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Research Article

Permanence of a Discrete Periodic Volterra Model with Mutual Interference

Lijuan Chen[1] and Liujuan Chen[2]

[1] *College of Mathematics and Computer Science, Fuzhou University, Fuzhou, Fujian 350002, China*
[2] *Department of Mathematics and Physics, Fujian Institute of Education, Fuzhou, Fujian 350001, China*

Correspondence should be addressed to Lijuan Chen, chenlijuan@fzu.edu.cn

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This paper discusses a discrete periodic Volterra model with mutual interference and Holling II type functional response. Firstly, sufficient conditions are obtained for the permanence of the system. After that, we give an example to show the feasibility of our main results.

1. Introduction

In 1971, Hassell introduced the concept of mutual interference between the predators and preys. Hassell [1] established a Volterra model with mutual interference as follows:

$$\begin{aligned} \dot{x}(t) &= xg(x) - \varphi(x)y^m, \\ \dot{y}(t) &= y\big(-d + k\varphi(x)y^{m-1} - q(y)\big), \end{aligned} \tag{1.1}$$

where m denote mutual interference constant and $0 < m \leq 1$.

Motivated by the works of Hassell [1], Wang and Zhu [2] considered the following Volterra model with mutual interference and Holling II type functional response:

$$\begin{aligned} \dot{x}(t) &= x(t)\big(r_1(t) - b_1(t)x(t)\big) - \frac{c_1(t)x(t)}{k + x(t)}y^m(t), \\ \dot{y}(t) &= y(t)\big(-r_2(t) - b_2(t)y(t)\big) + \frac{c_2(t)x(t)}{k + x(t)}y^m(t). \end{aligned} \tag{1.2}$$

Sufficient conditions which guarantee the existence, uniqueness, and global attractivity of positive periodic solution are obtained by employing Mawhin's continuation theorem and constructing suitable Lyapunov function.

On the other hand, it has been found that the discrete time models governed by difference equations are more appropriate than the continuous ones when the populations have nonoverlapping generations. Discrete time models can also provide efficient computational models of continuous models for numerical simulations (see [3–15]). However, to the best of the author's knowledge, until today, there are still no scholars propose and study a discrete-time analogue of system (1.2). Therefore, the main purpose of this paper is to study the following discrete periodic Volterra model with mutual interference and Holling II type functional response:

$$\begin{aligned} x(n+1) &= x(n)\exp\left\{r_1(n) - b_1(n)x(n) - \frac{c_1(n)}{k+x(n)}y^m(n)\right\}, \\ y(n+1) &= y(n)\exp\left\{-r_2(n) - b_2(n)y(n) + \frac{c_2(n)x(n)}{k+x(n)}y^{m-1}(n)\right\}, \end{aligned} \tag{1.3}$$

where $x(n)$ is the density of prey species at nth generation and $y(n)$ is the density of predator species at nth generation. Also, $r_1(n)$, $b_1(n)$ denote the intrinsic growth rate and density-dependent coefficient of the prey, respectively, $r_2(n)$, $b_2(n)$ denote the death rate and density-dependent coefficient of the predator, respectively, $c_1(n)$ denote the capturing rate of the predator and $c_2(n)$ represent the transformation from preys to predators. Further, m is mutual interference constant and k is a positive constant. In this paper, we always assume that $\{r_i(n)\}$, $\{b_i(n)\}$, $\{c_i(n)\}$, $i = 1,2$, are positive T-periodic sequences and $0 < m < 1$. Here, for convenience, we denote $\overline{f} = (1/T)\sum_{n=0}^{T-1} f(n)$, $f^M = \sup_{n\in I_T}\{f(n)\}$, and $f^L = \inf_{n\in I_T}\{f(n)\}$, where $I_T = \{0,1,2,\ldots,T-1\}$.

This paper is organized as follows. In Section 2, we will introduce a definition and establish several useful lemmas. The permanence of system (1.3) is then studied in Section 3. In Section 4, we give an example to show the feasibility of our main results.

From the view point of biology, we only need to focus our discussion on the positive solution of system (1.3). So it is assumed that the initial conditions of (1.3) are of the form

$$x(0) > 0, \qquad y(0) > 0. \tag{1.4}$$

One can easily show that the solution of (1.3) with the initial condition (1.4) are defined and remain positive for all $n \in N$ where $N = \{0,1,2,\ldots\}$.

2. Preliminaries

In this section, we will introduce the definition of permanence and several useful lemmas.

Definition 2.1. System (1.3) is said to be permanent if there exist positive constants x^*, y^*, x_*, y_*, which are independent of the solution of the system, such that for any positive

solution $(x(n), y(n))$ of system (1.3) satisfies

$$\begin{aligned} x_* &\le \liminf_{n\to\infty} x(n) \le \limsup_{n\to\infty} x(n) \le x^*, \\ y_* &\le \liminf_{n\to\infty} y(n) \le \limsup_{n\to\infty} y(n) \le y^*. \end{aligned} \tag{2.1}$$

Lemma 2.2. *Assume that $x(n)$ satisfies*

$$x(n+1) \le x(n)\exp\{a(n) - b(n)x(n)\} \quad \forall n \ge n_0, \tag{2.2}$$

where $\{a(n)\}$ and $\{b(n)\}$ are positive sequences, $x(n_0) > 0$ and $n_0 \in N$. Then, one has

$$\limsup_{n\to\infty} x(n) \le B, \tag{2.3}$$

where $B = \exp(a^M - 1)/b^L$.

Lemma 2.3. *Assume that $x(n)$ satisfies*

$$x(n+1) \ge x(n)\exp\{a(n) - b(n)x(n)\} \quad \forall n \ge n_0, \tag{2.4}$$

where $\{a(n)\}$ and $\{b(n)\}$ are positive sequences, $x(n_0) > 0$ and $n_0 \in N$. Also, $\limsup_{n\to\infty} x(n) \le B$ and $b^M B/a^L > 1$. Then, one has

$$\liminf_{n\to\infty} x(n) \ge A, \tag{2.5}$$

where $A = (a^L/b^M)\exp(a^L - b^M B)$.

Proof. The proofs of Lemmas 2.2 and 2.3 are very similar to that of [8, Lemmas 1 and 2], respectively. So, we omit the detail here. □

The following Lemma 2.4 is Lemma 2.2 of Fan and Li [12].

Lemma 2.4. *The problem*

$$x(n+1) = x(n)\exp\{a(n) - b(n)x(n)\}, \tag{2.6}$$

with $x(0) = x_0 > 0$ has at least one periodic positive solution $x^*(n)$ if both $b : Z \to R^+$ and $a : Z \to R$ are T-periodic sequences with $\overline{a} > 0$. Moreover, if $b(n) = b$ is a constant and $a^M < 1$, then $bx(n) \le 1$ for n sufficiently large, where $x(n)$ is any solution of (2.6).

The following comparison theorem for the difference equation is Theorem 2.1 of L. Wang and M. Q. Wang [15, page 241].

Lemma 2.5. *Suppose that $f : Z^+ \times [0,+\infty)$ and $g : Z^+ \times [0,+\infty)$ with $f(n,x) \le g(n,x)$ ($f(n,x) \ge g(n,x)$) for $n \in Z^+$ and $x \in [0,+\infty)$. Assume that $g(n,x)$ is nondecreasing with respect to the argument x. If $x(n)$ and $u(n)$ are solutions of*

$$x(n+1) = f(n,x(n)), \qquad u(n+1) = g(n,u(n)), \tag{2.7}$$

respectively, and $x(0) \le u(0)$ ($x(0) \ge u(0)$), then

$$x(n) \le u(n), \quad (x(n) \ge u(n)) \tag{2.8}$$

for all $n \ge 0$.

3. Permanence

In this section, we establish a permanent result for system (1.3).

Proposition 3.1. *If $(H_1) : (1-m)r_2^M < 1$ holds, then for any positive solution $(x(n), y(n))$ of system (1.3), there exist positive constants x^* and y^*, which are independent of the solution of the system, such that*

$$\limsup_{n\to\infty} x(n) \le x^*, \qquad \limsup_{n\to\infty} y(n) \le y^*. \tag{3.1}$$

Proof. Let $(x(n), y(n))$ be any positive solution of system (1.3), from the first equation of (1.3), it follows that

$$x(n+1) \le x(n) \exp\{r_1(n) - b_1(n)x(n)\}. \tag{3.2}$$

By applying Lemma 2.2, we obtain

$$\limsup_{n\to\infty} x(n) \le x^*, \tag{3.3}$$

where

$$x^* = \frac{1}{b_1^L} \exp\left(r_1^M - 1\right). \tag{3.4}$$

Denote $P(n) = (1/y(n))^{1-m}$. Then, from the second equation of (1.3), it follows that

$$P(n+1) = P(n) \exp\left\{(1-m)r_2(n) + \frac{(1-m)b_2(n)}{\sqrt[1-m]{P(n)}} - \frac{(1-m)c_2(n)x(n)}{k+x(n)}P(n)\right\}, \tag{3.5}$$

which leads to

$$P(n+1) \ge P(n) \exp\left\{(1-m)r_2(n) - (1-m)c_2^M P(n)\right\}. \tag{3.6}$$

Consider the following auxiliary equation:

$$Z(n+1) = Z(n)\exp\left\{(1-m)r_2(n) - (1-m)c_2^M Z(n)\right\}. \tag{3.7}$$

By Lemma 2.4, (3.7) has at least one positive T-periodic solution and we denote one of them as $Z^*(n)$. Now (H_1) and Lemma 2.4 imply $(1-m)c_2^M Z(n) \le 1$ for n sufficiently large, where $Z(n)$ is any solution of (3.7). Consider the following function:

$$g(n,Z) = Z\exp\left\{(1-m)r_2(n) - (1-m)c_2^M Z\right\}. \tag{3.8}$$

It is not difficult to see that $g(n,Z)$ is nondecreasing with respect to the argument Z. Then, applying Lemma 2.5 to (3.6) and (3.7), we easily obtain that $P(n) \ge Z^*(n)$. So $\liminf_{n\to\infty} P(n) \ge (Z^*(n))^L$, which together with that transformation $P(n) = (1/y(n))^{1-m}$, produces

$$\limsup_{n\to\infty} y(n) \le \frac{1}{\sqrt[1-m]{(Z^*(n))^L}} \triangleq y^*. \tag{3.9}$$

Thus, we complete the proof of Proposition 3.1. □

Proposition 3.2. *Assume that*

$$(H_2): \left(r_1(n) - \frac{c_1(n)}{k}(y^*)^m\right)^L > 0 \tag{3.10}$$

holds, then for any positive solution $(x(n), y(n))$ *of system* (1.3)*, there exist positive constants* x_* *and* y_**, which are independent of the solution of the system, such that*

$$\liminf_{n\to\infty} x(n) \ge x_*, \qquad \liminf_{n\to\infty} y(n) \ge y_*, \tag{3.11}$$

where y^* *can be seen in Proposition 3.1.*

Proof. Let $(x(n), y(n))$ be any positive solution of system (1.3). From (H_2), there exists a small enough positive constant ε such that

$$\left(r_1(n) - \frac{c_1(n)}{k}(y^*+\varepsilon)^m\right)^L > 0. \tag{3.12}$$

Also, according to Proposition 3.1, for above ε, there exists $N_1 > 0$ such that for $n \ge N_1$,

$$y(n) \le y^* + \varepsilon. \tag{3.13}$$

Then, from the first equation of (1.3), for $n \geq N_1$, we have

$$x(n+1) \geq x(n) \exp\left\{r_1(n) - \frac{c_1(n)}{k}\left(y^* + \varepsilon\right)^m - b_1(n)x(n)\right\}. \tag{3.14}$$

Let $a_1(n,\varepsilon) = r_1(n) - (c_1(n)/k)(y^* + \varepsilon)^m$, so the above inequality follows that

$$x(n+1) \geq x(n) \exp\left\{a_1(n,\varepsilon) - b_1(n)x(n)\right\}. \tag{3.15}$$

Because $(a_1(n,\varepsilon))^L < r_1^L$ and $b_1^L < b_1^M$, we have

$$\frac{b_1^M}{\left(a_1(n,\varepsilon)\right)^L} x^* > \frac{b_1^M}{r_1^L} \frac{\exp\left(r_1^M - 1\right)}{b_1^L} > 1. \tag{3.16}$$

Here, we use the fact $\exp(r_1^M - 1) > r_1^M$. From (3.12) and (3.15), by Lemma 2.3, we have

$$\liminf_{n\to\infty} x(n) \geq \frac{\left(a_1(n,\varepsilon)\right)^L}{b_1^M} \exp\left\{\left(a_1(n,\varepsilon)\right)^L - b_1^M x^*\right\}. \tag{3.17}$$

Setting $\varepsilon \to 0$ in the above inequality leads to

$$\liminf_{n\to\infty} x(n) \geq \frac{a_1^L}{b_1^M} \exp\left\{a_1^L - b_1^M x^*\right\} \triangleq x_*, \tag{3.18}$$

where

$$a_1(n) = r_1(n) - \frac{c_1(n)}{k}\left(y^*\right)^m. \tag{3.19}$$

For above ε, there exists $N_2 > N_1$ such that for $n \geq N_2$, $x(n) \geq x_* - \varepsilon$. So from (3.5), we obtain that

$$P(n+1) \leq P(n) \exp\left\{(1-m)\left(r_2(n) + b_2(n)\left(y^* + \varepsilon\right)\right) - \frac{(1-m)c_2(n)\left(x_* - \varepsilon\right)}{k + x_* - \varepsilon} P(n)\right\}. \tag{3.20}$$

Consider the following auxiliary equation:

$$W(n+1) = W(n) \exp\left\{(1-m)\left(r_2(n) + b_2(n)\left(y^* + \varepsilon\right)\right) - \frac{(1-m)c_2(n)\left(x_* - \varepsilon\right)}{k + x_* - \varepsilon} W(n)\right\}. \tag{3.21}$$

By Lemma 2.4, (3.21) has at least one positive T-periodic solution and we denote one of them as $W^*(n)$.

Let

$$R(n) = \ln(P(n)), \qquad Y(n) = \ln\left(W^*(n)\right). \tag{3.22}$$

Then,

$$\begin{aligned} R(n+1)-R(n) &\le (1-m)\left(r_2(n) + b_2(n)\left(y^*+\varepsilon\right)\right) - \frac{(1-m)c_2(n)\left(x_*-\varepsilon\right)}{k+x_*-\varepsilon}\exp\{R(n)\}, \\ Y(n+1)-Y(n) &= (1-m)\left(r_2(n) + b_2(n)\left(y^*+\varepsilon\right)\right) - \frac{(1-m)c_2(n)\left(x_*-\varepsilon\right)}{k+x_*-\varepsilon}\exp\{Y(n)\}. \end{aligned} \tag{3.23}$$

Set

$$U(n) = R(n) - Y(n). \tag{3.24}$$

Then,

$$U(n+1) - U(n) \le -\frac{(1-m)c_2(n)\left(x_*-\varepsilon\right)}{k+x_*-\varepsilon}\exp\{Y(n)\}[\exp\{U(n)\} - 1]. \tag{3.25}$$

In the following we distinguish three cases.

Case 1. $\{U(n)\}$ is eventually positive. Then, from (3.25), we see that $U(n+1) < U(n)$ for any sufficiently large n. Hence, $\lim_{n\to\infty} U(n) = 0$, which implies that

$$\limsup_{n\to\infty} P(n) \le \left(W^*(n)\right)^M. \tag{3.26}$$

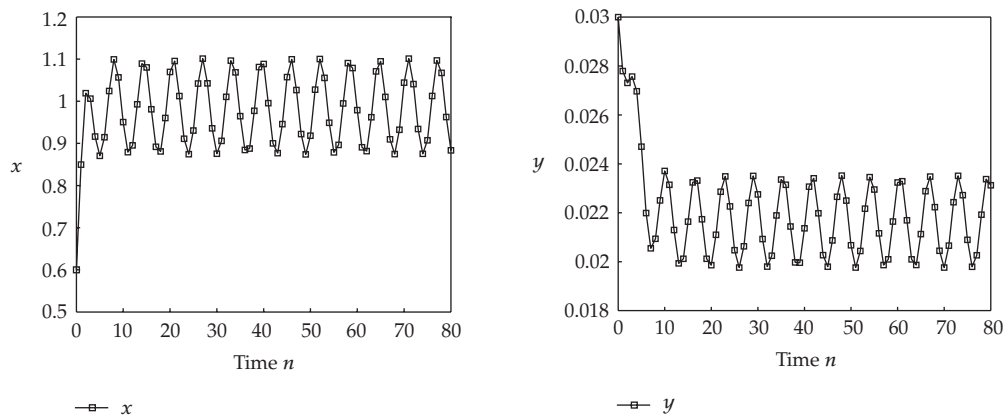
Case 2. $\{U(n)\}$ is eventually negative. Then, from (3.24), we can also obtain (3.26).

Case 3. $\{U(n)\}$ oscillates about zero. In this case, we let $\{U(n_{st})\}$ $(s,t \in N)$ be the positive semicycle of $\{U(n)\}$, where $U(n_{s1})$ denotes the first element of the sth positive semicycle of $\{U(n)\}$. From (3.25), we know that $U(n+1) < U(n)$ if $U(n) > 0$. Hence, $\limsup_{n\to\infty} U(n) = \limsup_{s\to\infty} U(n_{s1})$. From (3.25), and $U(n_{s1}-1) < 0$, we can obtain

$$\begin{aligned} U(n_{s1}) &\le \frac{(1-m)c_2(n_{s1})\left(x_*-\varepsilon\right)}{k+x_*-\varepsilon}\exp\left\{Y(n_{s1})\right\}\left[1-\exp\left\{U(n_{s1})\right\}\right], \\ &\le \frac{(1-m)c_2^M\left(x_*-\varepsilon\right)}{k+x_*-\varepsilon}\left(W^*(n)\right)^M. \end{aligned} \tag{3.27}$$

From (3.22) and (3.24), we easily obtain

$$\limsup_{n\to\infty} P(n) \le \left(W^*(n)\right)^M \exp\left\{\frac{(1-m)c_2^M\left(x_*-\varepsilon\right)}{k+x_*-\varepsilon}\left(W^*(n)\right)^M\right\}. \tag{3.28}$$



(a) x (b) y

Figure 1: Dynamics behavior of system (1.3) with initial condition $(x(0), y(0)) = (0.6, 0.03)$.

Setting $\varepsilon \to 0$ in the above inequality leads to

$$\limsup_{n\to\infty} P(n) \le \left(W^*(n)\right)^M \exp\left\{\frac{(1-m)c_2^M x_*}{k+x_*}\left(W^*(n)\right)^M\right\} \triangleq P^*, \tag{3.29}$$

which together with that transformation $P(n) = (1/y(n))^{1-m}$, we have

$$\liminf_{n\to\infty} y(n) \ge \frac{1}{\sqrt[1-m]{P^*}} \triangleq y_*. \tag{3.30}$$

Thus, we complete the proof of Proposition 3.2. □

Theorem 3.3. *Assume that* (H_1) *and* (H_2) *hold, then system* (1.3) *is permanent.*

It should be noticed that, from the proofs of Propositions 3.1 and 3.2, one knows that under the conditions of Theorem 3.3, the set $\Omega = \{(x, y) \mid x_* \le x \le x^*, y_* \le y \le y^*\}$ is an invariant set of system (1.3).

4. Example

In this section, we give an example to show the feasibility of our main result.

Example 4.1. Consider the following system

$$\begin{aligned} x(n+1) &= x(n)\exp\left\{0.7 + 0.1\sin(n) - 0.7x(n) - \frac{0.4y^{0.6}(n)}{2.4+x(n)}\right\}, \\ y(n+1) &= y(n)\exp\left\{-(0.8+0.1\cos(n)) - (1.1+0.1\sin(n))x(n) - \frac{0.6y^{-0.4}(n)}{2.4+x(n)}\right\}, \end{aligned} \tag{4.1}$$

where $r_1(n) = 0.7 + 0.1\sin(n)$, $b_1(n) = 0.7$, $c_1(n) = 0.4$, $r_2(n) = 0.8 + 0.1\cos(n)$, $b_2(n) = 1.1 + 0.1\sin(n)$, $c_2(n) = 0.6$, $m = 0.6$, $k = 2.4$.

By simple computation, we have $y^* \approx 1.1405$. Thus, one could easily see that

$$\left(r_1(n) - \frac{c_1(n)}{k}(y^*)^m\right)^L \approx 0.4197 > 0, \qquad (1-m)(r_2(n))^M = 0.36 < 1. \tag{4.2}$$

Clearly, conditions (H_1) and (H_2) are satisfied, then system (1.3) is permanent.

Figure 1 shows the dynamics behavior of system (1.3).

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