Research Article

Uniqueness and Multiplicity of Solutions for a Second-Order Discrete Boundary Value Problem with a Parameter

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This paper is concerned with the existence of unique and multiple solutions to the boundary value problem of a second-order difference equation with a parameter, which is a complement of the work by J. S. Yu and Z. M. Guo in 2006.

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1. Introduction and preliminaries

In this paper, we consider the existence, uniqueness, and multiplicity of solutions for a second-order discrete boundary value problem

$$\begin{aligned} p(n+1)u(n+1) + c(n)u(n) + p(n)u(n-1) &= \lambda f(n, u(n)), \quad n \in Z(1, k), \\ u(0) + \alpha u(1) &= A, \quad u(k+1) + \beta u(k) = B, \end{aligned} \quad (1.1)$$

where $\lambda \in \mathbb{R}$ is a parameter. Our technique is based on critical point theory, which is successfully used to deal with the existence of solutions for discrete problems (see [1–9]), especially in [7, 9]. Similarly to [7], we denote by $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{R}$ the sets of all natural numbers, integers, and real numbers, respectively. For $a, b \in \mathbb{Z}$, $Z(a) = \{a, a+1, \dots\}$, $Z(a, b) = \{a, a+1, \dots, b\}$ when $a \leq b$. We assume that $p(n)$ is nonzero and real-valued for each $n \in Z(1, k)$, $c(n)$ is real-valued for each $n \in Z(1, k)$, and $f(n, u)$ is real-valued for each $(n, u) \in Z(1, k) \times \mathbb{R}$ and continuous in u . Let $\mathbb{R}^k$ be the real Euclidean space with dimension k . For any $u, v \in \mathbb{R}^k$, $\|u\|$ and (u, v) , denote the usual norm and inner product in $\mathbb{R}^k$, respectively.

Consider the functional defined on $\mathbb{R}^k$,

$$J(u) = \frac{1}{2}(Mu, u) + (\eta, u) - \lambda F(u), \quad u = (u(1), u(2), \dots, u(k))^T \in \mathbb{R}^k, \quad (1.2)$$

where $(\cdot)^T$ is the transpose of a vector in $\mathbb{R}^k$,

$$M = \begin{pmatrix} c(1) - \alpha p(1) & p(2) & 0 & \cdots & 0 & 0 \\ p(2) & c(2) & p(3) & \cdots & 0 & 0 \\ 0 & p(3) & c(3) & \cdots & 0 & 0 \\ \cdots & & \cdots & \cdots & & \\ 0 & 0 & 0 & \cdots & c(k-1) & p(k) \\ 0 & 0 & 0 & \cdots & p(k) & c(k) - \beta p(k+1) \end{pmatrix},$$

$$\eta = \begin{pmatrix} p(1)A \\ 0 \\ \vdots \\ 0 \\ p(k+1)B \end{pmatrix}, \quad (1.3)$$

$$F(u) = \sum_{j=1}^k \int_0^{u(j)} f(j, s) ds.$$

It is easy to see that $J(u)$ is Fréchet differentiable with Fréchet derivative

$$J'(u) = Mu + \eta - \lambda f(u), \quad (1.4)$$

where $f(u) = (f(1, u(1)), f(2, u(2)), \dots, f(k, u(k)))^T$, and there is a one-to-one correspondence between the critical point of functional J and the solution of BVP (1.1). Furthermore, $u = (u(1), u(2), \dots, u(k))^T$ is a critical point of J if and only if $\{u(t)\}_{t=0}^{k+1} = (u(0), u(1), \dots, u(k+1))^T$ is a solution of (1.1), where $u(0) = A - \alpha u(1)$, $u(k+1) = B - \beta u(k)$ [7].

Recently, Yu and Guo [7] studied the BVP,

$$\begin{aligned} p(n+1)u(n+1) + c(n)u(n) + p(n)u(n-1) &= f(n, u(n)), \\ u(0) + \alpha u(1) &= A, \quad u(k+1) + \beta u(k) = B. \end{aligned} \quad (1.5)$$

They obtained some existence results for (1.5). One of the main results is as follows.

Theorem 1.1 (A_0). *Suppose that $f(n, z)$ satisfies the following assumption:*

(f3) *there exist constants $a_1 > 0$, $a_2 > 0$, $R > 0$, and $\beta > 2$ such that*

$$\sum_{j=1}^k \int_0^{u(j)} f(j, s) ds \geq a_1 \|u\|^\beta - a_2 \quad \text{for } u = (u(1), u(2), \dots, u(k))^T \in \mathbb{R}^k, \quad \|u\| \geq R. \quad (1.6)$$

Then BVP (1.5) has at least one solution.

Equation (1.6) shows that $F(u) = \sum_{j=1}^k \int_0^{u(j)} f(j, s) ds > 0$ for $\|u\|$ large enough. Since $F(u)$ may be negative, then the conclusion of Theorem 1.1 cannot be drawn, which motivates us to consider (1.1). Note that if we take $\lambda = -1$ in (1.1), then $F(u) < 0$. Under the similar condition to (1.6) when $\lambda < 0$, we not only obtain the existence of solutions for (1.1), but also the multiplicity.

Let E be a Banach space with a direct sum decomposition $E = X \oplus Y$. The functional $I \in C^1(E, \mathbb{R})$ has a local linking at 0 if for some $\rho > 0$, $I(u) \leq 0$, $u \in X$, $\|u\| \leq \rho$, and $I(u) \geq 0$, $u \in Y$, $\|u\| \leq \rho$. The functional $I \in C^1(E, \mathbb{R})$ is said to satisfy the (PS) condition if any sequence $x_n \subset X$ for which $I(x_n)$ is bounded, and $I'(x_n) \rightarrow 0$ ($n \rightarrow \infty$) possesses a convergent subsequence in E .

Theorem A (see[10]). *Let E be a Banach space. Suppose that $I \in C^1(E, \mathbb{R})$ satisfies the (PS) condition and has a local linking at 0. Assume that I is bounded below and $\inf_E I < 0$. Then I has at least two nontrivial critical points.*

Theorem B (see[6, 11]). *Let X be a real Banach space, $I \in C^1(X, \mathbb{R})$ with I even, bounded from below, and satisfying (PS) condition. Suppose $I(0) = 0$. There is a set $K \subset X$ such that K is homeomorphic to a unit sphere S^{n-1} in $\mathbb{R}^n$ ($n \in \mathbb{N}$) by an odd map, and $\sup_K I < 0$. Then I possesses at least n distinct pairs of critical points.*

2. Main results

Following conditions will be useful to prove our main results.

(H1) There exist numbers $\alpha_1 > 2$ and $a_1 > 0$ such that

$$F(u) \geq a_1 \|u\|^{\alpha_1} \quad \text{for } u = (u(1), u(2), \dots, u(k))^T \in \mathbb{R}^k. \quad (2.1)$$

(H2) $\lim_{u \rightarrow 0} (F(u) / \|u\|^2) = 0$ for $u = (u(1), u(2), \dots, u(k))^T \in \mathbb{R}^k$.

Theorem 2.1. *Suppose that M is positive definite, $f(j, 0) = 0$, $f(j, u)u < 0$ for $j \in Z(1, k)$ and $u \neq 0$, and that $p(1)A = p(k+1)B = 0$. Then (1.1) has only trivial solution for $\lambda > 0$.*

Proof. Note that

$$\begin{aligned} (J'(u), u) &= (Mu, u) + (\eta, u) - \lambda(f(u), u) \\ &\geq \lambda_1 \|u\|^2 - \lambda \sum_{j=1}^k f(j, u(j))u(j) \\ &\geq \lambda_1 \|u\|^2 > 0 \end{aligned} \quad (2.2)$$

for $u \neq 0$, where λ_1 is the least eigenvalue of M , which means that the Nahari manifold is empty. Thus (1.1) has only trivial solution. $\square$

Theorem 2.2. *Suppose that (H1) and (H2) hold, $p(1)A = p(k+1)B = 0$ and M is neither positive definite nor negative definite. Then (1.1) has at least two nontrivial solutions for $\lambda < 0$.*

Proof. We will prove that the functional $J(u)$ satisfies all conditions of Theorem A by two steps.

Step 1. J is bounded from below and satisfies (PS) condition. Let $\lambda_{-l}, \lambda_{-l+1}, \dots, \lambda_{-1}, \lambda_1, \lambda_2, \dots, \lambda_m$ denote all the eigenvalues of M , where $\lambda_{-l} \leq \lambda_{-l+1} \leq \dots \leq \lambda_{-1} < 0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_m$ and $l + m = k$. For any $j \in Z(-l, -1) \cup Z(1, m)$, set ξ_j to be an eigenvector of M corresponding to the eigenvalue λ_j , $j = -l, -l+1, \dots, -1, 1, 2, \dots, m$, such that

$$(\xi_i, \xi_j) = \begin{cases} 0 & \text{for } i \neq j, \\ 1 & \text{for } i = j. \end{cases} \quad (2.3)$$

Let X and Y be subspaces of $\mathbb{R}^k$ defined by

$$\begin{aligned} X &= \left\{ x \in \mathbb{R}^k : x = \sum_{j=1}^m x_j \xi_j, x_j \in \mathbb{R}, j \in Z(1, m) \right\}, \\ Y &= \left\{ y \in \mathbb{R}^k : y = \sum_{j=-l}^{-1} y_j \xi_j, y_j \in \mathbb{R}, j \in Z(-l, -1) \right\}, \end{aligned} \quad (2.4)$$

respectively. Then $\mathbb{R}^k$ has the direct sum decomposition $\mathbb{R}^k = X \oplus Y$. In view of (H1), we have

$$\begin{aligned} J(u) &\geq \frac{1}{2} \lambda_{-l} \|u\|^2 - \lambda a_1 \|u\|^{\alpha_1} \\ &\geq -\frac{1}{2} |\lambda_{-l}| (1 - 2/\alpha_1) (|\lambda_{-l}| / (-\alpha_1 \lambda a_1))^{2/(\alpha_1-2)}. \end{aligned} \quad (2.5)$$

The second inequality follows from the elementary inequality $-ax^2 + bx^q \geq -a(1 - 2/q)((2a)/(qb))^{2/(q-2)}$, where $a > 0$, $b > 0$, $x > 0$, and $q > 2$, which can be easily obtained by the fact that the function $h(x) = -ax^2 + bx^q$ ($a, b > 0$, $x \geq 0$) attains its minimum at $((2a)/(qb))^{1/(q-2)}$. Thus $J(u)$ is bounded from below.

Equation (2.5) shows that $J(u)$ is coercive, so we can obtain that any (PS) sequence must be bounded in $\mathbb{R}^k$ and, by a standard argument, has a convergent subsequence.

Step 2. J has a local linking at 0. Indeed, by (H2) for given $\lambda < 0$ and sufficiently small $\varepsilon > 0$ such that $-\lambda\varepsilon + (1/2)\lambda_{-1} < 0$, there exists $r > 0$ small enough such that for $\|u\| < r$,

$$F(u) < \varepsilon \|u\|^2 \quad (2.6)$$

holds. Then for $y \in Y$ such that $0 < \|y\| < r$, we have

$$\begin{aligned} J(y) &\leq \frac{1}{2} \lambda_{-1} \|y\|^2 - \lambda \varepsilon \|y\|^2 \\ &= \left(\frac{1}{2} \lambda_{-1} - \lambda \varepsilon \right) \|y\|^2 < 0. \end{aligned} \quad (2.7)$$

On the other hand, for $x \in X$ with $\|x\| < r$, we have

$$J(x) \geq \frac{1}{2} \lambda_1 \|x\|^2 - \lambda a_1 \|x\|^{\alpha_1} \geq 0. \quad (2.8)$$

The application of Theorem A finishes our proof. $\square$

Remark 2.3. By the above proof, we see that replacing (H1) with (f3), and by adding the condition that $F(u) \geq 0$ for $u \in \mathbb{R}^k$, Theorem 2.2 still holds, where (f3) is the same as in Theorem 1.1. Indeed, we have

$$J(u) \geq \begin{cases} \frac{1}{2}\lambda_{-l}R^2 & \text{for } \|u\| \leq R, \\ -\frac{1}{2}|\lambda_{-l}|(1-2/\beta)(|\lambda_{-l}|/(-\beta\lambda a_1))^{2/(\beta-2)} + \lambda a_2 & \text{for } \|u\| > R, \end{cases} \quad (2.9)$$

which means that $J(u)$ is bounded from below. The fact that $J(u)$ has local linking at 0 may be verified similarly.

If we further impose some condition on $f(u)$ and matrix M , then the following result can be derived.

Theorem 2.4. *Suppose (H1) and (H2) hold, $p(1)A = p(k+1)B = 0$, $f(u)$ is odd in u , that is, $f(j, -u) = -f(j, u)$ for $(j, u) \in Z(1, k) \times \mathbb{R}$, and that M is neither positive definite nor negative definite and has l distinct negative eigenvalues. Then (1.1) has at least l distinct pairs of solutions for $\lambda < 0$.*

Proof. By the proof of Theorem 2.2, $J(u)$ is bounded from below and satisfies (PS) condition. In addition, $J(0) = 0$, $J(u)$ is even. Consider the subset K of $\mathbb{R}^k$:

$$K = \left\{ y \in Y : y = \sum_{j=-l}^{-1} y_j \xi_j, \sum_{j=-l}^{-1} y_j^2 = \rho^2, y_j \in \mathbb{R}, j \in Z(-l, -1) \right\}, \quad (2.10)$$

where ρ is a positive number small enough to be determined later, Y is defined by Theorem 2.2 similarly. Define the mapping $T : K \rightarrow S^{l-1}$ by

$$T(y) = T\left(y = \sum_{j=-l}^{-1} y_j \xi_j\right) = \left(-\frac{y_{-l}}{\rho}, -\frac{y_{-l+1}}{\rho}, \dots, -\frac{y_{-1}}{\rho}\right), \quad (2.11)$$

where S^{l-1} is a unit sphere in $\mathbb{R}^l$. Then T is a homeomorphism between K and S^{l-1} , and K is a subset of the finite dimensional space Y equipped with the Euclidian norm. We can choose $\rho > 0$ and $\varepsilon > 0$ small enough such that $(1/2)\lambda_{-1} - \lambda\varepsilon < 0$ for $y \in K$, and then we have

$$J(y) \leq \frac{1}{2}\lambda_{-1}\|y\|^2 - \lambda F(u) \leq \frac{1}{2}\lambda_{-1}\rho^2 - \lambda\varepsilon\rho^2 \leq \left(\frac{1}{2}\lambda_{-1} - \lambda\varepsilon\right)\rho^2. \quad (2.12)$$

For above $\rho > 0$, we have

$$\sup_K J < 0, \quad (2.13)$$

which together with Theorem B concludes the proof. $\square$

Remark 2.5. The condition that $J(u)$ is bounded from below is crucial to prove both Theorems 2.2 and 2.4. As in Remark 2.3, if we replace (H1) by (f3) and the condition that $F(u) \geq 0$ for $u \in \mathbb{R}^k$, then Theorem 2.4 is also true.

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