Research Article

Existence of Monotone Solutions of a Difference Equation

Taixiang Sun,¹ Hongjian Xi,^{1,2} and Weizhen Quan^{1,3}

¹ Department of Mathematics, College of Mathematics and Information Science, Guangxi University, Nanning, Guangxi 530004, China

² Department of Mathematics, Guangxi College of Finance and Economics, Nanning, Guangxi 530003, China

³ Department of Mathematics, Zhanjiang Normal College, Zhanjiang, Guangdong 524048, China

Correspondence should be addressed to Taixiang Sun, stx1963@163.com

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We consider the nonlinear difference equation $x_{n+1} = f(x_{n-k}, x_{n-k+1}, \dots, x_n)$, $n = 0, 1, \dots$, where $k \in \{1, 2, \dots\}$ and the initial values $x_{-k}, x_{-k+1}, \dots, x_0 \in (0, +\infty)$. We give sufficient conditions under which this equation has monotone positive solutions which converge to the equilibrium, extending and including in this way some results of the literature.

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1. Introduction

In this paper, we study the existence of monotone positive solutions which converge to the equilibrium of a nonlinear difference equation. Recently, there has been a lot of interest in studying such solutions and the existence of some specific solutions (see [1–20]). In [8], Karakostas and Stević studied the boundedness, global attractivity, and oscillatory and asymptotic periodicity of the nonnegative solutions of the difference equation

$$x_{n+1} = B + \frac{x_{n-k}}{a_0 x_n + \dots + a_{k-1} x_{n-k+1} + \gamma}, \quad n = 0, 1, \dots, \quad (\text{E1})$$

where $B \geq 0$, $\gamma > 0$, $k \in \{1, 2, \dots\}$, and $a_i \geq 0$ for every $i \in \{0, \dots, k-1\}$ with $\sum_{i=0}^{k-1} a_i > 0$ and the initial conditions $x_{-k}, \dots, x_0 \in (0, +\infty)$. They proposed the following open problem.

Open problem A. Let $\gamma = 1$, $B = 0$, and $k \geq 2$. Is there a positive solution $\{x_n\}$ of (E1) such that $x_n \rightarrow 0$ as $n \rightarrow \infty$?

In [5], Devault et al. studied the boundedness, global stability, and periodic character of positive solutions of the difference equation

$$x_{n+1} = p + \frac{x_{n-k}}{x_n}, \quad n = 0, 1, \dots, \quad (\text{E2})$$

where $k \in \{1, 2, \dots\}$, $p \in (0, +\infty)$, and the initial conditions $x_{-k}, \dots, x_0 \in (0, +\infty)$. They proposed the following Open problem B (which has been solved in [1, 18] by quite different methods).

Open problem B. Do there exist nonoscillatory solutions of (E2)?

Recently, Stević [12] studied the following difference equations:

$$x_{n+1} = p + \frac{x_{n-k}}{\alpha_0 x_n + \dots + \alpha_{k-1} x_{n-k+1}}, \quad n = 0, 1, \dots, \quad (\text{E3})$$

$$x_{n+1} = \frac{1 + x_{n-k}}{\alpha_0 x_n + \dots + \alpha_{k-1} x_{n-k+1}}, \quad n = 0, 1, \dots, \quad (\text{E4})$$

$$x_{n+1} = \frac{\alpha + x_{n-k}}{1 + \alpha_0 x_n + \dots + \alpha_{k-1} x_{n-k+1}}, \quad n = 0, 1, \dots, \quad (\text{E5})$$

where $p > -1$, $\alpha > 0$, $k \in \{1, 2, \dots\}$, and $\alpha_i \geq 0$ for every $i \in \{0, \dots, k-1\}$ with $\sum_{i=0}^{k-1} \alpha_i = 1$ and the initial conditions $x_{-k}, \dots, x_0 \in (0, +\infty)$. He proved that (E3), (E4), and (E5) have positive solutions which decrease to the equilibrium.

The main theorem in this paper is motivated by the above studies and [17]. In this paper, we consider the following nonlinear difference equation:

$$x_{n+1} = f(x_{n-k}, x_{n-k+1}, \dots, x_n), \quad n = 0, 1, \dots, \quad (1.1)$$

where $k \in \{1, 2, \dots\}$, the initial values $x_{-k}, x_{-k+1}, \dots, x_0 \in (0, +\infty)$, and $f \in C(E^{k+1}, E)$, where $C(E^{k+1}, E)$ denotes the set of all continuous maps from E^{k+1} to E and $E = (0, +\infty)$ or $E = [0, +\infty)$. Using arguments similar to ones developed in the proof of main theorem in [18], we prove that under appropriate conditions (see (C₁)–(C₅) below) this difference equation has monotone solutions converging to the equilibrium $\bar{x}$.

2. Main result

In this section, we assume that f satisfies the following conditions.

(C₁) $f \in C(E^{k+1}, E)$ and $f(z_0, z_1, \dots, z_k)$ is increasing in z_0 (i.e., $f(a, z_1, \dots, z_k) > f(b, z_1, \dots, z_k)$ if $a > b$), where $E = (0, +\infty)$ or $E = [0, +\infty)$ and $k \geq 1$ is an integer.

(C₂) Equation (1.1) has the unique nonnegative equilibrium, denoted by $\bar{x}$.

(C₃) $A = \{(z_0, z_1, \dots, z_k) : z_0 \geq z_1 \geq \dots \geq z_k \geq f(z_0, z_1, \dots, z_k) \geq \bar{x}\}$ is an unbounded connected (closed) set.

Now we formulate and prove the main result of this paper.

Theorem 2.1. Let f satisfy (C_1) – (C_3) . If there exists $g \in C([\bar{x}, +\infty)^{k+1}, [\bar{x}, +\infty))$ such that the following two conditions hold:

$$(C_4) \ A \subset B = \{(z_0, z_1, \dots, z_k) : g(z_0, z_1, \dots, z_k) \geq z_0 \geq z_1 \geq \dots \geq z_k \geq \bar{x}\},$$

$(C_5) \ z_k = f(g(z_0, z_1, \dots, z_k), z_0, z_1, \dots, z_{k-1})$ for any $(z_0, z_1, \dots, z_k) \in [\bar{x}, +\infty)^{k+1}$, then (1.1) has a monotone positive solution which converges to the equilibrium $\bar{x}$.

Proof. Define $F : A \rightarrow B$ by

$$F(z_0, z_1, \dots, z_k) = (u_0, u_1, \dots, u_k) \equiv (z_1, z_2, \dots, z_k, f(z_0, z_1, \dots, z_k)), \quad (2.1)$$

for all $(z_0, z_1, \dots, z_k) \in A$.

Claim 1. F is well defined.

Proof of Claim 1. From (2.1) and the definition of A , we have

$$\begin{aligned} u_i &= z_{i+1}, \quad \text{for } i \in \{0, 1, \dots, k-1\}, \\ u_k &= f(z_0, z_1, \dots, z_k) \geq \bar{x}. \end{aligned} \quad (2.2)$$

It follows from (2.2) and (C_5) that

$$f(z_0, u_0, \dots, u_{k-1}) = u_k = f(g(u_0, u_1, \dots, u_k), u_0, u_1, \dots, u_{k-1}), \quad (2.3)$$

which with (C_1) implies

$$g(u_0, u_1, \dots, u_k) = z_0 \geq u_0 \geq \dots \geq u_k \geq \bar{x}. \quad (2.4)$$

Thus, $(u_0, u_1, \dots, u_k) \in B$. Claim 1 is proved. $\square$

Claim 2. F is a bijection from A to B .

Proof of Claim 2. Let $z = (z_0, z_1, \dots, z_k)$ and $y = (y_0, y_1, \dots, y_k) \in A$ with $z \neq y$. If $z_i \neq y_i$ for some $i \in \{1, \dots, k\}$, then $F(y) \neq F(z)$. If $z_0 \neq y_0$ and $z_i = y_i$ for every $i \in \{1, \dots, k\}$, then from (C_1) we have

$$f(z_0, z_1, \dots, z_k) \neq f(y_0, z_1, \dots, z_k), \quad (2.5)$$

which also implies $F(y) \neq F(z)$.

On the other hand, for any $u = (u_0, u_1, \dots, u_k) \in B$, we have

$$g(u_0, u_1, \dots, u_k) \geq u_0 \geq u_1 \geq \dots \geq u_k \geq \bar{x}. \quad (2.6)$$

Choose

$$z = (z_0, z_1, \dots, z_k) \equiv (g(u_0, u_1, \dots, u_k), u_0, u_1, \dots, u_{k-1}). \quad (2.7)$$

It follows from (2.6), (2.7), and (C₅) that

$$z_k = u_{k-1} \geq u_k = f(g(u_0, u_1, \dots, u_k), u_0, u_1, \dots, u_{k-1}) = f(z_0, z_1, \dots, z_k) \geq \bar{x}, \quad (2.8)$$

which implies $z \in A$. From (2.1) and (C₅), we obtain

$$\begin{aligned} F(z) &= (z_1, \dots, z_k, f(z_0, z_1, \dots, z_k)) \\ &= (u_0, \dots, u_{k-1}, f(g(u_0, u_1, \dots, u_k), u_0, u_1, \dots, u_{k-1})) \\ &= (u_0, u_1, \dots, u_k) = u. \end{aligned} \quad (2.9)$$

Claim 2 is proved. $\square$

Furthermore, since $F^{-1}(u_0, u_1, \dots, u_k) = (g(u_0, u_1, \dots, u_k), u_0, u_1, \dots, u_{k-1})$ is continuous, F is a homeomorphism from A to B .

Since $A \subset B$ and F is a homeomorphism from A onto B , it follows that $F^{-1}(A) \subset F^{-1}(B) = A$. By induction, we have

$$\bar{x} = (\bar{x}, \bar{x}, \dots, \bar{x}) \in F^{-n}(A) \subset F^{-n+1}(A) \quad (2.10)$$

for every positive integer n . Because A is an unbounded connected closed set, we know that $F^{-n}(A)$ is an unbounded connected closed set for every positive integer n . Let

$$S = \bigcap_{i=0}^{\infty} F^{-i}(A). \quad (2.11)$$

Claim 3. S is an unbounded connected set.

Proof of Claim 3. Indeed, if S is a bounded connected closed set, then there exists $\beta > 0$ such that $S \subset B(\bar{x}, \beta) \equiv \{x \in E^{k+1} : d(\bar{x}, x) < \beta\}$. Since $F^{-n}(A)$ is an unbounded connected closed set for every positive integer n , it follows that $K_n = [\{x : d(\bar{x}, x) \leq 2\beta\} - B(\bar{x}, \beta)] \cap F^{-n}(A) \neq \emptyset$ and K_n is a bounded closed set. Let $x_n \in K_n$, then there exist the positive integers $n_1 < n_2 < \dots < n_k < \dots$ and a point $v \in \{x : d(\bar{x}, x) \leq 2\beta\} - B(\bar{x}, \beta)$ such that $\lim_{k \rightarrow \infty} x_{n_k} = v$. Notice that $v \notin S$. On the other hand, for every positive integer n , there exists N such that $x_{n_k} \in F^{-n}(A)$ if $n_k > N$, which implies $v \in F^{-n}(A)$. Thus $v \in S$, which is a contradiction. Claim 3 is proved. $\square$

Now suppose that $\{x_n\}_{n=-k}^{\infty}$ is a positive solution of (1.1) with $(x_{-k}, \dots, x_0) \in S - \bar{x}$; we can show that for all positive integer n ,

$$F^n(x_{-k}, \dots, x_0) = (x_{n-k}, x_{n-k+1}, \dots, x_n) \in A. \quad (2.12)$$

Thus, $\{x_n\}_{n=-k}^{\infty}$ is a monotone positive solution. Let

$$\lim_{n \rightarrow \infty} x_n = a, \quad (2.13)$$

then

$$a = f(a, a, \dots, a) \geq \bar{x}. \quad (2.14)$$

It follows from (C_2) that $a = \bar{x}$. Thus, $\{x_n\}_{n=-k}^\infty$ is a nontrivial monotone positive solution which converges to $\bar{x}$. Theorem 2.1 is proved. $\square$

Remark 2.2. From the proof of Theorem 2.1, we can conclude that (1.1) has infinitely many monotone positive solutions which converge to the equilibrium $\bar{x}$.

Remark 2.3. In [21], Stević gave another proof of Claim 3 of Theorem 2.1 for the case of equation $x_n = x_{n-k} / (1 + x_{n-1} + \dots + x_{n-(k-1)})$.

3. Example and some remarks

In this section, we will give an application of Theorem 2.1 and some remarks.

Example 3.1. Consider the equation

$$x_{n+1} = p + \frac{a + x_{n-k}}{b + \sum_{i=0}^{k-1} a_i x_{n-i}}, \quad n = 0, 1, \dots, \quad (3.1)$$

where $k \in \{1, 2, \dots\}$ and $a_i \geq 0$ for every $i \in \{0, \dots, k-1\}$ with $s = \sum_{i=0}^{k-1} a_i > 0$ and the initial conditions $x_{-k}, \dots, x_0 \in (0, +\infty)$. If $a, b, p \in [0, +\infty)$ satisfy one of the following conditions:

- (i) $a = 0$,
- (ii) $b/s \geq a > 0$,

then (3.1) has monotone positive solutions which converge to the unique nonnegative equilibrium.

Proof. Let $E = (0, +\infty)$ if $b = 0$ and let $E = [0, +\infty)$ if $b > 0$. Define $f \in C(E^{k+1}, E)$ by

$$f(z_0, z_1, \dots, z_k) = p + \frac{a + z_0}{b + \sum_{i=0}^{k-1} a_i z_{k-i}}, \quad (3.2)$$

for all $(z_0, z_1, \dots, z_k) \in E^{k+1}$. Then, (3.1) has the unique nonnegative equilibrium

$$\bar{x} = \frac{1 + ps - b + \sqrt{(1 + ps - b)^2 + 4s(pb + a)}}{2s} \geq p. \quad (3.3)$$

Let $A = \{(z_0, z_1, \dots, z_k) : z_0 \geq z_1 \geq \dots \geq z_k \geq f(z_0, z_1, \dots, z_k) \geq \bar{x}\}$ and define

$$g(z_0, z_1, \dots, z_k) = (z_k - p) \left(b + \sum_{i=0}^{k-1} a_i z_{k-i-1} \right) - a, \quad (3.4)$$

for all $(z_0, z_1, \dots, z_k) \in [\bar{x}, +\infty)^{k+1}$; then

$$g(z_0, z_1, \dots, z_k) \geq (\bar{x} - p)(b + s\bar{x}) - a = \frac{a + \bar{x}}{b + s\bar{x}}(b + s\bar{x}) - a = \bar{x}. \quad (3.5)$$

Thus $g \in C([\bar{x}, +\infty)^{k+1}, [\bar{x}, +\infty))$.

It is easy to check that the conditions (C_1) , (C_2) , and (C_5) hold. Now we show that A is an unbounded connected set.

It follows from conditions (i) and (ii) that $b \geq as$; then

$$\begin{aligned} f(x, x, \dots, x) &= p + \frac{a + x}{b + xs} = p + \frac{1}{s} - \frac{b - as}{s(b + sx)}, \\ F(x) = x - f(x, x, \dots, x) &= \frac{sx^2 - (ps - b + 1)x - pb - a}{b + sx} \end{aligned} \quad (3.6)$$

are increasing in x in $[\bar{x}, +\infty)$. Thus $c \geq c \geq \dots \geq c \geq f(c, c, \dots, c) \geq \bar{x}$ for any $c \geq \bar{x}$, which implies that $(c, \dots, c) \in A$ and A is unbounded.

Let $(z_0, z_1, \dots, z_k) \in A$ and $A_i = \{(z_0, \dots, z_0, tz_0 + (1 - t)z_i, z_{i+1}, \dots, z_k) : 0 \leq t \leq 1\}$ for $0 \leq i \leq k$; then A_i is a connected set. Since

$$\begin{aligned} z_0 &\geq z_1 \geq \dots \geq z_k \geq f(z_0, z_1, \dots, z_k) \geq \bar{x}, \\ f(x, x, \dots, x) &= p + \frac{a + x}{b + xs} \end{aligned} \quad (3.7)$$

are increasing in x , we know that

$$\begin{aligned} z_0 &\geq \dots \geq z_0 \geq tz_0 + (1 - t)z_i \geq z_{i+1} \geq \dots \geq z_k \geq f(z_0, z_1, \dots, z_k) \\ &\geq f(z_0, \dots, z_0, tz_0 + (1 - t)z_i, z_{i+1}, \dots, z_k) \geq f(z_0, \dots, z_0) \geq \bar{x}, \end{aligned} \quad (3.8)$$

from which it follows that $A_i \subset A$. Again since $(z_0, \dots, z_0, z_{i+1}, \dots, z_k) \in A_i \cap A_{i+1}$ for any $0 \leq i \leq k - 1$, $\cup_{i=0}^k A_i \cup \{(c, c, \dots, c) : c \geq \bar{x}\}$ is a connected subset of A and $(z_0, z_1, \dots, z_k) \in A_0$, which implies that A is an unbounded connected set. Thus, the condition (C_3) holds.

On the other hand, let $z = (z_0, z_1, \dots, z_k) \in A$, then

$$z_0 \geq z_1 \geq \dots \geq z_k \geq f(z_0, z_1, \dots, z_k) = p + \frac{a + z_0}{b + \sum_{i=0}^{k-1} a_i z_{k-i}} \geq \bar{x}. \quad (3.9)$$

It follows from (3.9) that

$$g(z_0, z_1, \dots, z_k) = (z_k - p) \left(b + \sum_{i=0}^{k-1} a_i z_{k-i-1} \right) - a \geq \frac{a + z_0}{b + \sum_{i=0}^{k-1} a_i z_{k-i}} \left(b + \sum_{i=0}^{k-1} a_i z_{k-i} \right) - a = z_0, \quad (3.10)$$

which implies $z \in B = \{(z_0, z_1, \dots, z_k) : g(z_0, z_1, \dots, z_k) \geq z_0 \geq z_1 \geq \dots \geq z_k \geq \bar{x}\}$. Thus, condition (C_4) holds.

By Theorem 2.1, we know that (3.1) has monotone positive solutions which converge to the unique nonnegative equilibrium $\bar{x} = [1 + ps - b + \sqrt{(1 + ps - b)^2 + 4s(pb + a)}] / 2s$. $\square$

Remark 3.2. (i) Let $a = b = 0$ and $a_0 = 1 > a_1 = \dots = a_{k-1} = 0$, then (3.1) reduces to (E2).

(ii) Take $a = p = 0$ and $b = 1$ in Example 3.1; then we have solved Open problem A.

Example 3.3. Consider the following equations:

$$x_{n+1} = f(x_{n-k}, x_{n-k+1}, \dots, x_n) = \frac{x_{n-k}^3}{x_n}, \quad n = 0, 1, \dots, \quad (3.11)$$

$$x_{n+1} = f(x_{n-k}, x_{n-k+1}, \dots, x_n) = \frac{x_n^2}{1 + \sum_{i=0}^{k-1} a_i x_{n-i}}, \quad n = 0, 1, \dots, \quad (3.12)$$

$$x_{n+1} = f(x_{n-k}, x_{n-k+1}, \dots, x_n) = \frac{x_{n-k}}{1/2 + \sum_{i=0}^{k-1} a_i x_{n-i}}, \quad n = 0, 1, \dots, \quad (3.13)$$

where $k \in \{1, 2, \dots\}$ and $a_i \geq 0$ for every $i \in \{0, \dots, k-1\}$ with $\sum_{i=0}^{k-1} a_i = 1$ and the initial conditions $x_{-k}, \dots, x_0 \in (0, +\infty)$. Then

(i) equation (3.11) satisfies conditions (C_1) and (C_2) , but $A = \{(z_0, z_1, \dots, z_k) : z_0 \geq z_1 \geq \dots \geq z_k \geq f(z_0, z_1, \dots, z_k) \geq \bar{x}\} = \{(\bar{x}, \dots, \bar{x})\}$ since $z_0 \geq z_1 \geq \dots \geq z_k \geq z_0^3/z_k \geq \bar{x} = 1$ implies $z_0 = z_1 = \dots = z_k = 1$; thus condition (C_3) does not hold;

(ii) using arguments similar to ones developed in the proof of Example 3.1, it is easy to check that conditions (C_2) and (C_3) hold, but $f(z_0, z_1, \dots, z_k)$ is decreasing in z_0 , which implies that condition (C_1) does not hold for (3.12);

(iii) equation (3.13) satisfies condition (C_1) and has two nonnegative equilibria: $\bar{x}_1 = 0$ and $\bar{x}_2 = 1/2$, which implies that condition (C_2) does not hold; using arguments similar to ones developed in the proof of Example 3.1, it is easy to check that $A = \{(z_0, z_1, \dots, z_k) : z_0 \geq z_1 \geq \dots \geq z_k \geq f(z_0, z_1, \dots, z_k) \geq \bar{x}_2\}$ is an unbounded connected set.

Remark 3.4. From Example 3.3, we see that all the conditions (C_1) , (C_2) , and (C_3) are necessary, in the sense that no pair of such conditions implies the remaining condition.

Remark 3.5. If $k = 0$ and the conditions (C_1) – (C_3) are satisfied, then automatically the difference equation $x_{n+1} = f(x_n)$ has monotone positive solutions converging to $\bar{x}$.

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