Research Article

On the Solutions of the System of Difference Equations

\[ x_{n+1} = \max \left\{ \frac{A}{x_n}, \frac{y_n}{x_n} \right\}, \]
\[ y_{n+1} = \max \left\{ \frac{A}{y_n}, \frac{x_n}{y_n} \right\}, \]

Dağistan Simsek,1 Bilal Demir,2 and Cengiz Cinar2

1 Department of Industrial Engineering, Engineering-Architecture Faculty, Selcuk University, Kampüs, 42075 Konya, Turkey
2 Mathematics Department, Education Faculty, Selcuk University, Meram Yeni Yol, 42090 Konya, Turkey

Correspondence should be addressed to Dağistan Simsek, dsimsek@selcuk.edu.tr

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Recommended by Guang Zhang

We study the behavior of the solutions of the following system of difference equations

\[ x_{n+1} = \max \left\{ \frac{A}{x_n}, \frac{y_n}{x_n} \right\}, \]
\[ y_{n+1} = \max \left\{ \frac{A}{y_n}, \frac{x_n}{y_n} \right\}, \]

where the constant \( A \) and the initial conditions are positive real numbers.

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1. Introduction

Recently, there has been a great interest in studying the periodic nature of nonlinear difference equations. Although difference equations are relatively simple in form, it is, unfortunately, extremely difficult to understand thoroughly the periodic behavior of their solutions. The periodic nature of nonlinear difference equations of the max type has been investigated by many authors. See, for example [1–24].

In this paper we study the behavior of the solutions of the following system of difference equations:

\[ x_{n+1} = \max \left\{ \frac{A}{x_n}, \frac{y_n}{x_n} \right\}, \]
\[ y_{n+1} = \max \left\{ \frac{A}{y_n}, \frac{x_n}{y_n} \right\}, \]  

(1.1)

where the constant \( A \) and the initial conditions are positive real numbers.
2. Main Result

Definition 2.1. Fibonacci sequence is \( f_1 = 1, f_2 = 1 \) and for \( n \geq 3, f_n = f_{n-1} + f_{n-2} \).

Definition 2.2. The symbol \([ ]\) symbolizes the greatest integer function.

Definition 2.3. The sequence of \( a(n) \equiv n \pmod{2} \).

Definition 2.4. The sequence of \( k(n) = \begin{cases} n, & n = 0, 2, 4, \\ n + 1, & n = 1, 3, 5, \ldots \end{cases} \quad (2.1) \)

Theorem 2.5. Let \( (x_n, y_n) \) be the solution of the system of difference equations (1.1) for \( A < x_0 < y_0 \) and \( y_0/x_0 > A \).

If \( n \geq 1 \), then

\[
x_n = \left( \frac{A f_{k(n)-1-a(n)} x_0^{f_{k(n)}}}{y_0^{f_{k(n)}}} \right)^{(-1)^n}, \quad (2.2)
\]

\( y_1 = x_0/y_0 \), and if \( n \geq 2 \)

\[
y_n = \left( \frac{y_0^{f_{k(n-1)+1}}}{A f_{k(n-1)+a(n)-1} x_0^{f_{k(n-1)+1}}} \right)^{(-1)^n}. \quad (2.3)
\]

Proof. Let \( y_0 > A \), then

\[
x_1 = \max \left\{ \frac{A}{x_0}, \frac{y_0}{x_0} \right\} = \frac{y_0}{x_0} > A,
\]

\[
y_1 = \max \left\{ \frac{A}{y_0}, \frac{x_0}{y_0} \right\} = \frac{x_0}{y_0} < A,
\]

\[
x_2 = \max \left\{ \frac{A}{x_1}, \frac{y_1}{x_1} \right\} = \frac{A x_0}{y_0} < A,
\]

\[
y_2 = \max \left\{ \frac{A}{y_1}, \frac{x_1}{y_1} \right\} = \frac{x_1}{y_1} = \frac{y_0^2}{x_0^2} > A,
\]

\[
x_3 = \max \left\{ \frac{A}{x_2}, \frac{y_2}{x_2} \right\} = \frac{y_2}{x_2} = \frac{y_0^3}{Ax_0^3} > A,
\]

\[
y_3 = \max \left\{ \frac{A}{y_2}, \frac{x_2}{y_2} \right\} = \frac{x_2}{y_2} = \frac{A x_0^2}{y_0^2} < A.
\]
\[
x_4 = \max \left\{ \frac{A}{x_3}, \frac{y_3}{y_3} \right\} = \frac{A^3 x_0^3}{y_0^3} < A,
\]
\[
y_4 = \max \left\{ \frac{A}{y_3}, \frac{x_3}{x_3} \right\} = \frac{x_3}{y_3} = \frac{y_0^3}{A^2 x_0^3} > A,
\]
\[
x_5 = \max \left\{ \frac{A}{x_4}, \frac{y_4}{x_3} \right\} = \frac{y_4}{x_4} = \frac{y_0^8}{A^3 x_0^8} > A,
\]
\[
y_5 = \max \left\{ \frac{A}{y_4}, \frac{x_4}{y_4} \right\} = \frac{x_4}{y_4} = \frac{A^3 x_0^5}{y_0^5} < A,
\]
\[
\vdots
\]
\[
(2.4)
\]

\[n \geq 1 \text{ then} \]
\[
x_n = \left( \frac{A^{f_k(n-1)-a(n)} x_0^{f_k(n)}}{y_0^{f_k(n)}} \right)^{(-1)^n}.
\]
\[
(2.5)
\]
\[
y_1 = x_0 / y_0 \text{ then } n \geq 2,
\]
\[
y_n = \left( \frac{y_0^{f_k(n-1)+1}}{A^{f_k(n-1)+a(n)-1} x_0^{f_k(n-1)+1}} \right)^{(-1)^n}.
\]
\[
(2.6)
\]

**Theorem 2.6.** Let \( (x_n, y_n) \) be the solution of the system of difference equations (1.1) for \( A < y_0 < x_0 \).
\( x_1 = x_0 / y_0 \) and if \( n \geq 2 \),
\[
x_n = \left( \frac{A^{f_k(n-1)+a(n)-1} x_0^{f_k(n-1)+1}}{y_0^{f_k(n-1)+1}} \right)^{(-1)^n}.
\]
\[
(2.7)
\]

If \( n \geq 1 \), then
\[
y_n = \left( \frac{y_0^{f_k(n)}}{A^{f_k(n)-a(n)} x_0^{f_k(n)}} \right)^{(-1)^n}.
\]
\[
(2.8)
\]

**Proof.** Similarly we can obtain the proof as the proof of Theorem 2.5. \( \Box \)
Theorem 2.7. Let \((x_n, y_n)\) be the solution of the system of difference equations (1.1) for \(A < x_0 < y_0\) and \((y_0/x_0) > A\).

\[
\begin{align*}
\text{(a) } & \lim_{n \to \infty} x_{2n} = 0, \\
& \lim_{n \to \infty} y_{2n} = \infty. \\
\text{(b) } & \lim_{n \to \infty} x_{2n+1} = \infty, \\
& \lim_{n \to \infty} y_{2n+1} = 0. 
\end{align*}
\]

(2.9)

Proof. (a) We obtain that

\[
\begin{align*}
\lim_{n \to \infty} x_{2n} &= \lim_{n \to \infty} \left( \frac{A f_{2n+1} - a f_{2n} - 1}{y_0 f_{2n}} \right) \\
&= \lim_{n \to \infty} \left( \frac{A f_{2n+1} - a f_{2n} - 1}{y_0 f_{2n}} \right) \\
&= \lim_{n \to \infty} \left( \frac{A f_{2n+1} - a f_{2n} - 1}{y_0 f_{2n}} \right) \\
&= 0, \\
\lim_{n \to \infty} y_{2n} &= \lim_{n \to \infty} \left( \frac{y_0}{A f_{2n+1} - a f_{2n} - 1} \right) \\
&= \lim_{n \to \infty} \left( \frac{y_0}{A f_{2n+1} - a f_{2n} - 1} \right) \\
&= \lim_{k \to \infty} \left( \frac{y_0}{A f_{2n+1} - a f_{2n} - 1} \right) \\
&= \infty.
\end{align*}
\]

(2.10)

(b) Similarly we can obtain the proof of (b) as the proof of (a). \qed
Theorem 2.8. Let \((x_n, y_n)\) be the solution of the system of difference equations (1.1) for \(A < y_0 < x_0\) and \((x_0/y_0) > A\).

(a) \[ \lim_{n \to \infty} x_{2n} = \infty, \]
    \[ \lim_{n \to \infty} y_{2n} = 0. \]  
(b) \[ \lim_{n \to \infty} x_{2n+1} = 0, \]
    \[ \lim_{n \to \infty} y_{2n+1} = \infty. \]  

\[ (2.11) \]

Proof. Similarly we can obtain the proof as the proof of Theorem 2.7. \(\square\)

Theorem 2.9. Let \((x_n, y_n)\) be the solution of the system of difference equations (1.1) for \(1 < x_0 < y_0 < A\).

If \(n \geq 1\), then

\[ x_n = \left( \frac{x_0}{\frac{A}{y_0}} \right)^{(1)^n}. \]  
\[ (2.12) \]

If \(n \geq 1\), then

\[ y_n = \left( \frac{y_0}{\frac{A}{y_0}} \right)^{(1)^n}. \]  
\[ (2.13) \]

Proof. Let

\[ x_1 = \max \left\{ \frac{A}{x_0}, \frac{y_0}{x_0} \right\} = \frac{A}{x_0} < A, \]
\[ y_1 = \max \left\{ \frac{A}{y_0}, \frac{x_0}{y_0} \right\} = \frac{A}{y_0} < A, \]
\[ x_2 = \max \left\{ \frac{A}{x_1}, \frac{y_1}{x_1} \right\} = \frac{A x_0}{A} = x_0 < A, \]
\[ y_2 = \max \left\{ \frac{A}{y_1}, \frac{x_1}{y_1} \right\} = \frac{A}{y_1} = y_0 < A, \]
\[ x_3 = \max \left\{ \frac{A}{x_2}, \frac{y_2}{x_2} \right\} = \frac{A}{x_2} = \frac{A}{x_0} < A, \]
\[ y_3 = \max \left\{ \frac{A}{y_2}, \frac{x_2}{y_2} \right\} = \frac{A}{y_2} = \frac{A}{y_0} < A, \]
\[ x_4 = \max \left\{ \frac{A}{x_3}, \frac{y_3}{x_3} \right\} = \frac{A}{x_3} = x_0 < A, \]
\[ y_4 = \max \left\{ \frac{A}{y_3}, \frac{x_3}{y_3} \right\} = \frac{A}{y_3} = y_0 < A, \]
\[ x_5 = \max \left\{ \frac{A}{x_4}, \frac{y_4}{x_4} \right\} = \frac{A}{x_4} = \frac{A}{x_0} < A, \]
\[ y_5 = \max \left\{ \frac{A}{y_4}, \frac{x_4}{y_4} \right\} = \frac{A}{y_4} = \frac{A}{y_0} < A, \]
\[ \vdots \]
\[ (2.14) \]

\[ n \geq 1, \text{ then} \]
\[ x_n = \left( \frac{x_0}{A^{a(n)}} \right)^{(-1)^n}, \]
\[ y_n = \left( \frac{y_0}{A^{a(n)}} \right)^{(-1)^n}. \]
\[ (2.15) \]

**Theorem 2.10.** Let \((x_n, y_n)\) be the solution of the system of difference equations (1.1) for \(1 < x_0 < y_0 < A\).

(a) \[ \lim_{n \to \infty} x_{2n} = x_0, \]
\[ \lim_{n \to \infty} y_{2n} = y_0. \]

(b) \[ \lim_{n \to \infty} x_{2n+1} = \frac{A}{x_0}, \]
\[ \lim_{n \to \infty} y_{2n+1} = \frac{A}{y_0}. \]
\[ (2.16) \]

**Proof.** (a) We obtain that
\[ \lim_{n \to \infty} x_{2n} = \lim_{n \to \infty} \left( \frac{x_0}{A^{a(2n)}} \right)^{(-1)^{2n}} = \lim_{n \to \infty} \left( \frac{x_0}{A^0} \right)^{(-1)^{2n}} = \lim_{n \to \infty} \left( \frac{x_0}{A^0} \right) = x_0, \]
\[ \lim_{n \to \infty} y_{2n} = \lim_{n \to \infty} \left( \frac{y_0}{A^{a(2n)}} \right)^{(-1)^{2n}} = \lim_{n \to \infty} \left( \frac{y_0}{A^0} \right)^{(-1)^{2n}} = \lim_{n \to \infty} \left( \frac{y_0}{A^0} \right) = y_0. \]
\[ (2.17) \]

(b) Similarly we can obtain the proof of (b) as the proof of (a).

**Lemma 2.11.** Let \((x_0, y_0)\) be the initial condition of (1.1) for \(0 < x_0 < 1 < y_0 < A\); there is at least an \(i_0 \in \mathbb{N}\) such that every \(n \in \mathbb{N}\) for \(n > i_0\), \(y_0/x_0^n > A\).

**Proof.** We consider that \(x_0 < 1\) hence \(\lim_{n \to \infty} (y_0/x_0^n) = \infty\) and that proofs the existing of \(i_0\) defined in hypothesis.

Theorem 2.12. Let \((x_n, y_n)\) be the solution of the system of difference equations (1.1) for \(0 < x_0 < 1 < y_0 < A\), and \(i_0\) is the number, defined by Lemma 2.11.

\[1 \leq n \leq i_0\]

\[x_n = A^{a(n)}(x_0)^{(-1)^n}, \quad (2.18)\]

\[1 \leq n \leq i_0\]

\[y_n = A^{a(n)} \left( \frac{y_0}{x_0^{[n/2]}} \right)^{(-1)^n}, \quad (2.19)\]

and when \(n > i_0\), the solutions will be different for every different constant \(A\).

Proof. Let \(y_0 < A\), then

\[x_1 = \max \left\{ \frac{A}{x_0}, \frac{y_0}{x_0} \right\} = \frac{A}{x_0} > 1,\]

\[y_1 = \max \left\{ \frac{A}{y_0}, \frac{x_0}{y_0} \right\} = \frac{A}{y_0} > 1,\]

\[x_2 = \max \left\{ \frac{A}{y_1}, \frac{x_1}{y_1} \right\} = \frac{A}{x_1} = x_0 < 1,\]

\[y_2 = \max \left\{ \frac{A}{x_1}, \frac{x_1}{y_1} \right\} = \frac{x_1}{y_1} = \frac{y_0}{x_0} > 1,\]

\[x_3 = \max \left\{ \frac{A}{x_2}, \frac{x_2}{y_2} \right\} = \frac{A}{x_2} = \frac{A}{x_0} > 1,\]

\[y_3 = \max \left\{ \frac{A}{x_3}, \frac{x_3}{y_3} \right\} = \frac{x_3}{y_3} = \frac{y_0}{x_0},\]

\[x_4 = \max \left\{ \frac{A}{x_4}, \frac{x_4}{y_4} \right\} = \frac{A}{x_4} = x_0 < 1,\]

\[y_4 = \max \left\{ \frac{A}{y_4}, \frac{x_4}{y_4} \right\} = \frac{x_4}{y_4} = \frac{A}{x_0},\]

\[x_5 = \max \left\{ \frac{A}{x_5}, \frac{x_5}{y_5} \right\} = \frac{A}{x_5} = \frac{A}{x_0} > 1,\]

\[y_5 = \max \left\{ \frac{A}{y_5}, \frac{x_5}{y_5} \right\} = \frac{x_5}{y_5} = \frac{A}{x_0},\]
\[1 \leq n \leq i_0,\]
\[x_n = A^{a(n)}(x_0)^{(-1)^n},\]
\[y_n = A^{a(n)}\left(\frac{y_0}{x_0^{\lfloor n/2 \rfloor}}\right)^{(-1)^n}.\]  
\[(2.21)\]

**Lemma 2.13.** Let \((x_0, y_0)\) be the initial condition of (1.1) for \(0 < y_0 < 1 < x_0 < A\); there is at least an \(i_0 \in N\) such that every \(n \in N\) for \(n > i_0, x_n/y_n^n > A\).

**Proof.** Similarly we can obtain the proof as the proof of Lemma 2.11.

**Theorem 2.14.** Let \((x_n, y_n)\) be the solution of the system of difference equations (1.1) for \(0 < y_0 < 1 < x_0 < A\), and \(i_0\) is the number, defined by Lemma 2.13.

\[1 \leq n \leq i_0\]
\[x_n = A^{a(n)}\left(\frac{x_0}{y_0^{\lfloor n/2 \rfloor}}\right)^{(-1)^n},\]  
\[(2.22)\]
\[y_n = A^{a(n)}(y_0)^{(-1)^n},\]  
\[(2.23)\]

and when \(n > i_0\), the solutions will be different for every different constant \(A\).

**Proof.** Similarly we can obtain the proof of be as the proof of Theorem 2.12.

**Lemma 2.15.** Let \((x_0, y_0)\) be the initial condition of (1.1) for \(0 < y_0 < x_0 < 1\); there is at least an \(i_0 \in N\) such that every \(n \in N\) for \(n > i_0, (x_0/y_0)^n > A\).

**Proof.** We consider that \(y_0 < x_0\) hence \(\lim_{n \to \infty}(x_0/y_0)^n = \infty\) and that proofs the existing of \(i_0\) defined in hypothesis.

**Theorem 2.16.** Let \((x_n, y_n)\) be the solution of the system of difference equations (1.1) for \(0 < y_0 < x_0 < 1, A > 1\), and \(i_0\) is the number, defined by Lemma 2.15.

\[x_1 = A/x_0\] and if \(1 < n \leq i_0,\)
\[x_n = A^{a(n)}\left(\frac{x_0}{y_0^{\lfloor n/2 \rfloor}}\right)^{\lfloor n/2 \rfloor(-1)^n},\]  
\[(2.24)\]
\[y_1 = A/y_0,\] and if \(1 < n \leq i_0,\)
\[y_n = A^{a(n)}\left(\frac{y_0}{x_0}\right)^{(-1)^n},\]  
\[(2.25)\]

and when \(n > i_0\), the solutions will be different for every different constant \(A\).
Proof. Let \( y_0 < A \), then

\[
x_1 = \max \left( \frac{A}{x_0}, \frac{y_0}{x_0} \right) = \frac{A}{x_0} > 1,
\]

\[
y_1 = \max \left( \frac{y_0}{y_0}, \frac{x_0}{y_0} \right) = \frac{A}{y_0} > 1,
\]

\[
x_2 = \max \left( \frac{A}{x_1}, \frac{y_1}{x_1} \right) = \frac{y_1}{x_1} = \frac{x_1}{y_0} > 1,
\]

\[
y_2 = \max \left( \frac{y_1}{y_1}, \frac{x_1}{y_1} \right) = \frac{x_1}{y_1} = \frac{y_0}{x_0} < 1,
\]

\[
x_3 = \max \left( \frac{A}{x_2}, \frac{y_2}{y_2} \right) = \frac{A}{x_2} = \frac{A}{x_0} > 1,
\]

\[
y_3 = \max \left( \frac{x_2}{x_2}, \frac{y_3}{x_2} \right) = \frac{x_3}{x_2} = \frac{x_2}{y_0} > 1,
\]

\[
x_4 = \max \left( \frac{A}{x_3}, \frac{y_3}{x_3} \right) = \frac{y_3}{x_3} = \frac{y_0}{x_0} < 1,
\]

\[
y_4 = \max \left( \frac{A}{y_3}, \frac{x_3}{y_3} \right) = \frac{A}{y_3} = \frac{y_0}{x_0} < 1,
\]

\( n > 1 \) then

\[
x_n = A^{a(n)} \left( \frac{x_0}{y_0} \right)^{\left\lceil \frac{n}{2} \right\rceil} (-1)^n, \tag{2.26}
\]

\( n \geq 1 \) then

\[
y_n = A^{a(n)} \left( \frac{y_0}{x_0} \right)^{(-1)^n}. \tag{2.28}
\]

Lemma 2.17. Let \((x_0, y_0)\) be the initial condition of (1.1) for \( 0 < y_0 < x_0 < 1 \); there is at least an \( i_0 \in \mathbb{N} \) such that every \( n \in \mathbb{N} \) for \( n > i_0 \), \( (y_0/x_0)^n > A \).

Proof. Similarly we can obtain the proof as the proof of Lemma 2.15. \( \square \)

Theorem 2.18. Let \((x_n, y_n)\) be the solution of the system of difference equations (1.1) for \( 0 < x_0 < y_0 < 1 \), \( A > 1 \), and \( i_0 \) is the number, defined by Lemma 2.17.

\( x_1 = A/x_0 \), and if \( 1 < n \leq i_0 \),

\[
x_n = A^{a(n)} \left( \frac{x_0}{y_0} \right)^{(-1)^n}, \tag{2.29}
\]
\[ y_1 = \frac{A}{y_0}, \text{ and if } 1 < n \leq i_0, \]
\[ y_n = A^{a(n)} \left( \frac{y_0}{x_0} \right)^{([n/2])(-1)^n}, \]  
(2.30)

and when \( n > i_0, \) the solutions will be different for every different constant \( A. \)

\textbf{Proof.} Similarly we can obtain the proof as the proof of Theorem 2.16, which completes the proofs of theorems. \qed

\textbf{References}


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