Research Article

Existence of Solutions to Boundary Value Problems for the Discrete Generalized Emden-Fowler Equation

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We consider the existence of solutions to boundary value problems for the discrete generalized Emden-Fowler equation. By means of the minimax methods in the critical point theory, some new results are obtained. Two examples are also given to illustrate the main results.

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1. Introduction and Statements of Main Results

Let $\mathbf{Z}$ and $\mathbf{R}$ be the sets of all integers and real numbers, respectively. For $a, b \in \mathbf{Z}$, define $\mathbf{Z}[a] = \{a, a+1, \dots\}$, $\mathbf{Z}[a, b] = \{a, a+1, \dots, b\}$ when $a \leq b$.

In this paper, we consider the following boundary value problem (BVP for short) consisting of the discrete generalized Emden-Fowler equation:

$$\Delta[p(t)\Delta u(t-1)] + q(t)u(t) = f(t, u(t)), \quad t \in \mathbf{Z}[1, k], \quad (1.1)$$

and the boundary value conditions:

$$u(0) + \alpha u(1) = A, \quad u(k+1) + \beta u(k) = B, \quad (1.2)$$

where k is a positive integer, α, β, A , and B are constants, and $\Delta u(t) = u(t+1) - u(t)$ is the forward difference operator. We assume that $p(t)$ is nonzero and realvalued for each $t \in \mathbf{Z}[1, k+1]$. We also assume that $q(t)$ is realvalued for each $t \in \mathbf{Z}[1, k]$, $f(t, x)$ is realvalued for each $(t, x) \in \mathbf{Z}[1, k] \times \mathbf{R}$, and $f(t, x)$ is continuous in the second variable x .

Equation (1.1) has been extensively studied by many authors; for example, see [1–9] concerning its disconjugacy, disfocality, oscillation, asymptotic behaviour, existence of periodic solutions, and solutions to boundary value problem.

Recently, Yu and Guo in [10] employed the critical point theory to obtain the existence of solutions to the BVP (1.1)-(1.2). Motivated by this and the results in [11], the main purpose of this paper is to give some new sufficient conditions for the existence of solutions to the BVP (1.1)-(1.2) by applying the Saddle Point Theorem and the Least Action Principle.

Before giving the main results, we first set

$$F(t, x) = \int_0^x f(t, s) ds, \quad c(t) = q(t) - p(t) - p(t+1),$$

$$M = \begin{pmatrix} c(1) - \alpha p(1) & p(2) & 0 & \cdots & 0 & 0 \\ p(2) & c(2) & p(3) & \cdots & 0 & 0 \\ 0 & p(3) & c(3) & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & c(k-1) & p(k) \\ 0 & 0 & 0 & \cdots & p(k) & c(k) - \beta p(k+1) \end{pmatrix}, \quad (1.3)$$

$$\eta = (\eta(1), \eta(2), \dots, \eta(k))^T = \begin{pmatrix} p(1)A \\ 0 \\ \vdots \\ 0 \\ p(k+1)B \end{pmatrix}.$$

The main results are as follows.

Theorem 1.1. *Suppose that f , M satisfy the following assumptions.*

(F₁) *There are constants $C_1 > 0$, $C_2 > 0$, $1/2 \leq \theta < 1$ such that for all $(t, x) \in \mathbf{Z}[1, k] \times \mathbf{R}$,*

$$|f(t, x)| \leq C_1 |x|^\theta + C_2. \quad (1.4)$$

(F₂) *One has*

- (i) *either $\|u\|^{-2\theta} \sum_{t=1}^k F(t, u(t)) \rightarrow -\infty$ as $\|u\| \rightarrow \infty$, or*
- (ii) *$\|u\|^{-2\theta} \sum_{t=1}^k F(t, u(t)) \rightarrow +\infty$ as $\|u\| \rightarrow \infty$,*

where for all $u = (u(1), u(2), \dots, u(k))^T \in \mathbf{R}^k$, $\|u\| = (\sum_{t=1}^k |u(t)|^2)^{1/2}$.

(P₁) *The matrix M is singular.*

Then the BVP (1.1)-(1.2) has at least one solution.

Remark 1.2. There are functions $p(t)$, $q(t)$, and $f(t, x)$ satisfying our Theorem 1.1 and not satisfying the results in [10]. For example, let A, B be arbitrary constants, $k = 4$, $\alpha = \beta = -1$, $\theta = 1/2$ and

$$p(t) \equiv 1, \quad q(t) \equiv 0, \quad f(t, x) = g(t) \frac{x}{\sqrt[4]{1+x^2}} + 1, \quad (1.5)$$

where $g(t) < 0$ for every $t \in \mathbf{Z}[1, 4]$. It is easy to verify that (F_1) , $(F_2)(i)$, and (P_1) are satisfied. Then the BVP (1.1)-(1.2) has at least one solution. And it is easy to see that this solution is a nonzero solution since $f(t, 0) \neq 0$.

Theorem 1.3. Suppose that f, M satisfy the following assumptions.

(F_3) For any $t \in \mathbf{Z}[1, k]$, $\lim_{|x| \rightarrow \infty} (f(t, x)/x) = 0$.

(P_2) The matrix M is nonsingular.

Then the BVP (1.1)-(1.2) has at least one solution.

Corollary 1.4. Suppose that f, M satisfy (F_1) and (P_2) . Then the BVP (1.1)-(1.2) has at least one solution.

Remark 1.5. Since for all $(t, x) \in \mathbf{Z}[1, k] \times \mathbf{R}$, $|f(t, x)| \leq C_2$ implies $|f(t, x)| \leq C_1|x|^\theta + C_2$, our Corollary 1.4 extends Theorem 3.2 in [10].

Theorem 1.6. Suppose that f, M satisfy the following assumptions.

(F_4) For any $t \in \mathbf{Z}[1, k]$, $\lim_{|x| \rightarrow \infty} (F(t, x)/x^2) = 0$.

(F_5) One has

(i) either $2F(t, x) - (f(t, x) + \eta(t))x \rightarrow -\infty$ as $|x| \rightarrow \infty$ for all $t \in \mathbf{Z}[1, k]$, or

(ii) $2F(t, x) - (f(t, x) + \eta(t))x \rightarrow +\infty$ as $|x| \rightarrow \infty$ for all $t \in \mathbf{Z}[1, k]$.

(P_3) The matrix M is singular and indefinite.

Then the BVP (1.1)-(1.2) has at least one solution.

Remark 1.7. There are functions $p(t)$, $q(t)$, and $f(t, x)$ satisfying our Theorem 1.6 and not satisfying the results in [10]. For example, let A, B be arbitrary constants, $k = 4$, $\alpha = 1$, $\beta = 0$ and

$$p(1) = p(3) = -1, \quad p(2) = p(4) = 1, \quad p(5) = -2, \quad q(t) \equiv 0, \quad (1.6)$$

$$f(t, x) = g(t) \left(\frac{x}{\sqrt{1+x^2}} \ln(1+x^2) + \sqrt{1+x^2} \frac{2x}{1+x^2} + 1 \right),$$

where $g(t) < 0$ for every $t \in \mathbf{Z}[1, 4]$. It is easy to verify that (F_4) , $(F_5)(i)$ and (P_3) are satisfied. Then the BVP (1.1)-(1.2) has at least one solution. And it is easy to see that this solution is a nonzero solution since $f(t, 0) \neq 0$.

Corollary 1.8. Assume that (F_4) holds. If one of the following conditions is satisfied: (H_1) the matrix M is negative semi-definite, and $(F_5)(i)$ holds, (H_2) the matrix M is positive semi-definite, and $(F_5)(ii)$ holds, then, the BVP (1.1)-(1.2) has at least one solution.

2. Variational Structure and Two Basic Lemmas

Let $\mathbf{R}^k$ be the real Euclidean space with dimension k . For any $u, v \in \mathbf{R}^k$, $\|u\|$ and (u, v) denote the usual norm and inner product in $\mathbf{R}^k$, respectively.

Define the functional J on $\mathbf{R}^k$ as follows:

$$J(u) = \frac{1}{2}(Mu, u) + (\eta, u) - \sum_{t=1}^k F(t, u(t)), \quad \forall u = (u(1), u(2), \dots, u(k))^T \in \mathbf{R}^k. \quad (2.1)$$

It is well known that $u = (u(1), u(2), \dots, u(k))^T \in \mathbf{R}^k$ is a critical point of J if and only if $\{u(t)\}_{t=0}^{k+1} = (u(0), u(1), u(2), \dots, u(k), u(k+1))^T$ is a solution of the BVP (1.1)-(1.2), where $u(0) = A - \alpha u(1)$ and $u(k+1) = B - \beta u(k)$. For details, see [10]. It follows from the continuity of f that J is continuously differentiable on $\mathbf{R}^k$. Moreover, one has

$$(J'(u), h) = (Mu, h) + (\eta, h) - \sum_{t=1}^k f(t, u(t)) \cdot h(t), \quad \forall u, h \in \mathbf{R}^k. \quad (2.2)$$

When the matrix M is singular and indefinite, we suppose that $0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_l$ and $0 > -\mu_1 \geq -\mu_2 \geq \dots \geq -\mu_s$ are the positive and negative eigenvalues of M , respectively, and $l + s = k - 1$. We also suppose that ς_i , $1 \leq i \leq l$ and ξ_j , $1 \leq j \leq s$ are the eigenvectors of M corresponding to eigenvalues λ_i , $1 \leq i \leq l$ and $-\mu_j$, $1 \leq j \leq s$ satisfying

$$(\varsigma_{i_1}, \varsigma_{i_2}) = \begin{cases} 0, & i_1 \neq i_2, \\ 1, & i_1 = i_2, \end{cases} \quad (\varsigma_i, \xi_j) = 0, \quad (\xi_{j_1}, \xi_{j_2}) = \begin{cases} 0, & j_1 \neq j_2, \\ 1, & j_1 = j_2, \end{cases} \quad (2.3)$$

where $i, i_1, i_2 \in \mathbf{Z}[1, l]$, $j, j_1, j_2 \in \mathbf{Z}[1, s]$. We denote

$$R^+ = \text{span}\{\varsigma_i \mid i \in \mathbf{Z}[1, l]\}, \quad R^- = \text{span}\{\xi_j \mid j \in \mathbf{Z}[1, s]\}, \quad (2.4)$$

$$R^0 = (R^+ \oplus R^-)^\perp. \quad (2.5)$$

Then $\mathbf{R}^k$ has the direct sum decomposition $\mathbf{R}^k = R^+ \oplus R^0 \oplus R^-$. So, for each $u \in \mathbf{R}^k$, u can be expressed by

$$u = u^+ + u^0 + u^-, \quad (2.6)$$

where $u^* \in R^*$, $*$ is $+$, 0 , or $-$, respectively. Furthermore, we have the following estimates:

$$\begin{aligned} \lambda_1 \|u^+\|^2 &\leq (Mu^+, u^+) \leq \lambda_l \|u^+\|^2, \quad u^+ \in R^+, \\ -\mu_s \|u^-\|^2 &\leq (Mu^-, u^-) \leq -\mu_1 \|u^-\|^2, \quad u^- \in R^-. \end{aligned} \quad (2.7)$$

Set

$$\|u\|_\infty = \max_{t \in \mathbb{Z}[1,k]} |u(t)|, \quad \|u\|_p = \left(\sum_{t=1}^k |u(t)|^p \right)^{1/p}, \quad (2.8)$$

where $u = (u(1), u(2), \dots, u(k))^T \in \mathbb{R}^k$ and $\infty > p \geq 1$. Then for any $u \in \mathbb{R}^k$,

$$\frac{1}{\sqrt{k}} \|u\| \leq \|u\|_\infty \leq \|u\|, \quad \frac{1}{\sqrt{k}} \|u\| \leq \|u\|_p \leq k^{1/p} \|u\|. \quad (2.9)$$

We will make use of the least action principle and saddle point theorem to obtain the critical points of J . Let us first recall these theorems.

Lemma 2.1 (the least action principle, see [12]). *Let X be a real Banach space, and assume that $J \in C^1(X, \mathbb{R})$ is bounded from below in X and satisfies the Palais-Smale condition ((PS) condition for short). Then $c = \inf_{u \in X} J(u)$ is a critical value of J .*

Lemma 2.2 (saddle point theorem, see [13]). *Let X be a real Hilbert space, $X = X_1 \oplus X_2$, where $X_1 \neq \{0\}$ and is finite dimensional. Suppose that $J \in C^1(X, \mathbb{R})$ satisfies the (PS) condition and*

(φ_1) there exist constants $\sigma, \rho > 0$ such that $J|_{\partial B_\rho \cap X_1} \leq \sigma$, where $B_\rho = \{u \in X \mid \|u\| < \rho\}$, ∂B_ρ denotes the boundary of B_ρ ;

(φ_2) there exist $e \in B_\rho \cap X_1$ and a constant $\omega > \sigma$ such that $J|_{e+X_2} \geq \omega$.

Then J possesses a critical value $c \geq \omega$ and

$$c = \inf_{h \in \Phi} \max_{u \in B_\rho \cap X_1} J(h(u)), \quad (2.10)$$

where $\Phi = \{h \in C(\overline{B}_\rho \cap X_1, X) \mid h|_{\partial B_\rho \cap X_1} = id\}$.

Remark 2.3. As shown in [14], a deformation lemma can be proved with the weaker condition (C) replacing the usual (PS) condition, and it turns out that the saddle point theorem holds under condition (C).

3. Proofs of the Main Results

In order to prove Theorem 1.1, we need to prove the following lemma.

Lemma 3.1. *Assume that conditions (F_1) , (F_2) , and (P_3) hold. Then the functional J (see (2.1)) satisfies the (PS) condition; that is, for any sequence $\{u_m\}$ such that $J(u_m)$ is bounded and $J'(u_m) \rightarrow 0$ as $m \rightarrow \infty$, there exists a subsequence of $\{u_m\}$ which is convergent in $\mathbb{R}^k$.*

Proof. First suppose that (F₁), (F₂)(i), and (P₃) hold. Recall that $\mathbf{R}^k$ is a finite dimensional Hilbert space. Consequently, in order to prove that J satisfies the (PS) condition, we only need to prove that $\{u_m\}$ is bounded. Let $\{u_m\}$ be a sequence in $\mathbf{R}^k$ such that $J(u_m)$ is bounded and $J'(u_m) \rightarrow 0$ as $m \rightarrow \infty$. Then there exist $C_3 > 0$ and $m_0 \in \mathbf{Z}[1]$ such that

$$|J(u_m)| \leq C_3, \quad |(J'(u_m), h)| \leq \|h\| \quad (3.1)$$

for all $m > m_0$, $h \in \mathbf{R}^k$.

Since M is singular and indefinite, we write $u_m = u_m^+ + u_m^0 + u_m^-$ with $u_m^* \in R^*$, where $*$ = +, 0, -, respectively. By (F₁), (2.9), and Hölder's inequality ($p = 1/\theta, q = 1/(1 - \theta)$), we have

$$\begin{aligned} & \left| \sum_{t=1}^k f(t, u_m(t)) \cdot (u_m^-(t) - u_m^+(t)) \right| \\ & \leq \sum_{t=1}^k \left(C_1 |u_m^0(t) + (u_m^+(t) + u_m^-(t))|^\theta + C_2 \right) \cdot |u_m^-(t) - u_m^+(t)| \\ & \leq 2C_1 \|u_m^0\|_\infty^\theta \sum_{t=1}^k |u_m^-(t) - u_m^+(t)| + 2C_1 \sum_{t=1}^k |u_m^+(t) + u_m^-(t)|^\theta |u_m^-(t) - u_m^+(t)| \\ & \quad + C_2 \sum_{t=1}^k |u_m^-(t) - u_m^+(t)| \leq \frac{2C_1^2 k}{\tau_1} \|u_m^0\|_\infty^{2\theta} + \frac{\tau_1}{2} \|u_m^- - u_m^+\|^2 \\ & \quad + 2C_1 \|u_m^+ + u_m^-\|_1^\theta \|u_m^- - u_m^+\|_q + C_2 \sqrt{k} \|u_m^- - u_m^+\| \\ & \leq \frac{2C_1^2 k}{\tau_1} \|u_m^0\|_\infty^{2\theta} + \frac{\tau_1}{2} \|u_m^- - u_m^+\|^2 + 2C_1 k \|u_m^+ + u_m^-\|^\theta \|u_m^- - u_m^+\| \\ & \quad + C_2 \sqrt{k} \|u_m^- - u_m^+\|, \end{aligned} \quad (3.2)$$

where $\tau_1 = \min\{\lambda_1, \mu_1\}$. On the other hand, by the fact that u_m^+ and u_m^- are mutually orthogonal, one has $\|u_m^- - u_m^+\| = \|u_m^+ + u_m^-\|$. Hence we have

$$\begin{aligned} & \left| \sum_{t=1}^k f(t, u_m(t)) \cdot (u_m^-(t) - u_m^+(t)) \right| \leq \frac{2C_1^2 k}{\tau_1} \|u_m^0\|_\infty^{2\theta} + \frac{\tau_1}{2} \|u_m^+ + u_m^-\|^2 \\ & \quad + 2C_1 k \|u_m^+ + u_m^-\|^{\theta+1} + C_2 \sqrt{k} \|u_m^+ + u_m^-\|. \end{aligned} \quad (3.3)$$

Similarly to (3.3), we have

$$\begin{aligned} \left| \sum_{t=1}^k \left[F(t, u_m(t)) - F(t, u_m^0(t)) \right] \right| &\leq \frac{2C_1^2 k}{\mu_s} \|u_m^0\|^{2\theta} + \frac{\mu_s}{2} \|u_m^+ + u_m^-\|^2 \\ &\quad + 2C_1 k \|u_m^+ + u_m^-\|^{\theta+1} + C_2 \sqrt{k} \|u_m^+ + u_m^-\|. \end{aligned} \quad (3.4)$$

Take $h = u_m^- - u_m^+$ in (2.2). Then

$$-(Mu_m, u_m^- - u_m^+) = -(J'(u_m), u_m^- - u_m^+) + (\eta, u_m^- - u_m^+) - \sum_{t=1}^k f(t, u_m(t)) (u_m^-(t) - u_m^+(t)). \quad (3.5)$$

We know

$$-(Mu_m, u_m^- - u_m^+) \geq \lambda_1 \|u_m^+\|^2 + \mu_1 \|u_m^-\|^2 \geq \tau_1 \|u_m^+ + u_m^-\|^2. \quad (3.6)$$

Thus, by (3.1) and (3.3), we have

$$\begin{aligned} \tau_1 \|u_m^+ + u_m^-\|^2 &\leq (1 + \|\eta\|) \|u_m^+ + u_m^-\| + \frac{2C_1^2 k}{\tau_1} \|u_m^0\|^{2\theta} + \frac{\tau_1}{2} \|u_m^+ + u_m^-\|^2 \\ &\quad + 2C_1 k \|u_m^+ + u_m^-\|^{\theta+1} + C_2 \sqrt{k} \|u_m^+ + u_m^-\|, \end{aligned} \quad (3.7)$$

that is,

$$\frac{\tau_1}{2} \|u_m^+ + u_m^-\|^2 - 2C_1 k \|u_m^+ + u_m^-\|^{\theta+1} - (1 + \|\eta\| + C_2 \sqrt{k}) \|u_m^+ + u_m^-\| \leq \frac{2C_1^2 k}{\tau_1} \|u_m^0\|^{2\theta}. \quad (3.8)$$

It follows from (3.8) and $1/2 \leq \theta < 1$ that

$$\|u_m^+ + u_m^-\| \leq C_4 \|u_m^0\|^\theta + C_5 \quad (3.9)$$

for all $m > m_0$ and some positive constants C_4, C_5 .

By (3.1), (2.9), (3.4), and (3.9), we have

$$\begin{aligned}
C_3 &\geq J(u_m) = \frac{1}{2}(Mu_m, u_m) + (\eta, u_m) - \sum_{t=1}^k F(t, u_m(t)) \\
&\geq \frac{1}{2}(\lambda_1 \|u_m^+\|^2 - \mu_s \|u_m^-\|^2) - \|\eta\|(\|u_m^0\| + \|u_m^+ + u_m^-\|) \\
&\quad - \sum_{t=1}^k (F(t, u_m(t)) - F(t, u_m^0(t))) - \sum_{t=1}^k F(t, u_m^0(t)) \\
&\geq -\frac{\mu_s}{2} \|u_m^+ + u_m^-\|^2 - \|\eta\| \|u_m^0\| - \frac{2C_1^2 k}{\mu_s} \|u_m^0\|^{2\theta} - \frac{\mu_s}{2} \|u_m^+ + u_m^-\|^2 \\
&\quad - 2C_1 k \|u_m^+ + u_m^-\|^{\theta+1} - (\|\eta\| + C_2 \sqrt{k}) \|u_m^+ + u_m^-\| - \sum_{t=1}^k F(t, u_m^0(t)) \\
&\geq -\mu_s (C_4^2 \|u_m^0\|^{2\theta} + 2C_4 C_5 \|u_m^0\|^\theta + C_5^2) - \|\eta\| \|u_m^0\| - \frac{2C_1^2 k}{\mu_s} \|u_m^0\|^{2\theta} \\
&\quad - 2C_1 k (C_4 \|u_m^0\|^\theta + C_5)^{\theta+1} - (\|\eta\| + C_2 \sqrt{k}) (C_4 \|u_m^0\|^\theta + C_5) - \sum_{t=1}^k F(t, u_m^0(t)) \\
&= -\left(\mu_s C_4^2 + \frac{2C_1^2 k}{\mu_s}\right) \|u_m^0\|^{2\theta} - \|\eta\| \|u_m^0\| - 2C_1 k (C_4 \|u_m^0\|^\theta + C_5)^{\theta+1} \\
&\quad - 2\mu_s C_4 C_5 \|u_m^0\|^\theta - (\|\eta\| + C_2 \sqrt{k}) (C_4 \|u_m^0\|^\theta + C_5) - \mu_s C_5^2 - \sum_{t=1}^k F(t, u_m^0(t)).
\end{aligned} \tag{3.10}$$

Since $1/2 \leq \theta < 1$, we deduce

$$-C_3 \leq \|u_m^0\|^{2\theta} \left(\|u_m^0\|^{-2\theta} \sum_{t=1}^T F(t, u_m^0(t)) + C_6 \right) + C_7 \tag{3.11}$$

for all $m > m_0$ and some positive constants C_6, C_7 . The above inequality and (F₂)(i) imply that $\{u_m^0(t)\}$ is bounded. Then it follows from (3.9) that $\{u_m^+ + u_m^-\}$ is bounded. Thus we conclude that $\{u_m\}$ is bounded, and the (PS) condition is verified.

Now, suppose that (F₁), (F₂)(ii), and (P₃) hold. By a similar argument as above, we know also that J satisfies the (PS) condition. The proof is complete. $\square$

Proof of Theorem 1.1. Assume that (F₁), (F₂)(i), and (P₁) hold. The proof for the case when (F₁), (F₂)(ii), and (P₁) hold is similar and will be omitted here. Since $p(t)$ is nonzero for each $t \in \mathbf{Z}$, the singular symmetric matrix M has at least one nonzero eigenvalue and we will give the proof in three cases.

- (i) Suppose that the matrix M is singular and indefinite. Then $\mathbf{R}^k$ has the direct sum decomposition: $\mathbf{R}^k = R^+ \oplus R^0 \oplus R^-$. In view of Lemma 3.1, we only to check that conditions (φ_1) and (φ_2) in the saddle point theorem hold. To this end, let $X_1 =$

$R^+ \oplus R^0$, $X_2 = R^-$. For any $u = u^+ + u^0 \in X_1$, by (F₁), (2.9), and the mean value theorem, we have

$$\begin{aligned}
J(u) &\geq \frac{\lambda_1}{2} \|u^+\|^2 - \|\eta\| (\|u^0\| + \|u^+\|) - \sum_{t=1}^k f(t, u^0(t) + \theta_1(t) u^+(t)) \cdot u^+(t) - \sum_{t=1}^k F(t, u^0(t)) \\
&\geq \frac{\lambda_1}{2} \|u^+\|^2 - \|\eta\| (\|u^0\| + \|u^+\|) - 2C_1 \sqrt{k} \|u^0\|^\theta \|u^+\| \\
&\quad - 2C_1 k \|u^+\|^{\theta+1} - C_2 \sqrt{k} \|u^+\| - \sum_{t=1}^k F(t, u^0(t)) \\
&\geq \frac{\lambda_1}{2} \|u^+\|^2 - \|\eta\| (\|u^0\| + \|u^+\|) - \frac{4C_1^2 k}{\lambda_1} \|u^0\|^{2\theta} - \frac{\lambda_1}{4} \|u^+\|^2 \\
&\quad - 2C_1 k \|u^+\|^{\theta+1} - C_2 \sqrt{k} \|u^+\| - \sum_{t=1}^k F(t, u^0(t)) \\
&= \frac{\lambda_1}{4} \|u^+\|^2 - 2C_1 k \|u^+\|^{\theta+1} - (\|\eta\| + C_2 \sqrt{k}) \|u^+\| \\
&\quad - \|u^0\|^{2\theta} \left(\|u^0\|^{-2\theta} \sum_{t=1}^k F(t, u^0(t)) + \frac{4C_1^2 k}{\lambda_1} + \|\eta\| \|u^0\|^{-2\theta+1} \right),
\end{aligned} \tag{3.12}$$

where $0 < \theta_1(t) < 1$, $t = 1, 2, \dots, k$. Since $1/2 \leq \theta < 1$ and (F₂)(i), we have

$$J(u) \longrightarrow +\infty \quad \text{as } \|u\| \longrightarrow \infty \text{ in } X_1. \tag{3.13}$$

On the other hand, for any $u = u^- \in X_2$, by (F₁), (2.9), and the mean value theorem, we have

$$\begin{aligned}
J(u) &\leq -\frac{1}{2} \mu_1 \|u\|^2 + \|\eta\| \|u\| - \sum_{t=1}^k f(t, \theta_2(t) u(t)) \cdot u(t) - \sum_{t=1}^k F(t, 0) \\
&\leq -\frac{1}{2} \mu_1 \|u\|^2 + \|\eta\| \|u\| + \sum_{t=1}^k C_1 |u(t)|^{\theta+1} + C_2 \sqrt{k} \|u\| \\
&\leq -\frac{1}{2} \mu_1 \|u\|^2 + C_1 k \|u\|^{\theta+1} + (\|\eta\| + C_2 \sqrt{k}) \|u\|,
\end{aligned} \tag{3.14}$$

where $0 < \theta_2(t) < 1$, $t = 1, 2, \dots, k$. Since $1/2 \leq \theta < 1$, we can obtain

$$J(u) \longrightarrow -\infty \quad \text{as } \|u\| \longrightarrow \infty \text{ in } X_2. \tag{3.15}$$

Let $e = 0$, then it follows from (3.13) and (3.15) that (φ_1) and (φ_2) are satisfied. By the saddle point theorem, J has at least one critical point.

- (ii) Suppose that $0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_l$ are the positive eigenvalues of M and $l = k - 1$. Then $\mathbf{R}^k$ has the direct sum decomposition: $\mathbf{R}^k = R^+ \oplus R^0$, where R^+ and R^0 are defined as in (2.4) and (2.5), respectively. By a similar argument as in the proof of Lemma 3.1, we see that J satisfies the (PS) condition. By (3.13), J is bounded from below. Then, by the least action principle, $c_1 = \inf_{u \in \mathbf{R}^k} J(u)$ is a critical value of J .
- (iii) Suppose that $0 > -\mu_1 \geq -\mu_2 \geq \dots \geq -\mu_s$ are the negative eigenvalues of M and $s = k - 1$. Then $\mathbf{R}^k$ has the direct sum decomposition: $\mathbf{R}^k = R^0 \oplus R^-$. Following almost the same procedure as the proof of Lemma 3.1, we know also that J satisfies the (PS) condition in this case.

For any $u = u^- \in R^-$, (3.15) holds. For any $u = u^0 \in R^0$, since $Mu = 0$, we have

$$J(u) \geq -\|\eta\| \|u^0\| - \sum_{t=1}^k F(t, u^0(t)) = -\|u^0\|^{2\theta} \left(\|u^0\|^{-2\theta} \sum_{t=1}^k F(t, u^0(t)) + \|\eta\| \|u^0\|^{-2\theta+1} \right). \quad (3.16)$$

Due to (F₂)(i) and $1/2 \leq \theta < 1$,

$$J(u) \longrightarrow +\infty \quad \text{as } \|u\| \longrightarrow \infty \text{ in } R^0. \quad (3.17)$$

By the saddle point theorem, there exists at least one critical point of J .

Since J has at least one critical point in all three cases, the BVP (1.1)-(1.2) has at least one solution. $\square$

Proof of Theorem 1.3. Since the matrix M is nonsingular, we will give the proof in three cases.

- (i) Suppose that $0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_l$ and $0 > -\mu_1 \geq -\mu_2 \geq \dots \geq -\mu_s$ are the positive and negative eigenvalues of M , respectively, and $l + s = k$. Then $\mathbf{R}^k$ has the direct sum decomposition: $\mathbf{R}^k = R^+ \oplus R^-$. Denote $\tau_1 = \min\{\lambda_1, \mu_1\}$. It follows from (F₃) that there exists a positive constant a_1 such that

$$|f(t, x)| \leq \frac{1}{4} \tau_1 |x| + a_1 \quad (3.18)$$

for any $(t, x) \in \mathbf{Z}[1, k] \times \mathbf{R}$.

We now prove that the functional J satisfies the (PS) condition. Let $\{u_m\}$ be a sequence in $\mathbf{R}^k$ such that $J(u_m)$ is bounded and $J'(u_m) \rightarrow 0$ as $m \rightarrow \infty$. Write $u_m = u_m^+ + u_m^-$, where $u_m^+ \in R^+$, $u_m^- \in R^-$. Similarly to (3.8), we have, by (3.18),

$$\frac{3\tau_1}{4} \|u_m^+ + u_m^-\|^2 \leq (1 + \|\eta\| + a_1 \sqrt{k}) \|u_m^+ + u_m^-\|. \quad (3.19)$$

Thus $\{u_m\}$ is bounded, and the (PS) condition is verified.

For any $u = u^+ \in R^+$, by (3.18) and the mean value theorem, we have

$$\begin{aligned} J(u) &\geq \frac{1}{2}\lambda_1\|u\|^2 - \|\eta\|\|u\| - \sum_{t=1}^k f(t, \theta_3(t)u(t)) \cdot u(t) - \sum_{t=1}^k F(t, 0) \\ &\geq \left(\frac{\lambda_1}{2} - \frac{\tau_1}{4}\right)\|u\|^2 - \|\eta\|\|u\| - a_1\sqrt{k}\|u\|, \end{aligned} \quad (3.20)$$

where $0 < \theta_3(t) < 1$, $t = 1, 2, \dots, k$. For any $u = u^- \in R^-$, Similarly to (3.20), we have

$$J(u) \leq -\left(\frac{\mu_1}{2} - \frac{\tau_1}{4}\right)\|u\|^2 + \|\eta\|\|u\| + a_1\sqrt{k}\|u\|. \quad (3.21)$$

Let $e = 0$, then it follows from (3.20) and (3.21) that (φ_1) and (φ_2) are satisfied. By the saddle point theorem, J has at least one critical point.

- (ii) Suppose that $0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_l$ are the positive eigenvalues of M and $l = k$. Then E has the direct sum decomposition: $\mathbf{R}^k = R^+$. It follows from (3.20) that J satisfies the (PS) condition and is bounded from below. Then, by the least action principle, $c_2 = \inf_{u \in \mathbf{R}^k} J(u)$ is a critical value of J .
- (iii) Suppose that $0 > -\mu_1 \geq -\mu_2 \geq \dots \geq -\mu_s$ are the negative eigenvalues of M and $s = k$. Then $\mathbf{R}^k$ has the direct sum decomposition: $\mathbf{R}^k = R^-$. It follows from (3.21) that J satisfies the (PS) condition and is bounded from above. Then, by the least action principle, $c_3 = -\inf_{u \in \mathbf{R}^k} J(u)$ is a critical value of $-J$.

Since J has a critical point in all three cases, the BVP (1.1)-(1.2) has at least one solution. $\square$

Proof of Corollary 1.4. This is immediate from Theorem 1.3. $\square$

The following lemma is useful for proving Theorem 1.6 and Corollary 1.8.

Lemma 3.2. *Under the condition (F_5) , the functional J satisfies condition (C) ; that is, for any sequence $\{u_m\}$ such that $J(u_m)$ is bounded and $\|J'(u_m)\|(1 + \|u_m\|) \rightarrow 0$ as $m \rightarrow \infty$, there exists a subsequence of $\{u_m\}$ which is convergent in $\mathbf{R}^k$.*

Proof. First suppose that $(F_5)(i)$ holds. Let $\{u_m\}$ be a sequence in $\mathbf{R}^k$ such that $J(u_m)$ is bounded and $\|J'(u_m)\|(1 + \|u_m\|) \rightarrow 0$ as $m \rightarrow \infty$. Then there exists a constant $L_1 > 0$ such that

$$|J(u_m)| \leq L_1, \quad \|J'(u_m)\|(1 + \|u_m\|) \leq L_1 \quad (3.22)$$

for all $m \in \mathbf{Z}[1]$. Hence, we have

$$\begin{aligned} -3L_1 &\leq -\|J'(u_m)\| \cdot (1 + \|u_m\|) - 2J(u_m) \leq (J'(u_m), u_m) - 2J(u_m) \\ &= \sum_{t=1}^k [2F(t, u_m(t)) - (f(t, u_m(t)) + \eta(t))u_m(t)]. \end{aligned} \quad (3.23)$$

Then, $\{u_m\}$ is bounded. In fact, if $\{u_m\}$ is unbounded, there exist a subsequence of $\{u_m\}$ (still denoted by $\{u_m\}$) and $t_0 \in \mathbf{Z}[1, k]$ such that $|u_m(t_0)| \rightarrow \infty$ as $m \rightarrow \infty$. By (F₅)(i), we have

$$2F(t_0, u_m(t_0)) - (f(t_0, u_m(t_0)) + \eta(t_0))u_m(t_0) \rightarrow -\infty \quad \text{as } m \rightarrow \infty. \quad (3.24)$$

The continuity of $2F(t, x) - (f(t, x) + \eta(t))x$ with respect to x and (F₅)(i) implies that there exists a constant $L_2 > 0$ such that for any $(t, x) \in \mathbf{Z}[1, k] \times \mathbf{R}$, $2F(t, x) - (f(t, x) + \eta(t))x \leq L_2$. Then, we get

$$\begin{aligned} & \sum_{t=1}^k [2F(t, u_m(t)) - (f(t, u_m(t)) + \eta(t))u_m(t)] \\ & \leq 2F(t_0, u_m(t_0)) - (f(t_0, u_m(t_0)) + \eta(t_0))u_m(t_0) + (k-1)L_2. \end{aligned} \quad (3.25)$$

Thus,

$$\sum_{t=1}^k [2F(t, u_m(t)) - (f(t, u_m(t)) + \eta(t))u_m(t)] \rightarrow -\infty \quad \text{as } m \rightarrow \infty, \quad (3.26)$$

which contradicts (3.23). Therefore, $\{u_m\}$ is bounded in $\mathbf{R}^k$ and J satisfies condition (C).

Now, suppose that (F₅)(ii) holds. By a similar argument as above, we know also that J satisfies condition (C). The proof is complete. $\square$

Proof of Theorem 1.6. Assume that (F₄), (F₅)(i), and (P₃) hold. The proof for the case when (F₄), (F₅)(ii), and (P₃) hold is similar and will be omitted here. Due to (P₃), $\mathbf{R}^k$ has the direct sum decomposition: $\mathbf{R}^k = R^+ \oplus R^0 \oplus R^-$, where R^+ , R^- and R^0 are defined as in (2.4) and (2.5), respectively. We claim that for every $t \in \mathbf{Z}[1, k]$,

$$F(t, x) - \eta(t)x \rightarrow -\infty \quad \text{as } |x| \rightarrow \infty. \quad (3.27)$$

Indeed, according to (F₅)(i), we can obtain that for any given $\varepsilon > 0$, there exists a positive constant G_1 such that for $t \in \mathbf{Z}[1, k]$, $|x| > G_1$,

$$2F(t, x) - (f(t, x) + \eta(t))x < -\frac{1}{\varepsilon}. \quad (3.28)$$

Obviously,

$$(f(t, sx) + \eta(t))sx - 2F(t, sx) > \frac{1}{\varepsilon}, \quad (3.29)$$

where $s > 1$, $|x| > G_1$. We have

$$\frac{d}{ds} \left(\frac{F(t, sx) - \eta(t)sx}{s^2} \right) = \frac{(f(t, sx) + \eta(t))sx - 2F(t, sx)}{s^3} \geq \frac{d}{ds} \left(-\frac{1}{2\varepsilon s^2} \right). \quad (3.30)$$

By integrating both sides of the above inequality from 1 to s , we get

$$\frac{F(t, sx) - \eta(t)sx}{s^2} - F(t, x) + \eta(t)x \geq -\frac{1}{2\varepsilon s^2} + \frac{1}{2\varepsilon}. \quad (3.31)$$

Let $s \rightarrow +\infty$ in the above inequality, and it follows from (F_4) that

$$F(t, x) - \eta(t)x \leq -\frac{1}{2\varepsilon} \quad (3.32)$$

for $t \in \mathbf{Z}[1, k]$, $|x| > G_1$. From the arbitrariness of ε , we can conclude that (3.27) holds, proving our claim.

Now we prove

$$J(u) \rightarrow +\infty \quad \text{as } \|u\| \rightarrow \infty \text{ in } R^+ \oplus R^0. \quad (3.33)$$

If (3.33) does not hold, there exist a constant $L_3 > 0$ and a sequence $\{u_m\}$ in $R^+ \oplus R^0$ such that $\|u_m\| \rightarrow \infty$ as $m \rightarrow \infty$ and

$$J(u_m) \leq L_3 \quad (3.34)$$

for all m . Since $\|u_m\| \rightarrow \infty$ as $m \rightarrow \infty$, there exist a subsequence of $\{u_m\}$ (still denoted by $\{u_m\}$) and $t_0 \in \mathbf{Z}[1, k]$ such that $|u_m(t_0)| \rightarrow \infty$ as $m \rightarrow \infty$. By (3.27), we have

$$F(t_0, u_m(t_0)) - \eta(t_0)u_m(t_0) \rightarrow -\infty \quad \text{as } m \rightarrow \infty. \quad (3.35)$$

The continuity of $F(t, x) - \eta(t)x$ with respect to x and (3.27) implies that there exists a constant $L_4 > 0$ such that for any $(t, x) \in \mathbf{Z}[1, k] \times \mathbf{R}$, $F(t, x) - \eta(t)x \leq L_4$. Then, we get

$$J(u_m) \geq \frac{\lambda_1}{2} \|u_m^+\|^2 - \sum_{t=1}^k [F(t, u_m(t)) - \eta(t)u_m(t)] \geq -F(t_0, u_m(t_0)) + \eta(t_0)u_m(t_0) - (k-1)L_4. \quad (3.36)$$

Thus,

$$J(u_m) \rightarrow +\infty \quad \text{as } m \rightarrow \infty, \quad (3.37)$$

which contradicts (3.34). Hence, (3.33) follows.

On the other hand, by (F_4) , there exists a positive constant a_2 such that for any $(t, x) \in \mathbf{Z}[1, k] \times \mathbf{R}$,

$$|F(t, x)| \leq \frac{1}{4}\mu_1|x|^2 + a_2. \quad (3.38)$$

Then we have

$$J(u) \leq -\frac{\mu_1}{2} \|u^-\|^2 + \|\eta\| \|u^-\| + \frac{\mu_1}{4} \|u^-\|^2 + a_2 k \quad (3.39)$$

for all $u = u^- \in R^-$. Thus, we can conclude that

$$J(u) \longrightarrow -\infty \quad \text{as } \|u\| \longrightarrow \infty \text{ in } R^-. \quad (3.40)$$

It follows from (3.33) and (3.40) that J satisfies conditions (φ_1) and (φ_2) . Hence Theorem 1.6 follows from Lemma 3.2, the saddle point theorem, and Remark 2.3. $\square$

The proof of Corollary 1.8 is similar to that of Theorem 1.6 and is omitted.

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