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Research Article

Spectrum and Generation of Solutions of the Toda Lattice

D. Barrios Rolanía[1] and J. R. Gascón Márquez[2]

[1] *Facultad de Informática, Universidad Politécnica de Madrid, 28660 Madrid, Spain*
[2] *Área de Matemática, Universidad Nacional Abierta, 1010 Caracas, Venezuela*

Correspondence should be addressed to D. Barrios Rolanía, dbarrios@fi.upm.es

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Sufficient conditions for constructing a set of solutions of the Toda lattice are analyzed. First, under certain conditions the invariance of the spectrum of $J(t)$ is established in the complex case. Second, given the tri-diagonal matrix $J(t)$ defining a Toda lattice solution, the dynamic behavior of zeros of polynomials associated to $J(t)$ is analyzed. Finally, it is shown by means of an example how to apply our results to generate complex solutions of the Toda lattice starting with a given solution.

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1. Introduction

We consider the Toda lattice given by the following equations:

$$\begin{aligned} \dot{\alpha}_n(t) &= \lambda_{n+1}^2(t) - \lambda_n^2(t), \\ \dot{\lambda}_{n+1}(t) &= \frac{1}{2}\lambda_{n+1}(t)(\alpha_{n+1}(t) - \alpha_n(t)), \end{aligned} \qquad n \in \mathbb{N}\ (\lambda_1 \equiv 0), \tag{1.1}$$

where $\lambda_n(t)$, $\alpha_n(t)$ are complex and differentiable functions of one real variable, $\dot{\alpha}_n(t)$, $\dot{\lambda}_n(t)$ denote its derivatives, and we assume $\lambda_n(t) \neq 0$ for each $t \in \mathbb{R}$ and $n \geq 2$. It is well known (see [1, page 705]) that (1.1) can be expressed in the Lax pair form as

$$\dot{J}(t) = [J(t), K(t)], \tag{1.2}$$

where $[A, B] = AB - BA$ is the commutator of the operators A and B, and $J(t)$, $K(t)$ are the operators for which matrix representation is given, respectively, by

$$J(t) = \begin{pmatrix} \alpha_1(t) & \lambda_2(t) & & \\ \lambda_2(t) & \alpha_2(t) & \lambda_3(t) & \\ & \lambda_3(t) & \alpha_3(t) & \ddots \\ & & \ddots & \ddots \end{pmatrix}, \qquad K(t) = \frac{1}{2}\begin{pmatrix} 0 & -\lambda_2(t) & & \\ \lambda_2(t) & 0 & -\lambda_3(t) & \\ & \lambda_3(t) & 0 & \ddots \\ & & \ddots & \ddots \end{pmatrix} \tag{1.3}$$

with respect to the canonical basis $\{e_i\}, i \geq 0$. (In what follows, we identify an operator and its matrix representation respect to the canonical basis.)

In the particular case that $\lambda_n(t)$, and $\alpha_n(t)$ are real functions, under certain conditions $J(t)$ is a self-adjoint operator. This property has several consequences in the study of system (1.1). For instance, in this situation the unitary equivalence between operators $J(t), t \in \mathbb{R}$, was established in [2]. In that paper, the existence of unitary operators $U(t)$ such that

$$J(t) = U(t)^{-1} J(0) U(t) \tag{1.4}$$

for each $t \in \mathbb{R}$ was proved. As it is well known that under these conditions the spectrum $\sigma(J(t))$ of each operator $J(t)$ verifies

$$\sigma(J(t)) = \sigma(J(0)), \quad t \in \mathbb{R}. \tag{1.5}$$

In other words, the spectrum does not depend on $t \in \mathbb{R}$. This fact permits to use the self-adjoint operator theory to analyze the integrability of system (1.1) (see [3, 4]). These tools can be used, also, in more general systems (see [5–7]). Then, due to some properties of the real Toda lattice (see, e.g., [1]), the associated Cauchy problem can be solved, recovering the solution $J(t)$ from the initial values defined by $J(0)$.

If $\lambda_n(t)$ and $\alpha_n(t)$ are complex functions, then the operator $J(t)$ given in (1.3) is not any longer a self-adjoint operator. Therefore, some of the assumptions of [2] are not verified. To the best of our knowledge there is no proof of the invariance of the spectrum of $J(t)$ in the complex case, so we would like to establish that result in the more general possible situation, that is, when $J(t)$ is not necessarily a bounded operator. However, we think that some advance, in this sense, is a relevant contribution in the study of solutions of the Toda lattice. This is related with our first result.

Theorem 1.1. *Let $\{\alpha_n(t), \lambda_{n+1}(t)\}, n \in \mathbb{N}$ be a solution of* (1.1) *such that the sequence $\{\lambda_{n+1}(t)\}, n \in \mathbb{N}$, is bounded for each $t \in \mathbb{R}$. Then one has*

$$\sigma(J(t)) = \sigma(J(t_0)), \quad \forall t,\ t_0 \in \mathbb{R}; \tag{1.6}$$

that is, the spectrum $\sigma(J(t))$ of $J(t)$ is invariant on $t \in \mathbb{R}$.

System (1.1) is a particular case of the generalized Toda lattice of order p;

$$\begin{aligned} \dot{J}_{nn}(t) &= J_{n,n+1}(t)J^p_{n,n+1}(t) - J_{n-1,n}(t)J^p_{n-1,n}(t), \\ \dot{J}_{n,n+1}(t) &= \frac{1}{2}J_{n,n+1}(t)\left[J^p_{n+1,n+1}(t) - J^p_{n,n}(t)\right], \end{aligned} \qquad n = 0, 1, \ldots, \tag{1.7}$$

where we denote by $J_{ij}(t)$ (resp., $J^p_{ij}(t)$) the entry of $J(t)$ (resp., $J^p(t)$) corresponding to the row i and the column j (see [8, 9]). The sequence $\{P_n(t,z)\}$ of polynomials defined by the three-term recurrence relation

$$\begin{gathered} P_{n+1}(t,z) = (z - \alpha_{n+1}(t))P_n(t,z) - \lambda^2_{n+1}(t)P_{n-1}(t,z), \quad n \geq 0 \\ P_{-1}(t,z) \equiv 0, \qquad P_0(t,z) \equiv 1 \end{gathered} \tag{1.8}$$

is an important tool in the study of complex solutions of (1.7) (see [10, 11]). From $\{P_n(t,z)\}$ we can define the sequence $\{\widehat{p}_n(t,z)\}$ by

$$\widehat{p}_n(t,z) = \frac{P_n(t,z)}{\lambda_2(t)\cdots\lambda_{n+1}(t)}, \quad n \in \mathbb{N}. \tag{1.9}$$

(Obviously, the zeros of $\widehat{p}_n(t,z)$ and $P_n(t,z)$ are the same.) Beside some other results, the bases of a method for obtaining new solutions of (1.7) from a given solution were established in [11]. In that paper, the location of zeros of the sequence $\{\widehat{p}_n(t,z)\}$ plays an important role, and the relevance of finding a point $C \in \mathbb{C}$ which is not a root of any polynomial $\widehat{p}_n(t,z)$, $n \in \mathbb{N}$, $t \in \mathbb{R}$, was showed. Hence, our interest is in knowing the dynamic behaviour of $\widehat{p}_n(t,z)$ and, also, some bound for its zeros.

Denote by $J_n(t)$ the finite-dimensional matrix of order n defined by the first n rows and columns of $J(t)$ (see [12]). From (1.8), it can be easily established that

$$P_n(t,z) = \det(zI_n - J_n(t)) \tag{1.10}$$

(see, i.e., [13]). Thus, for any $t \in \mathbb{R}$ and $n \in \mathbb{N}$, the set of zeros of $P_n(t,z)$ coincides with the spectrum $\sigma(J_n(t))$ of $J_n(t)$. When $J(t)$ is a self-adjoint operator, then the spectrum $\sigma(J_n(t))$ of each main section $J_n(t)$ is contained in $\sigma(J(t))$. So, for this kind of operators, using (1.5) and the relationship between $\sigma(J_n(t))$ and $\sigma(J(t))$ it is possible to deduce some bound for the set of zeros of $P_n(t,z)$ in terms of $\sigma(J(0))$.

If $J(t)$ is not a self-adjoint, to get some knowledge about the behaviour of solutions of (1.7) as well as (1.1) is very difficult. This is due to the lack of a general result about the relationship between $\sigma(J_n(t))$ and $\sigma(J(t))$ (see [11, 14]). For general banded matrices, the relation between the spectrum $\sigma(A)$ of a band infinite matrix A and the spectrum $\sigma(A_n)$ of its main sections was analyzed, under certain conditions, in [15]. More precisely, the representation $A = \Re A + i\Im A$ was used, assuming $\Re A$ self-adjoint and $\Im A$ bounded. In our case, if we suppose that $J(t)$ verifies this restriction, then we have

$$J(t) = \Re J(t) + i\Im J(t), \quad t \in \mathbb{R}, \tag{1.11}$$

where $\Re J(t)$ is a self-adjoint operator and $\Im J(t)$ is bounded. For $C \in \mathbb{R}$ verifying

$$d(C, \sigma(\Re J_n(t))) > \|\Im J_n(t)\|, \tag{1.12}$$

from [15, Lemmas 1, 2] we know $P_n(t, C) \neq 0$ or, what is the same, $C \notin \sigma(J_n(t))$. Moreover, taking into account that $\|\Im J(t)\| \geq \|\Im J_n(t)\|$, from these results we can deduce $P_n(t, C) \neq 0$ for any $n \in \mathbb{N}$ when $C \in \mathbb{C}$ is such that $d(C, \sigma(\Re J(t))) > \|\Im J(t)\|$. In this way, the zeros of each sequence of polynomials $\{P_n(t, z)\}$, $n \in \mathbb{N}$, are located in the neighborhood of $\sigma(\Re J(t))$ given by

$$\{z : d(z, \sigma(\Re J(t))) \leq \|\Im J_n(t)\|\}. \tag{1.13}$$

Besides the above comments on Theorem 1.1 importance, this fact justifies our interest in obtaining relationship between $\sigma(J(t))$ for different values of $t \in \mathbb{R}$, because bounding the zeros in a certain region of the complex plane permits to work with the method given in [11] in the complement of the region zeros free.

On the other hand, in conditions under which there are not any information about the dynamic behaviour of the spectrum, our following result gives complementary information about the knowledge and the dynamic behavior of zeros of $P_n(t, z)$.

Theorem 1.2. *Let $z_{n1}(t), z_{n2}(t), \ldots, z_{nn}(t)$ be the roots of $P_n(t, z)$, nonnecessary distinct. Then one has*

$$\dot{z}_{nk}(t) = \frac{\left(\widehat{p}_{n-1}(t, z_{nk}(t))\right)^2}{\sum_{j=0}^{n-1} \left(\widehat{p}_j(t, z_{nk}(t))\right)^2}, \tag{1.14}$$

understanding that $\dot{z}_{nk}(t) = \infty$ when the multiplicity of $z_{nk}(t)$ as a zero of $P_n(t, z)$ would be $m(z_{nk}(t)) > 1$.

We stress that, in Theorem 1.2, we do not need additional conditions about the operator $J(t)$. Theorems 1.1 and 1.2 are complementary results, in the sense that both can be used for determining conditions to obtain some new solutions of (1.1) and (1.7).

Section 2 is devoted to prove Theorems 1.1 and 1.2. After the existence of $C \in \mathbb{C}$ such that $P_n(t, C) \neq 0$ for any $n \in \mathbb{N}$, $t \in \mathbb{R}$, can be guaranteed, we will show, in Section 3, how to construct a new solution of (1.1) from a given solution.

2. Invariance of Spectrum versus Variation of Zeros of Polynomials

2.1. Proof of Theorem 1.1

We define the antilinear operator $\mathcal{C}$ such that $\mathcal{C}e_i = e_i$ for each vector $e_i, i = 0, 1, \ldots,$ in the canonical base. Thus, for any $x \in \ell^2$ we have

$$x = \sum_{i \geq 0} x_i e_i, \qquad \mathcal{C}x = \sum_{i \geq 0} \overline{x_i} e_i. \tag{2.1}$$

In [16], antilinear operators were introduced in order to study symmetric complex operators. In our case, we have the following auxiliaryy result for $\mathcal{C}$, which justifies the definition of transpose operator (see [16, page 2]). We recall that we identify an operator with its matrix representation.

Lemma 2.1. *(a) Let A be a linear operator and let A^* be the adjoint operator of A. Then, the matrix representation of $\mathcal{C}A^*\mathcal{C}$ is A^T, that is, $A^T = \mathcal{C}A^*\mathcal{C}$.*

(b) $J(t)$ is a symmetric complex operator, that is, $J(t) = \mathcal{C}J(t)^*\mathcal{C}$ for each $t \in \mathbb{R}$.

(c) $K(t)$ is an antisymmetric operator, that is, $K(t) = -\mathcal{C}K(t)^*\mathcal{C}$ for each $t \in \mathbb{R}$.

Proof. Given a linear operator B, it is obvious that $\mathcal{C}B\mathcal{C}$ is also a linear operator. So, it is sufficient to prove the enunciated equalities for each basic vector e_i. In (a), for $A = (a_{ks})_{k,s=0}^{\infty}$, the column i of A^* is given by

$$A^* e_i = \sum_{k\geq 0} \overline{a_{ik}} e_k, \tag{2.2}$$

and therefore,

$$\mathcal{C}A^*\mathcal{C}e_i = \sum_{k\geq 0} a_{ik} e_k \tag{2.3}$$

is the i column of the transpose matrix A^T. For proving (b) and (c), it is sufficient to take in account the following expressions,

$$\begin{aligned} J(t)e_i &= \lambda_{i+1}(t)e_{i-1} + \alpha_{i+1}(t)e_i + \lambda_{i+2}(t)e_{i+1}, \\ 2K(t)e_i &= -\lambda_{i+1}(t)e_{i-1} + \lambda_{i+2}(t)e_{i+1}, \end{aligned} \qquad i = 0, 1, \ldots, \tag{2.4}$$

where we understand $e_{-1} = 0$. In other words, (b) and (c) can be obtained directly as a consequence of the structure of the matrices $J(t)$ and $K(t)$. □

Now, we consider the following matrix initial value problem:

$$\begin{aligned} \dot{Q}(t) &= Q(t)K(t) \\ Q(0) &= I. \end{aligned} \tag{2.5}$$

Under the restrictions of Theorem 1.1, the operator $K(t)$ given by (1.3) is bounded. Hence, we assume that $K(t)$ is a bounded operator in the rest of the section. Moreover, assuming continuous solutions for the Toda lattice, the operator $K(t)$ is a continuous function on $t \in \mathbb{R}$. It is known that we can consider different kinds of continuity for a operator-value function $t \mapsto A(t)$. In our case, the function $t \mapsto K(t)$ of a real variable is continuous in norm (see [17, page 152]). Therefore, the existence of a solution $Q(t)$ of (2.5) can be guaranteed (see [18, page 123]).

We have the following auxiliary result.

Lemma 2.2. *Let $Q(t)$ be a solution of* (2.5). *Then*

$$Q(t)Q(t)^T = Q(t)^T Q(t) = I; \tag{2.6}$$

that is, $Q(t)$ is an invertible matrix and $Q(t)^{-1} = Q(t)^T$

Proof. Transposing the equations (2.5), since Lemma 2.1 we arrive to

$$\begin{gathered} \dot{Q}(t)^T = -K(t)Q(t)^T \\ Q(0)^T = I. \end{gathered} \tag{2.7}$$

In other words, $Q(t)^T$ is a solution of differential equation $R(t) = -K(t)R(t)$, verifying the same initial condition given by (2.5). To see this one has the following.

(1) First of all, we show $Q(t)Q(t)^T = I$. Using (2.5) and (2.7) we obtain

$$\frac{d}{dt}\left(Q(t)Q(t)^T\right) = \dot{Q}(t)Q(t)^T + Q(t)\dot{Q}(t)^T = 0, \tag{2.8}$$

then $Q(t)Q(t)^T$ is independent on $t \in \mathbb{R}$. From this fact and $Q(0) = I$, we deduce $Q(0)Q(0)^T = I$ and the first part of (2.6) is proved.

(2) Following [18, pages 123-124], we can write

$$Q^T(t) = I - \int_0^t K(\tau)d\tau + \sum_{n\geq 2}(-1)^n \int_0^t\int_0^{\tau_1}\cdots\int_0^{\tau_{n-1}} K(\tau_1)K(\tau_2)\cdots K(\tau_n)d\tau_n\cdots d\tau_2 d\tau_1. \tag{2.9}$$

Due to the continuity in norm of $K(t)$, the series given in the right-hand side of (2.9) converges in norm. Even more,

$$\left\|Q^T(t)\right\| \leq e^{t\max_{[0,t]}\|K(\tau)\|}. \tag{2.10}$$

In a similar way, the series given in the right-hand side of

$$R(t) = I + \int_0^t K(\tau)d\tau + \sum_{n\geq 2}(-1)^n \int_t^0\int_t^{\tau_1}\cdots\int_t^{\tau_{n-1}} K(\tau_1)K(\tau_2)\cdots K(\tau_n)d\tau_n\cdots d\tau_2 d\tau_1 \tag{2.11}$$

converges in norm. Then, the above series defines the bounded operator $R(t)$ for each $t \in \mathbb{R}$. From straightforward computations, we obtain

$$Q^T(t)R(t) = R(t)Q^T(t) = I, \quad t \in \mathbb{R}. \tag{2.12}$$

Then, from $Q(t)Q^T(t) = I$ we get that $R(t) = Q(t)$ and, finally, $Q^T(t)Q(t) = I$. □

Now, we will finish the proof of Theorem 1.1. For this purpose, take the solution $Q(t)$ of (2.5). Using (1.2), (2.5), and (2.7), we immediately arrive to

$$\frac{d}{dt}\left(Q(t)J(t)Q^T(t)\right) = \dot{Q}(t)J(t)Q(t)^T + Q(t)\dot{J}(t)Q(t)^T + Q(t)J(t)\dot{Q}(t)^T = 0. \tag{2.13}$$

Then, taking into account the initial condition in (2.5) and (2.7),

$$Q(t)J(t)Q(t)^T = Q(0)J(0)Q(0)^T = J(0). \tag{2.14}$$

From this and Lemma 2.2,

$$J(t) = Q(t)^T J(0)Q(t). \tag{2.15}$$

Therefore, $J(0)$ and $J(t)$ are equivalent operators, and we have, as a consequence,

$$\sigma(J(t)) = \sigma(J(0)) \tag{2.16}$$

for each $t \in \mathbb{R}$. So, $\sigma(J(t))$ is independent on $t \in \mathbb{R}$, as we wanted to prove.

2.2. Proof of Theorem 1.2

Taking $p = 1$ in (2.6) of [11, Theorem 2], we obtain

$$\dot{P}_n(t,z) = -\lambda_{n+1}^2(t)P_{n-1}(t,z) \tag{2.17}$$

for each $n \in \mathbb{N}$ and all $z \in \mathbb{C}$. Then, writing

$$P_n(t,z) = \prod_{i=1}^{n}(z - z_{ni}(t)) \tag{2.18}$$

and taking derivatives with respect to t, we have

$$\dot{P}_n(t,z) = -\sum_{i=1}^{n}\dot{z}_{ni}(t)\prod_{j\neq i}\left(z - z_{nj}(t)\right). \tag{2.19}$$

With the notation established in Section 1, for each fixed zero $z = z_{nk}(t)$ of $P_n(t,z)$ the right-hand side of (2.17) is not zero. As a matter of fact, we have $\lambda_{n+1}(t) \neq 0$, and, if we suppose $P_n(t, z_{nk}(t)) = P_{n-1}(t, z_{nk}(t)) = 0$, then using the recurrence relation (1.8) we will arrive to $P_{n-2}(t, z_{nk}(t)) = 0$ and, iterating, to $P_0(t, z_{nk}(t)) = 0$, which is not possible being $P_0 \equiv 1$.

Comparing (2.17) and (2.19) for $z = z_{nk}(t)$, we see

$$\sum_{i=1}^{n}\dot{z}_{ni}(t)\prod_{j\neq i}\left(z_{nk}(t) - z_{nj}(t)\right) = \lambda_{n+1}^2(t)P_{n-1}(t, z_{nk}(t)), \quad k = 1,\ldots,n. \tag{2.20}$$

Moreover, $\prod_{j\neq i}(z_{nk}(t)-z_{nj}(t))=0$ when $i\neq k$. Therefore, from (2.20) we have

$$\dot{z}_{nk}(t)\prod_{j\neq k}\left(z_{nk}(t)-z_{nj}(t)\right)=\lambda_{n+1}^{2}(t)P_{n-1}(t,z_{nk}(t)),\quad k=1,\ldots,n, \tag{2.21}$$

and, consequently,

$$\dot{z}_{nk}(t)\prod_{j\neq k}\left(z_{nk}(t)-z_{nj}(t)\right)\neq 0,\quad k=1,\ldots,n. \tag{2.22}$$

We will take in consideration the-two possible cases following.

(i) If the multiplicity of $z_{nk}(t)$ as a zero of $P_n(t,z)$ is $m_{nk}(t)>1$, then the factor $z_{nk}(t)-z_{nk}(t)$ is in the left-hand side of (2.22), so $\dot{z}_{nk}(t)=\infty$.

(ii) If $z_{nk}(t)$ is a simple zero of $P_n(t,z)$, then, from (2.23), we obtain

$$\dot{z}_{nk}(t)=\frac{\lambda_{n+1}^{2}(t)P_{n-1}(t,z_{nk}(t))}{\prod_{j\neq k}\left(z_{nk}(t)-z_{nj}(t)\right)}. \tag{2.23}$$

On the other hand, writing

$$P_n(t,z(t))=\prod_{i=1}^{n}(z-z_{ni}(t)) \tag{2.24}$$

and taking derivatives with respect to z,

$$P_n'(t,z)=\sum_{i=1}^{n}\prod_{j\neq i}\left(z-z_{nj}(t)\right). \tag{2.25}$$

So,

$$P_n'(t,z_{nk}(t))=\prod_{j\neq k}\left(z_{nk}(t)-z_{nj}(t)\right). \tag{2.26}$$

Moreover, the following formula is well known:

$$\sum_{j=0}^{n-1}\left(\widehat{p}_j(t,z_{nk}(t))\right)^2=\frac{P_n'(t,z_{nk}(t))P_{n-1}(t,z_{nk}(t))}{\left(\lambda_2(t)\cdots\lambda_{n+1}(t)\right)^2} \tag{2.27}$$

(see [13, page 24]).

Finally, from (2.23), (2.26), and (2.27) we arrive to (1.14).

We point out that (2.27) also holds in the case (i) when $z_{nk}(t)$ is not a simple zero and the denominator in (1.14) is zero.

Remark 2.3. (i) It follows, from Theorem 1.2, that the zeros of each polynomial $P_n(t,z)$ depend on $t \in \mathbb{R}$ because its derivatives are not zero. Moreover, in the case of real Toda lattices, that is, when the coefficients $\alpha_n(t), \lambda_n(t)$ in (1.8) are real functions, we have $\dot{z}_{nk}(t) > 0, k = 1,2,\ldots,n$. Then, in this case $z_{nk}(t), k = 1,2,\ldots,n$, are monotonically increasing functions of $t \in \mathbb{R}$. For each fixed $n, z_{nk}(t), k = 1,2,\ldots,n$, are simple zeros of $P_n(t,z)$. Then, $z_{nk}(t) \neq z_{nk'}(t)$ for $k \neq k', k, k' = 1,\ldots,n$, and, therefore, the curves $\{z_{nk}(t) : k = 1,2,\ldots,n\}$ have no points in common.

(ii) Let $J(t)$ be a bounded operator. It is a consequence of Theorem 1.1 that $\|J(t)\|$ is independent on $t \in \mathbb{R}$. Then, for each $n \in \mathbb{N}$,

$$|z_{nk}(t)| \leq \|J(t)\| \leq M, \quad k = 1,\ldots,n, \ n \in \mathbb{N}. \tag{2.28}$$

From this fact and (i), we deduce

$$\lim_{t\to\infty} z_{nk}(t) = m_k \in \mathbb{R}, \quad k = 1,\ldots,n; \tag{2.29}$$

that is, each curve $z_{nk}(t), t \in \mathbb{R}$, has an asymptotic line $z = m_k$ in the (t,z)—plane.

(iii) When the entries of $J(t)$ are not real functions, then we do not know the multiplicity of $z_{nk}(t)$ as a zero of $P_n(t,z)$. Therefore, in the complex case it is possible that (i) and (ii) are not longer true.

3. Obtaining Some New Solutions of Toda Lattice

Consider the following solution of (1.1):

$$\begin{aligned} \alpha_n(t) &= e^t + n - 1, \\ \lambda_n(t) &= \sqrt{(n-1)e^t}, \end{aligned} \quad n \in \mathbb{N}. \tag{3.1}$$

With the notation employed in the above sections, we have

$$J(t) = \begin{pmatrix} e^t & \sqrt{e^t} & & \\ \sqrt{e^t} & e^t + 1 & \sqrt{2e^t} & \\ & \sqrt{2e^t} & e^t + 2 & \ddots \\ & & \ddots & \ddots \end{pmatrix}. \tag{3.2}$$

Since $\sum_{n\geq 1}(1/\sqrt{ne^t})$ is a divergent series, the Carleman condition ([19, page 59]) indicates that $J(t)$ is a self-adjoint operator. Moreover, it is easy to see that $\det(J_n(t)) = e^{nt}, n \in \mathbb{N}$, and therefore, $J(t)$ is a positive-definite operator. From both issues, we can get that

$$\sigma(J_n(t)) \subset [0,+\infty). \tag{3.3}$$

Then, (3.1) is an example of solution of (1.1) for which the associated polynomials $\{P_n(t,z)\}$ have all their zeros in $[0,+\infty)$. The dynamic behavior of these zeros was determined in Theorem 1.2 and Remark 2.3.

From (3.1), it is possible to obtain some complex solutions of (1.1). For this purpose we take $C \in \mathbb{C} \setminus [0,+\infty)$ and we apply the method given in [11]. Here, we explain and illustrate that method. Let

$$J^{(1)}(t) := \begin{pmatrix} e^t & e^t & & \\ 1 & e^t+1 & 2e^t & \\ & 1 & e^t+2 & \ddots \\ & & \ddots & \ddots \end{pmatrix}. \tag{3.4}$$

Due to the fact that $P_n(t,C) \neq 0$, we have

$$\det(J_n(t) - CI_n) = \det\left(J_n^{(1)}(t) - CI_n\right) \neq 0. \tag{3.5}$$

Thus, $J^{(1)}(t) - CI$ admits the formal representation given by

$$J^{(1)}(t) - CI = L(t)U(t), \tag{3.6}$$

([20, Theorem 1, page 35]), where

$$L(t) := \begin{pmatrix} l_{11}(t) & & & \\ l_{21}(t) & l_{22}(t) & & \\ & l_{32}(t) & l_{33}(t) & \\ & & \ddots & \ddots \end{pmatrix}, \qquad U(t) := \begin{pmatrix} 1 & u_{12}(t) & & \\ & 1 & u_{23}(t) & \\ & & 1 & u_{34}(t) \\ & & & \ddots & \ddots \end{pmatrix}. \tag{3.7}$$

(Despite the fact that the entries in both matrices depend on C, in order to simplify our notation we do not write down explicitly this dependence.) More precisely, for each $m \in \mathbb{N}$ we obtain

$$\begin{gathered} l_{mm}(t) = e^t + m - 1 - C - \frac{(m-1)e^t}{e^t + m - 2 - C - (m-2)e^t/\ddots - e^t/(e^t - C)}, \\ l_{m+1,m}(t) = 1, u_{m,m+1}(t) = \frac{me^t}{l_{mm}(t)}. \end{gathered} \tag{3.8}$$

Then, the new obtained solution, generated from $J(t)$ and C, is given as $\{\tilde{\alpha}_n(t), \tilde{\lambda}_{n+1}(t)\}, n \in \mathbb{N}$, being

$$U(t)L(t) := \begin{pmatrix} \tilde{\alpha}_1(t) - C & \left(\tilde{\lambda}_2(t)\right)^2 & & & \\ 1 & \tilde{\alpha}_2(t) - C & \left(\tilde{\lambda}_3(t)\right)^2 & & \\ & 1 & \tilde{\alpha}_3(t) - C & \ddots & \\ & & \ddots & \ddots & \end{pmatrix}. \tag{3.9}$$

In other words, for each $C \in \mathbb{C} \setminus [0, +\infty)$ a new solution of the Toda lattice can be generated from the product $U(t)L(t)$. In this way, a sequence of solutions can be obtained iterating this process. In our example, the new complex solution is given by

$$\begin{gathered} \tilde{\alpha}_1(t) = e^t + \frac{e^t}{e^t - C}, \\ \tilde{\alpha}_2(t) = e^t + 1 - \frac{e^t}{e^t - C} + \frac{2e^t}{e^t + 1 - C - e^t/(e^t - C)}, \cdots \\ \tilde{\lambda}_2(t) = \frac{e^t}{e^t - C}\left(e^t + 1 - C - \frac{e^t}{e^t - C}\right), \\ \tilde{\lambda}_3(t) = \frac{2e^t}{e^t + 1 - C - e^t/(e^t - C)}\left(e^t + 2 - C\frac{2e^t}{e^t + 1 - C - e^t/(e^t - C)}\right), \cdots . \end{gathered} \tag{3.10}$$

Because our initial solution $\{\alpha_n(t), \lambda_{n+1}(t)\}, n \in \mathbb{N}$, is a real solution, we know $\sigma(J(t)) = \sigma(J(t_0))$ for any $t, t_0 \in \mathbb{R}$. Moreover,

$$\sigma\left(\tilde{J}(t)\right) \setminus \{C\} = \sigma(J(t)) \setminus \{C\} \tag{3.11}$$

(see [21, Proposition 3.6, page 225]). Thus

$$\sigma\left(\tilde{J}(t)\right) \setminus \{C\} = \sigma(J(t_0)) \setminus \{C\}. \tag{3.12}$$

Because $\{\lambda_n(t)\}, n \in \mathbb{N}$, is not a bounded sequence, we cannot apply Theorem 1.1. However, from (3.12) we conjecture that Theorem 1.1 could be extended to a more general situation.

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References

[1] H. Flaschka, "On the Toda lattice. II. *Inverse-scattering solution*," *Progress of Theoretical Physics*, vol. 51, pp. 703–716, 1974.

[2] P. D. Lax, "Integrals of nonlinear equations of evolution and solitary waves," *Communications on Pure and Applied Mathematics*, vol. 21, pp. 467–490, 1968.

[3] J. Moser, "Three integrable Hamiltonian systems connected with isospectral deformations," *Advances in Mathematics*, vol. 16, pp. 197–220, 1975.

[4] J. Moser, "Finitely many mass points on the line under the influence of an exponential potential—an integrable system," in *Dynamical Systems, Theory and Applications*, vol. 38 of *Lecture Notes in Physics*, pp. 467–497, Springer, Berlin, Germany, 1975.

[5] J. Coussement, A. B. J. Kuijlaars, and W. Van Assche, "Direct and inverse spectral transform for the relativistic Toda lattice and the connection with Laurent orthogonal polynomials," *Inverse Problems*, vol. 18, no. 3, pp. 923–942, 2002.

[6] J. Coussement and W. Van Assche, "An extension of the Toda lattice: a direct and inverse spectral transform connected with orthogonal rational functions," *Inverse Problems*, vol. 20, no. 1, pp. 297–318, 2004.

[7] J. Coussement and W. Van Assche, "A continuum limit of the relativistic Toda lattice: asymptotic theory of discrete Laurent orthogonal polynomials with varying recurrence coefficients," *Journal of Physics A*, vol. 38, no. 15, pp. 3337–3366, 2005.

[8] G. Teschl, *Jacobi Operators and Completely Integrable Nonlinear Lattices*, vol. 72 of *Mathematical Surveys and Monographs*, American Mathematical Society, Providence, RI, USA, 2000.

[9] F. Gesztesy, H. Holden, J. Michor, and G. Teschl, *Soliton Equations and Their Algebro-Geometric Solutions*, vol. 114 of *Cambridge Studies in Advanced Mathematics*, Cambridge University Press, Cambridge, UK, 2008.

[10] A. I. Aptekarev and A. Branquinho, "Padé approximants and complex high order Toda lattices," *Journal of Computational and Applied Mathematics*, vol. 155, no. 2, pp. 231–237, 2003.

[11] D. Barrios Rolanía and A. Branquinho, "Complex high order Toda and Volterra lattices," *Journal of Difference Equations and Applications*, vol. 15, no. 2, pp. 197–213, 2009.

[12] D. Barrios Rolanía and G. López Lagomasino, "Asymptotic behavior of solutions of general three term recurrence relations," *Advances in Computational Mathematics*, vol. 26, no. 1-3, pp. 9–37, 2007.

[13] T. S. Chihara, *An Introduction to Orthogonal Polynomials*, Gordon and Breach Science, New York, NY, USA, 1978.

[14] F. Gesztesy, H. Holden, and G. Teschl, "The algebro-geometric Toda hierarchy initial value problem for complex-valued initial data," *Revista Mathemática Iberoamericana*, vol. 24, no. 1, pp. 117–182, 2008.

[15] D. Barrios, G. López, A. Martínez-Finkelshtein, and E. Torrano, "Finite-dimensional approximations of the resolvent of an infinite band matrix, and continued fractions," *Matematicheskii Sbornik*, vol. 501, no. 4, pp. 23–42, 1999, English translation in *Sbornik: Mathematics*, vol. 190, no. 3-4, pp. 501–509, 1999.

[16] S. R. Garcia, "Conjugation and Clark operators," *Contemporary Mathematics*, vol. 393, pp. 67–112, 2006.

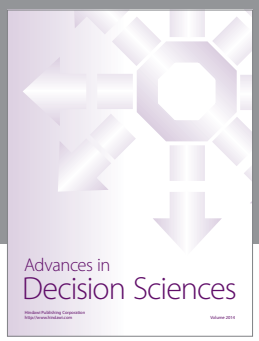
[17] T. Kato, *Perturbation Theory for Linear Operators*, Springer, Berlin, Germany, 1995.

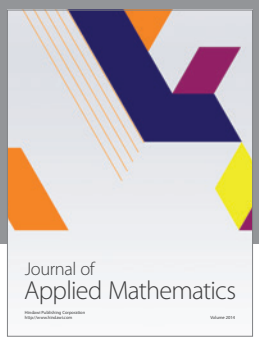
[18] N. Y. Vilenkin, E. A. Gorin, A. G. Kostyuchenko, et al., *Functional Analysis*, Wolters-Noordhoff, Groningen, The Netherlands, 1972.

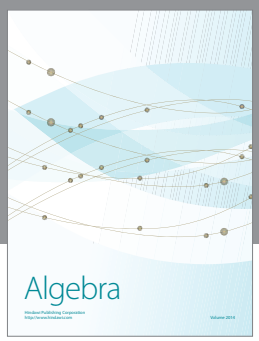
[19] J. A. Shohat and J. D. Tamarkin, *The Problem of Moments*, American Mathematical Society, Providence, RI, USA, 1963.

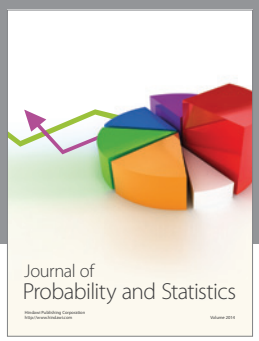
[20] F. R. Gantmacher, *The Theory of Matrices*, AMS Chelsea, Providence, RI, USA, 2000.

[21] M. I. Bueno and F. Marcellán, "Darboux transformation and perturbation of linear functionals," *Linear Algebra and Its Applications*, vol. 384, pp. 215–242, 2004.









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