Research Article

On the Asymptotic Behavior of a Difference Equation with Maximum

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We study the asymptotic behavior of positive solutions to the difference equation $x_n = \max\{A/x_{n-1}^\alpha, B/x_{n-2}^\beta\}$, $n = 0, 1, \dots$, where $0 < \alpha, \beta < 1$, $A, B > 0$. We prove that every positive solution to this equation converges to $x^* = \max\{A^{1/(\alpha+1)}, B^{1/(\beta+1)}\}$.

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1. Introduction

Recently, there has been a considerable interest in studying, the so-called, max-type difference equations, see for example, [1–21] and the references cited therein. The max-type operators arise naturally in certain models in automatic control theory (see [9, 11]). The investigation of the difference equation

$$x_n = \max \left\{ \frac{A_1}{x_{n-1}}, \frac{A_2}{x_{n-2}}, \dots, \frac{A_p}{x_{n-p}} \right\}, \quad n = 0, 1, \dots, \quad (1.1)$$

where $p \in \mathbb{N}$, A_i , $i = 1, \dots, p$, are real numbers such that at least one of them is different from zero and the initial values $x_{-1}, \dots, x_{-p}$ are different from zero, was proposed in [6]. Some results about (1.1) and its generalizations can be found in [1, 3–5, 7, 8, 10, 12, 17–19] (see also the references therein). The study of max-type equations whose some terms contain nonconstant numerators was initiated by Stević, see for example, [2, 14–16]. For some closely related papers, see also [20, 21].

Motivated by the aforementioned papers and by computer simulations, in this paper we study the asymptotic behavior of positive solutions to the difference equation

$$x_n = \max \left\{ \frac{A}{x_{n-1}^\alpha}, \frac{B}{x_{n-2}^\beta} \right\}, \quad n = 0, 1, \dots, \quad (1.2)$$

where $0 < \alpha, \beta < 1$, $A, B > 0$. We prove that every positive solution of this equation converges to $x^* = \max\{A^{1/(\alpha+1)}, B^{1/(\beta+1)}\}$.

2. Main results

In this section, we will prove the following result concerning (1.2).

Theorem 2.1. *Let (x_n) be a positive solution to (1.2).*

Then

$$x_n \longrightarrow \max\{A^{1/(\alpha+1)}, B^{1/(\beta+1)}\} \quad \text{as } n \longrightarrow \infty. \quad (2.1)$$

In order to establish Theorem 2.1, we need the following lemma and its corollary which can be found in [13].

Lemma 2.2. *Let $(a_n)_{n \in \mathbb{N}}$ be a sequence of positive numbers which satisfies the inequality*

$$a_{n+k} \leq q \max\{a_{n+k-1}, a_{n+k-2}, \dots, a_n\}, \quad \text{for } n \in \mathbb{N}, \quad (2.2)$$

where $q > 0$ and $k \in \mathbb{N}$ are fixed. Then there exist $L \in \mathbb{R}_+$ such that

$$a_{km+r} \leq Lq^m \quad \forall m \in \mathbb{N}_0, 1 \leq r \leq k. \quad (2.3)$$

Corollary 2.3. *Let $(a_n)_{n \in \mathbb{N}}$ be a sequence of positive numbers as in Lemma 2.2. Then there exists $M > 0$ such that*

$$a_n \leq M(\sqrt[k]{q})^n, \quad n \in \mathbb{N}. \quad (2.4)$$

Now, we are in a position to prove Theorem 2.1.

Proof of Theorem 2.1. We proceed by distinguishing two possible cases.

Case 1 ($A^{1/(\alpha+1)} \geq B^{1/(\beta+1)}$). We prove $x_n \rightarrow A^{1/(\alpha+1)}$ as $n \rightarrow \infty$.

Set $x_n = y_n A^{1/(\alpha+1)}$, then (1.2) becomes

$$y_n = \max\left\{\frac{1}{y_{n-1}^\alpha}, \frac{C}{y_{n-2}^\beta}\right\}, \quad n = 0, 1, \dots, \quad (2.5)$$

where $C = B/A^{(\alpha+1)/(\beta+1)}$. Since $A^{1/(\alpha+1)} \geq B^{1/(\beta+1)}$, we have $C \leq 1$. To prove $x_n \rightarrow A^{1/(\alpha+1)}$ as $n \rightarrow \infty$, it suffices to prove $y_n \rightarrow 1$ as $n \rightarrow \infty$.

We proceed by two cases: $C = 1$ and $0 < C < 1$.

Case $C = 1$. In this case (2.5) is reduced to

$$y_n = \max\left\{\frac{1}{y_{n-1}^\alpha}, \frac{1}{y_{n-2}^\beta}\right\}, \quad n = 0, 1, \dots, \quad (2.6)$$

where $0 < \alpha, \beta < 1$. Choose a number D so that $0 < D < 1$. Let $y_n = D^{z_n}$, $n \geq -2$. Then, (z_n) is a solution to the difference equation

$$z_n = \min \{ -\alpha z_{n-1}, -\beta z_{n-2} \}, \quad n = 0, 1, \dots \quad (2.7)$$

To prove $y_n \rightarrow 1$ as $n \rightarrow \infty$, it suffices to prove $z_n \rightarrow 0$ as $n \rightarrow \infty$.

It can be easily proved that there is a positive integer N such that for all $n \geq 0$,

$$z_{3n+N} \geq 0, \quad z_{3n+N+1} \leq 0, \quad z_{3n+N+2} \leq 0. \quad (2.8)$$

By simple computation, we get that, for all $n \geq 0$,

$$z_{3n+N+2} = \min \{ -\alpha z_{3n+N+1}, -\beta z_{3n+N} \} = -\beta z_{3n+N}, \quad (2.9)$$

$$0 \leq z_{3n+N+3} = \min \{ -\alpha z_{3n+N+2}, -\beta z_{3n+N+1} \} = \min \{ \alpha \beta z_{3n+N}, -\beta z_{3n+N+1} \} \leq \alpha \beta z_{3n+N}, \quad (2.10)$$

$$z_{3n+N+4} = \min \{ -\alpha z_{3n+N+3}, -\beta z_{3n+N+2} \} = -\alpha z_{3n+N+3}. \quad (2.11)$$

Since $0 < \alpha \beta < 1$, (2.10) implies $z_{3n+N} \rightarrow 0$ as $n \rightarrow \infty$. From (2.9) and (2.11), it follows that $z_{3n+N+1} \rightarrow 0$, $z_{3n+N+2} \rightarrow 0$ as $n \rightarrow \infty$. This implies $z_n \rightarrow 0$.

Case $0 < C < 1$. Let $y_n = C^{z_n}$, then (z_n) is a solution to the difference equation

$$z_n = \min \{ -\alpha z_{n-1}, 1 - \beta z_{n-2} \}, \quad n = 0, 1, \dots \quad (2.12)$$

To prove $y_n \rightarrow 1$ as $n \rightarrow \infty$, it suffices to prove $z_n \rightarrow 0$ as $n \rightarrow \infty$. If $z_{-1} = 0$, $z_{-2} = 0$, then we have $z_n = 0$ for all $n \geq -2$. Next, we assume either $z_{-1} \neq 0$ or $z_{-2} \neq 0$. Then the following four claims are obviously true.

Claim 1. If $z_{n-1} \geq 0$ and $z_{n-2} \geq 0$ for some n , then

$$|z_n| \leq \max \{ \alpha |z_{n-1}|, \beta |z_{n-2}| - 1 \}. \quad (2.13)$$

Claim 2. If $z_{n-1} \leq 0$ and $z_{n-2} \leq 0$ for some n , then $|z_n| \leq \alpha |z_{n-1}|$.

Claim 3. If $z_{n-1} \geq 0$ and $z_{n-2} \leq 0$ for some n , then $|z_n| = \alpha |z_{n-1}|$.

Claim 4. If $z_{n-1} \leq 0$ and $z_{n-2} \geq 0$ for some n , then

$$|z_n| \leq \max \{ \alpha |z_{n-1}|, \beta |z_{n-2}| - 1 \}. \quad (2.14)$$

In general, we have

$$|z_n| \leq \max \{ \alpha |z_{n-1}|, \beta |z_{n-2}| - 1 \} \leq \max \{ \alpha |z_{n-1}|, \beta |z_{n-2}| \} \leq \gamma \max \{ |z_{n-1}|, |z_{n-2}| \}, \quad (2.15)$$

where $0 < \gamma = \max \{ \alpha, \beta \} < 1$. From (2.15) and Corollary 2.3, there exists $M > 0$ such that

$$|z_n| \leq M(\sqrt{\gamma})^n. \quad (2.16)$$

This implies $z_n \rightarrow 0$ as $n \rightarrow \infty$.

Case 2 ($A^{1/(\alpha+1)} < B^{1/(\beta+1)}$). We prove $x_n \rightarrow B^{1/(\beta+1)}$ as $n \rightarrow \infty$.

Similar to the proof of Case 1, we set $x_n = y_n B^{1/(\beta+1)}$, then (1.2) becomes

$$y_n = \max \left\{ \frac{C}{y_{n-1}^\alpha}, \frac{1}{y_{n-2}^\beta} \right\}, \quad n = 0, 1, \dots, \quad (2.17)$$

where $C = A/B^{(\alpha+1)/(\beta+1)} < 1$. To prove $x_n \rightarrow B^{1/(\beta+1)}$ as $n \rightarrow \infty$, it suffices to prove $y_n \rightarrow 1$ as $n \rightarrow \infty$. Let $y_n = C^{z_n}$, then (z_n) is a solution to the difference equation

$$z_n = \min \{1 - \alpha z_{n-1}, -\beta z_{n-2}\}, \quad n = 0, 1, \dots \quad (2.18)$$

To prove $y_n \rightarrow 1$ as $n \rightarrow \infty$, it suffices to prove $z_n \rightarrow 0$ as $n \rightarrow \infty$. If $z_{-1} = 0, z_{-2} = 0$, then we have $z_n = 0$ for all $n \geq -2$. Next, we assume either $z_{-1} \neq 0$ or $z_{-2} \neq 0$, then the following four claims are obviously true.

Claim 1. If $z_{n-1} \geq 0$ and $z_{n-2} \geq 0$ for some n , then

$$|z_n| \leq \max \{ \alpha |z_{n-1}| - 1, \beta |z_{n-2}| \}. \quad (2.19)$$

Claim 2. If $z_{n-1} \leq 0$ and $z_{n-2} \leq 0$ for some n , then $|z_n| \leq \beta |z_{n-2}|$.

Claim 3. If $z_{n-1} \geq 0$ and $z_{n-2} \leq 0$ for some n , then

$$|z_n| \leq \max \{ \alpha |z_{n-1}| - 1, \beta |z_{n-2}| \}. \quad (2.20)$$

Claim 4. If $z_{n-1} \leq 0$ and $z_{n-2} \geq 0$ for some n , then $|z_n| = \beta |z_{n-2}|$.

In general, we have

$$|z_n| \leq \max \{ \alpha |z_{n-1}| - 1, \beta |z_{n-2}| \} \leq \max \{ \alpha |z_{n-1}|, \beta |z_{n-2}| \} \leq \gamma \max \{ |z_{n-1}|, |z_{n-2}| \}, \quad (2.21)$$

where $0 < \gamma = \max\{\alpha, \beta\} < 1$. Then the rest of the proof is similar to the proof of Case 1 and will be omitted. The proof is complete. $\square$

Theorem 2.4. Every solution to the difference equation $x_n = A/x_{n-m}^\alpha$, $0 < \alpha < 1$, $A > 0$ converges to $x^* = A^{1/(\alpha+1)}$.

Proof. Let $x_n = y_n A^{1/(\alpha+1)}$, then the equation becomes

$$y_n = \frac{1}{y_{n-m}^\alpha} = y_{n-2m}^{\alpha^2} = y_{n-4m}^{\alpha^4} = \dots = y_{n-2[n/2m]m}^{\alpha^{2[n/2m]}}. \quad (2.22)$$

From this and the condition $0 < \alpha < 1$, it follows that $y_n \rightarrow 1$ as $n \rightarrow \infty$ which implies $x_n \rightarrow A^{1/(\alpha+1)}$ as $n \rightarrow \infty$. $\square$

3. Conclusions and remarks

This paper examines the asymptotic behavior of positive solutions to the difference equation (1.2) with $0 < \alpha, \beta < 1$, $A, B > 0$. The method used in this work may provide insight into the asymptotic behavior of positive solutions to the generic difference equation

$$x_n = \max \left\{ \frac{A_1}{x_{n-1}^{\alpha_1}}, \frac{A_2}{x_{n-2}^{\alpha_2}}, \dots, \frac{A_p}{x_{n-p}^{\alpha_p}} \right\}, \quad n = 0, 1, \dots, \quad (3.1)$$

where $0 < \alpha_i < 1$, $A_i > 0$, $i = 1, \dots, p$. We close this work by proposing the following conjecture.

Conjecture 3.1. Assume that (x_n) is a positive solution to (3.1). Then $x_n \rightarrow \max_{1 \leq i \leq p} \{A_i^{1/(\alpha_i+1)}\}$ as $n \rightarrow \infty$.

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