (2.6), and Lemma 2.2 that

$$h_{p-2} \leq h_{p-2}|_{i_k-t} < \frac{(p-1)}{\sum_{i=1,j \neq t}^p (1/a_i)} |_{i_k-i_t} \leq \max_{1 \leq i \leq p, i \neq t} \left\{ a_i \right\} |_{i_k-i_t} \leq \max \left\{ M, \frac{1}{m} \right\}. \quad (2.8)$$

A contradiction occurs. Whereas if $t \in \{p+1,\ldots,2p-1\} - \{i_1,\ldots,i_r\}$, it follows by (2.5), (2.7), and Lemma 2.3 that

$$h_{p-2} \leq h_{p-2}|_{i_k-t} < \frac{2^p}{\sum_{i=p+1,j \neq t} (1/a_i)} |_{i_k-i_t} \leq \max_{p+1 \leq i \leq 2p-2, i \neq t} \left\{ \frac{1}{a_i} \right\} |_{i_k-i_t} \leq \max \left\{ M, \frac{1}{m} \right\}. \quad (2.9)$$

Again a contradiction occurs.

Case 2. For each $t \in \{1,\ldots,2p-1\} - \{i_1,\ldots,i_r\}$, $h_{p-2}|_{i_k-t}$ is constant in $a_t$.

First, let us show that $\{1,\ldots,p\} \subseteq \{i_1,\ldots,i_r\}$. Otherwise, there is $t \in \{1,\ldots,p\} - \{i_1,\ldots,i_r\}$. By Lemma 2.2, we have

$$h_{p-2}|_{i_k-t} = \frac{(p-1)}{\sum_{i=1,j \neq t}^p (1/a_i)} |_{i_k-i_t}. \quad (2.10)$$

If there is $s \in \{1,\ldots,p\} - \{i_1,\ldots,i_t,t\}$, it follows from (2.10) that $h_{p-2}|_{i_k-t}$ is strictly increasing in $a_s$, a contradiction occurs. So, $\{1,\ldots,p\} - \{i_1,\ldots,i_r\} = \{t\}$ and thus

$$\max \left\{ M, \frac{1}{m} \right\} = h_{p-2} \leq h_{p-2}|_{i_k-t} = h_{p-2}(a_1,\ldots,a_{2p-1})|_{a_1=\cdots=a_t=m} = m < M, \quad (2.11)$$

from which a contradiction follows. So, $\{1,\ldots,p\} \subseteq \{i_1,\ldots,i_r\}$. \hfill $\Box$

According to the previous argument, there is $t \in \{p+1,\ldots,2p-1\} - \{i_1,\ldots,i_r\}$. By Lemma 2.3, we get

$$h_{p-2}|_{i_k-t} = \frac{2^p}{\sum_{i=p+1,j \neq t} (1/a_i)} |_{i_k-i_t}. \quad (2.12)$$

If there is $s \in \{p+1,\ldots,2p-1\} - \{i_1,\ldots,i_t,t\}$, it follows from (2.12) that $h_{p-2}|_{i_k-t}$ is strictly decreasing in $a_s$, a contradiction. So, $\{p+1,\ldots,2p-1\} - \{i_1,\ldots,i_r\} = \{t\}$ and thus

$$a_1 = \cdots = a_{t-1} = a_{t+1} = \cdots = a_{2p-1} = m. \quad (2.13)$$

By (2.13) and (2.2), we get

$$h_{p-2} = h_{p-2}|_{i_k-t} = \frac{(p-1)m^3 + m + (p-2)a_t}{pm^2 + (p-2)ma_t}. \quad (2.14)$$
Since \( h_{p-2}\mid_{a_i} \) is constant in \( a_i \) and \((d/da_i)h_{p-2}\mid_{a_i} = ((p - 1)(p - 2)m^2(1 - m^2))/\left[pm^2 + (p - 2)ma_i\right]^2\), we derive \( m = 1. \) From (2.12) and (2.13), we get \( h_{p-2}\mid_{a_i} = 1/\max\{M, 1/m\} \), all equalities in chains (2.5) and (2.7) hold. These plus \( m = 1 \) yield \( h_{p-2}\mid_{a_i} = 1/m = 1 \geq M, \) from which we derive \( M = m = 1. \) So, \( a_i = 1 = m. \) This is a contradiction. Claim 1 is proved.

By Claim 1 and \( h_{p-2} = \max\{M, 1/m\} \), all equalities in (2.4) must hold. This plus Lemma 2.2 yields \( a_1 = \cdots = a_{2p-1} = m \) and \( h_{p-2}(m, \ldots, m) = (m + 1/m)/2 = m. \) This implies \( a_1 = \cdots = a_{2p-1} = 1. \)

**Theorem 2.5.** Let \( p \geq 3, a_1, \ldots, a_{2p-1} > 0. \) Then, \( \min_{1\leq i\leq 2p-1}\{a_i, 1/a_i\} \leq h_{p-1} \leq \max_{1\leq i\leq 2p-1}\{a_i, 1/a_i\}. \) One of the two equalities holds if and only if \( a_1 = \cdots = a_p = 1/\frac{a_{p+1} = \cdots = 1}{a_{2p-1}}. \)

**Proof.** By Lemma 2.1 and (2.2), we get

\[
\begin{align*}
h_{p-1} &\leq \max \left\{ \frac{a_1}{a_p}, \ldots, \frac{1}{a_{2p-1}} \right\} \leq \max_{1\leq i\leq 2p-1}\left\{ \frac{a_i}{1} \right\}, \\
h_{p-1} &\geq \min \left\{ \frac{a_1}{a_p}, \ldots, \frac{1}{a_{2p-1}} \right\} \geq \min_{1\leq i\leq 2p-1}\left\{ \frac{a_i}{1} \right\}.
\end{align*}
\]

(2.15)

The second claim follows immediately from Lemma 2.1.

We are ready to present the main result of this paper. \( \square \)

**Theorem 2.6.** Let \( p \geq 3, p - 2 \leq w \leq p - 1, a_1, \ldots, a_{2p-1} > 0. \) Let

\[
a_k = h_w(a_{k-2p+1}, \ldots, a_{k-1}), \quad k = 2p, 2p + 1, \ldots.
\]

Then \( \min_{1\leq i\leq 2p-1}\{a_i, 1/a_i\} \leq a_k \leq \max_{1\leq i\leq 2p-1}\{a_i, 1/a_i\}, k = 2p, 2p + 1, \ldots. \) If \( k \geq 2p + 1, \) then one of the two equalities holds if and only if \( a_1 = \cdots = a_{2p-1} = 1. \)

**Proof.** Regard \( h_w \) as a linear fractional function in \( w, \) which is monotone in \( w. \) By Theorems 2.4 and 2.5, we obtain

\[
\begin{align*}
a_{2p} &\geq \min \left\{ h_{p-2}(a_1, \ldots, a_{2p-1}), h_{p-1}(a_1, \ldots, a_{2p-1}) \right\} \geq \min_{1\leq i\leq 2p-1}\left\{ \frac{a_i}{1} \right\}, \\
a_{2p} &\leq \max \left\{ h_{p-2}(a_1, \ldots, a_{2p-1}), h_{p-1}(a_1, \ldots, a_{2p-1}) \right\} \leq \max_{1\leq i\leq 2p-1}\left\{ \frac{a_i}{1} \right\},
\end{align*}
\]

(2.17)

\[
\begin{align*}
a_{2p+1} &\geq \min \left\{ h_{p-2}(a_2, \ldots, a_{2p}), h_{p-1}(a_2, \ldots, a_{2p}) \right\} \geq \min_{2\leq i\leq 2p}\left\{ \frac{a_i}{1} \right\} \geq \min_{1\leq i\leq 2p-1}\left\{ \frac{a_i}{1} \right\}, \\
a_{2p+1} &\leq \max \left\{ h_{p-2}(a_2, \ldots, a_{2p}), h_{p-1}(a_2, \ldots, a_{2p}) \right\} \leq \max_{2\leq i\leq 2p}\left\{ \frac{a_i}{1} \right\} \leq \max_{1\leq i\leq 2p-1}\left\{ \frac{a_i}{1} \right\}.
\end{align*}
\]

Working inductively, we conclude that for \( k = 2p, 2p + 1, \ldots, \)

\[
\begin{align*}
a_k &\geq \min \left\{ h_{p-2}(a_{k-2p+1}, \ldots, a_k), h_{p-1}(a_{k-2p+1}, \ldots, a_k) \right\} \geq \min_{1\leq i\leq 2p-1}\left\{ \frac{a_i}{1} \right\}, \\
a_k &\leq \max \left\{ h_{p-2}(a_{k-2p+1}, \ldots, a_k), h_{p-1}(a_{k-2p+1}, \ldots, a_k) \right\} \leq \max_{1\leq i\leq 2p-1}\left\{ \frac{a_i}{1} \right\}.
\end{align*}
\]

(2.18) \( \square \)
Claim 2. If \( a_{2p+1} = \max_{1 \leq i \leq 2p-1} \{ a_i, 1/a_i \} \), then \( a_1 = \cdots = a_{2p-1} = 1 \).

Proof of Claim 2. By (2.19), we get
\[
a_{2p+1} = \max \{ h_{p-2}(a_2, \ldots, a_{2p}), h_{p-1}(a_2, \ldots, a_{2p}) \} = \max_{1 \leq i \leq 2p-1} \left\{ a_i, \frac{1}{a_i} \right\}.
\]
(2.20)

Here, we encounter two possible cases.

Case 1. \( a_{2p+1} = h_{p-2}(a_2, \ldots, a_{2p}) = \max_{1 \leq i \leq 2p-1} \{ a_i, 1/a_i \} \). By Theorem 2.4, we get \( a_2 = \cdots = a_{2p} = 1 \) and, hence, \( a_{2p+1} = 1 \). Then \( 1 = a_{2p+1} = \max_{1 \leq i \leq 2p-1} \{ a_i, 1/a_i \} = \max\{ a_1, 1/a_1 \} \), implying \( a_1 = 1 \).

Case 2. \( a_{2p+1} = h_{p-1}(a_2, \ldots, a_{2p}) = \max_{1 \leq i \leq 2p-1} \{ a_i, 1/a_i \} \). By Theorem 2.5, we get
\[
a_2 = \cdots = a_{p+1} = \frac{1}{a_{p+2}} = \cdots = \frac{1}{a_{2p}},
\]
(2.21)
and consequently,
\[
a_{2p+1} = h_{p-1}(a_2, \ldots, a_{2p}) = a_2.
\]
(2.22)

Then,
\[
\max_{1 \leq i \leq 2p-1} \left\{ a_i, \frac{1}{a_i} \right\} = a_{2p+1} = \frac{1}{a_{2p}} \leq \frac{1}{\min \{ h_{p-2}(a_1, \ldots, a_{2p-1}), h_{p-1}(a_1, \ldots, a_{2p-1}) \}} \leq \max_{1 \leq i \leq 2p-1} \left\{ a_i, \frac{1}{a_i} \right\}.
\]
(2.23)

Hence, all equalities in this chain hold. In particular, we have
\[
\min \{ h_{p-2}(a_1, \ldots, a_{2p-1}), h_{p-1}(a_1, \ldots, a_{2p-1}) \} = \min_{1 \leq i \leq 2p-1} \left\{ a_i, \frac{1}{a_i} \right\}.
\]
(2.24)

If \( h_{p-2}(a_1, \ldots, a_{2p-1}) = \min_{1 \leq i \leq 2p-1} \{ a_i, 1/a_i \} \), it follows from Theorem 2.4 that \( a_1 = \cdots = a_{2p-1} = 1 \). Now, assume that \( h_{p-1}(a_1, \ldots, a_{2p-1}) = \min_{1 \leq i \leq 2p-1} \{ a_i, 1/a_i \} \). By Theorem 2.5, we get
\[
a_1 = \cdots = a_p = \frac{1}{a_{p+1}} = \cdots = \frac{1}{a_{2p-1}}.
\]
(2.25)

Equations (2.21) and (2.25) imply that \( a_1 = \cdots = a_{2p-1} = 1 \). Claim 2 is proven.

By Claim 2 and working inductively, we get that if \( a_k = \max_{1 \leq i \leq 2p-1} \{ a_i, 1/a_i \} \) for some \( k \geq 2p+1 \), then \( a_1 = \cdots = a_{2p-1} = 1 \).

Similarly, we can show that \( a_1 = \cdots = a_{2p-1} = 1 \) if \( a_k = \min_{1 \leq i \leq 2p-1} \{ a_i, 1/a_i \} \) holds for some \( k \geq 2p+1 \).

As an application of Theorem 2.6, we have the following theorem.

Theorem 2.7. Let \( p \geq 3 \), \( p-2 \leq \omega \leq p-1 \). The difference equation
\[
x_n = h_\omega(x_{n-2p+1}, \ldots, x_{n-1}), \quad n = 1, 2, \ldots
\]
(2.26)
with positive initial conditions admits the globally asymptotically stable equilibrium \( c = 1 \).

The proof of this theorem is similar to those in [11, 13], and hence is omitted.
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References
