Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2009, Article ID 407913, 22 pages
doi:10.1155/2009/407913

Research Article

Impulsivity in Binary Choices and the Emergence of Periodicity

Gian Italo Bischi,[1] Laura Gardini,[1] and Ugo Merlone[2]

[1] *Department of Economics and Quantitative Methods, Faculty of Economics and Business, University of Urbino, 61029 Urbino, Italy*
[2] *Department of Statistics and Applied Mathematics "de Castro", Corso Unione Sovietica 218 bis, 10134 Torino, Italy*

Correspondence should be addressed to Gian Italo Bischi, gian.bischi@uniurb.it

Received 17 March 2009; Accepted 15 June 2009

Recommended by Xue-Zhong He

Binary choice games with externalities, as those described by Schelling (1973, 1978), have been recently modelled as discrete dynamical systems (Bischi and Merlone, 2009). In this paper we discuss the dynamic behavior in the case in which agents are impulsive; that is; they decide to switch their choices even when the difference between payoffs is extremely small. This particular case can be seen as a limiting case of the original model and can be formalized as a piecewise linear discontinuous map. We analyze the dynamic behavior of this map, characterized by the presence of stable periodic cycles of any period that appear and disappear through border-collision bifurcations. After a numerical exploration, we study the conditions for the creation and the destruction of periodic cycles, as well as the analytic expressions of the bifurcation curves.

1. Introduction

In many situations the consequences of the choices of an actor are affected by the actions of other actors, that is, the population of agents that form the social system as a whole. Systems characterized by such a trade-off between individual choices and collective behavior are ubiquitous and have been studied extensively in different fields. Among the different contributions the seminal work by Schelling [1] stands out on its own as it provides a simple model which can qualitatively explain a wealth of everyday life situations. Indeed, the model proposed in Schelling [1], and in the successive generalizations, is general enough to include several games, such as the well known n-players prisoner's dilemma or the minority games (e.g., [2]).

As remarked by Granovetter [3], Schelling [1] does not specify explicitly the time sequence, even if a dynamic adjustment is implicitly assumed in order to both analyze

the time evolution of the fraction of agents that make a binary choice, and provides arguments about existence and stability of equilibrium values. However, this implicit dynamic adjustment fails in describing some important phenomena observed in many real situations, such as oscillations caused by overshooting (or overreaction) of the actors involved in choices repeated over time, as well as problems of equilibrium selection when nonmonotonic payoff curves lead to the presence of several stable equilibria.

Recently Bischi and Merlone [4] presented an explicit discrete-time dynamic model which is based on the qualitative properties described by Schelling [1] and simulates an adaptive adjustment process of repeated binary choices of boundedly rational agents with social externalities. This permitted them to study the effects on the dynamic behavior of different kinds of payoff functions as well as the qualitative changes of the asymptotic dynamics induced by variations of the main parameters of the model. Moreover, even with monotonic payoff functions, their model allows them to detect the occurrence of oscillatory time series (periodic or chaotic), an outcome often observed in real economic and social systems (see, e.g., [5]).

In Bischi and Merlone [4], the adaptive process by which agents switch their decisions depends on the difference observed between their own payoffs and those associated with the opposite choice in the previous turn, and the switching intensity is modulated by a parameter λ representing the speed of reaction of agents—small values of λ imply more inertia, while, on the contrary, larger values of λ imply more reactive agents.

In this paper we reconsider the model presented in Bischi and Merlone [4] in order to study the dynamics when λ tends to infinity, that is, agents immediately switch their strategies even when the difference between payoffs is extremely small. This may be interpreted by saying that agents are impulsive (see, e.g., [6] for the meaning of impulsivity in the psychological and psychiatric literature) or it may be referred to the case of an automatic device used to determine a sudden switching, between two different kinds of behavior, according to a discrepancy observed between payoff functions, whatever is the measure of such a discrepancy.

From the point of view of the mathematical properties of the model, the limiting case obtained by setting the parameter λ to infinity corresponds to a change of the iterated map from continuous to discontinuous. This gives us the opportunity to investigate some particular properties of discontinuous dynamical systems. In this paper we numerically show the gradual changes induced by increasing values of λ, and in the limiting case we show that the asymptotic dynamics is characterized by the existence of periodic cycles of any period. Moreover, we explain how the periods observed depend on the values of the parameters according to analytically determined regions of the parameters' space, called regions of periodicity, or "periodicity tongues", in literature. These tongues are infinitely many, and their boundaries can be described by analytic equations obtained through the study of border collision bifurcations that cause the creation and destruction of periodic orbits.

The structure of the paper is the following. In Section 2 we summarize the formal dynamic model and its properties when the payoff functions have one single intersection. In Section 3 we analyze the system dynamics as the speed of reaction increases, as the dynamic behavior changes from regular (periodic) to chaotic, and give a formal analysis of the bifurcation curves in the asymptotic case. In Section 4 we analyze the system in the limiting case and provide the analytic expression of the bifurcation curves that bound the periodicity tongues. In fact we will see that whichever are the parameter values only one invariant attracting set can exist: a stable cycle, whose period may be any integer number, and also several different cycles with the same periods can exist. We will show that the analysis

of the limiting case (a discontinuous function) is very informative also of the dynamics occurring in the original piecewise continuous model. The last section is devoted to some concluding remarks.

2. The Dynamic Model

Following Schelling [1], Bischi and Merlone [4] propose a model where a population of players is assumed to be engaged in a game where they have to choose between two strategies, say A and B, respectively. They assume that the set of players is normalized to the interval $[0,1]$ and denote by the real variable $x \in [0,1]$ the fraction of players that choose strategy A. Then the payoffs are functions of x, say $A : [0,1] \to \mathbb{R}$, $B : [0,1] \to \mathbb{R}$, where $A(x)$ and $B(x)$ represent the payoff associated to strategies A and B, respectively. Obviously, since binary choices are considered, when fraction x is playing A, then fraction $1-x$ is playing B. As a consequence $x = 0$ means that the whole population of players is playing B and $x = 1$ means that all the agents are playing A. The basic assumption modeling the dynamic adjustment is the following: *x will increase whenever $A(x) > B(x)$ whereas it will decrease when the opposite inequality holds*.

Consistently with Schelling [1], this assumption, together with the constraint $x \in [0,1]$, implies that equilibria are located either in the points $x = x^*$ such that $A(x^*) = B(x^*)$, or in $x = 0$ (provided that $A(0) < B(0)$) or in $x = 1$ (provided that $A(1) > B(1)$). In the process of repeated binary choices which is considered in Bischi and Merlone [4], the agents update their binary choice at each time period $t = 0,1,2,\ldots$, and x_t represents the number of players playing strategy A at time period t. They assume that at time $(t+1)$ x_t becomes common knowledge, hence each agent is able to compute (or observe) payoffs $B(x_t)$ and $A(x_t)$. Finally, agents are homogeneous and myopic, that is, each of them is interested to increase its own next period payoff. In this discrete-time model, if at time t x_t players are playing strategy A and $A(x_t) > B(x_t)$ then a fraction of the $(1-x_t)$ agents that are playing B will switch to strategy A in the following turn; analogously, if $A(x_t) < B(x_t)$ then a fraction of the x_t players that are playing A will switch to strategy B. In other words, at any time period t agents decide their action for period $t+1$ comparing $B(x_t)$ and $A(x_t)$ according to

$$x_{t+1} = f(x_t) = \begin{cases} x_t + \delta_A g[\lambda(A(x_t) - B(x_t))](1-x_t), & \text{if } A(x_t) \geq B(x_t), \\ x_t - \delta_B g[\lambda(B(x_t) - A(x_t))]x_t, & \text{if } A(x_t) < B(x_t), \end{cases} \tag{2.1}$$

where δ_A, $\delta_B \in [0,1]$ are propensities to switch to the other strategy; $g : \mathbb{R}_+ \to [0,1]$ is a continuous and increasing function such that $g(0) = 0$ and $\lim_{z\to\infty} g(z) = 1$, λ is a positive real number. The function g modulates how the fraction of switching agents depends on the difference between the previous turn payoffs; the parameters δ_A and δ_B represent how many agents may switch to A and B, respectively (when $\delta_A = \delta_B$, there are no differences in the propensity to switch to either strategies) and the parameter λ represents the switching intensity (or speed of reaction) of agents as a consequence of the difference between payoffs. In other words, small values of λ imply more inertia, that is, anchoring attitude, of the actors involved, while, on the contrary, larger values of λ can be interpreted in terms of impulsivity. In fact, according to the Clinical Psychology literature [7] impulsivity can be separated in different components such as acting on the spur of the moment and lack of planning.

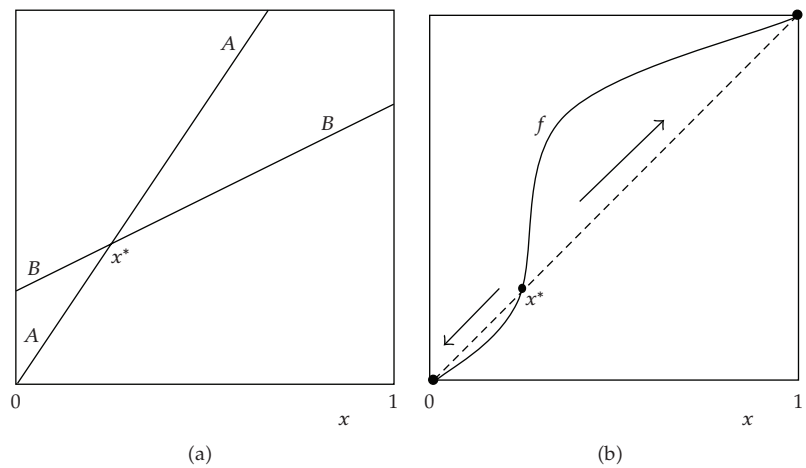

Figure 1: (a) Payoff functions $A(x) = 1.5x$ and $B(x) = 0.25 + 0.5x$. (b) Function f obtained with the same payoff functions as in (a) and parameters $\delta_A = 0.3$, $\delta_B = 0.7$, $\lambda = 20$ with switching function $g(\cdot) = (2/\pi)\arctan(\cdot)$.

Notice that if both the payoff functions are continuous, then also the map f is continuous in the whole interval [0,1], and its graph is contained in the strip bounded by two lines, being $(1-\delta_B)x \leqslant f(x) \leqslant (1-\delta_A)x + \delta_A$. However, even if $B(x)$ and $A(x)$ are smooth functions, the map f in general is not smooth in the considered interval, since f is not differentiable where the payoff functions intersect. When studying the dynamics, this does not allow us to use the usual approach which relies on the first derivative value in the equilibrium points. As a consequence an approach based on the left and right side slopes in the neighborhood of any interior fixed point is needed.

In this paper we consider payoff functions $A(x)$ and $B(x)$ with one and only one internal intersection $x^* \in (0,1)$ such that $A(x^*) = B(x^*)$. Assume, first, that the strategy A is preferred at the right of $x^*$and B is preferred at the left, as in Figure 1(a). This situation is discussed in Schelling [1] where a vivid example is also provided by assuming that B stands for carrying a visible weapon, A for going unarmed. One may prefer to be armed if everybody else is, but not if the rest is not (individuals may also be nations). According to the relative position of the two payoff curves, Schelling [1] concludes that there exist two stable equilibria, namely $x = 0$ and $x = 1$, where everybody is choosing B and everybody is choosing A respectively, whereas the inner equilibrium x^* is unstable. The reasons given by Schelling to prove these statements are based on the following arguments: at the equilibrium $x = 0$, where everybody is choosing B, nobody is motivated to choose A because $A(x) < B(x)$ in a right neighborhood of 0, and analogously at $x = 1$, where everybody is choosing A, nobody is motivated to choose B being $B(x) < A(x)$ in a left neighborhood of 1; instead, starting from the inner equilibrium x^*, where both choices coexist, if x is displaced in a right neighborhood of x^* by an exogenous force, there $A(x) > B(x)$ and a further increase of x will be generated by endogenous dynamics, whereas if x is displaced in a left neighborhood of x^*, where $A(x) < B(x)$, then a further decrease of x will be observed. These qualitative arguments of Schelling are confirmed by the following proposition, proved in Bischi and Merlone,[4] (see, also Figure 1(b)).

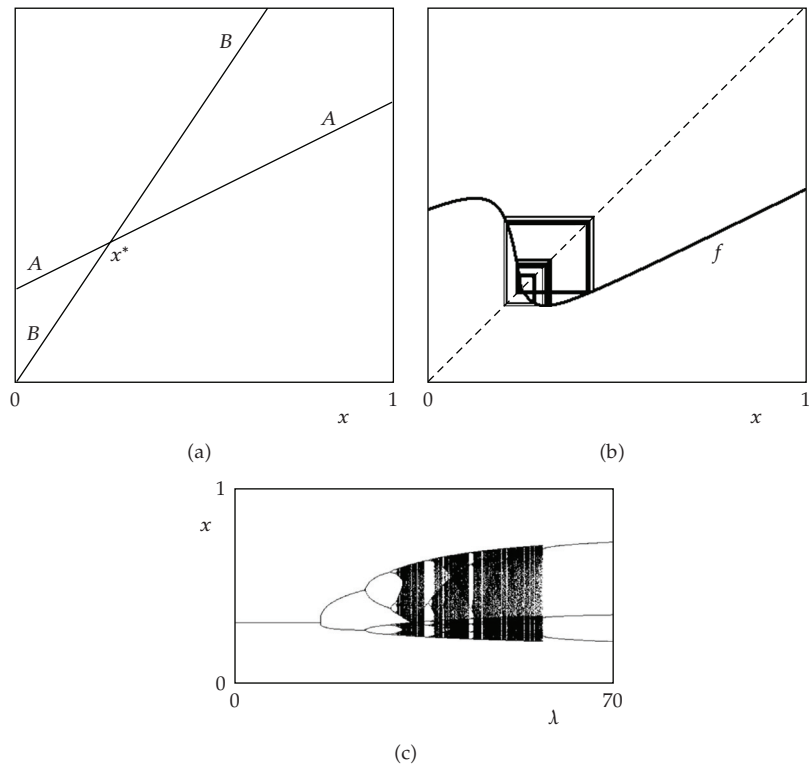

Figure 2: (a) Payoff functions $A(x) = 1.5x$, $B(x) = 0.25 + 0.5x$. (b) Function f obtained with $g(\cdot) = (2/\pi)\arctan(\cdot)$, the same payoffs as in (a) and parameters $\delta_A = \delta_B = 0.5$, $\lambda = 35$. The interior equilibrium is unstable and the generic trajectory converges to the attractor shown around x^*. (c) Bifurcation diagram obtained with the same values of parameters δ and payoff functions as in (b) and bifurcation parameter $\lambda \in (0, 70)$.

Proposition 2.1. *Assume that* $A : [0,1] \to \mathbb{R}$ *and* $B : [0,1] \to \mathbb{R}$ *are continuous functions such that*

(i) $A(0) < B(0)$,

(ii) $A(1) > B(1)$,

(iii) *there exists a unique* $x^* \in (0,1)$ *such that* $A(x^*) = B(x^*)$,

then the dynamical system (2.1) *has three fixed points,* $x = 0$, $x = x^*$, *and* $x = 1$, *where* x^* *is unstable and constitutes the boundary that separates the basins of attraction of the stable fixed points* 0 *and* 1. *All the dynamics generated by* (2.1) *converge to one of the two stable fixed points monotonically, decreasing if* $x_0 < x^*$, *increasing if* $x_0 > x^*$.

The situation is quite different when the payoff functions are switched, that is, $A(0) > B(0)$, and $A(1) < B(1)$, so that A is preferred at the left of the unique intersection x^* and B is preferred at the right (see, Figure 2(a)). In this case we have a unique equilibrium, given by the interior fixed point x^*.

Schelling [1] describes this case as well, and provides some real-life examples of collective binary choices with this kind of payoff functions. Among these examples one concerns the binary choice about whether using the car or not, depending to the traffic

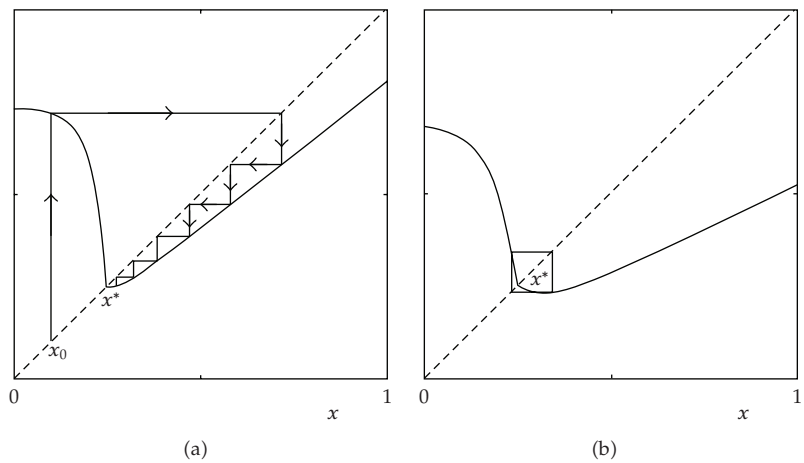

Figure 3: Same payoff functions as in Figure 2 (a) $\delta_A = 0.8$, $\delta_B = 0.2$, $\lambda = 30$, $f'_-(x^*) < -1$ and $0 < f'_+(x^*) < 1$. (b) $\delta_A = 0.8$, $\delta_B = 0.5$, $\lambda = 17$, $f'_-(x^*) < -1$ and $f'_+(x^*) < 0$ such that $f'_-(x^*)f'_+(x^*) > 1$.

congestion. Let A represent the strategy "staying at home" and B "using the car". If many individuals choose B (i.e., x is small), then A is preferred because of traffic congestion, whereas if many choose A (i.e., x is large), then B is preferred as the roads are empty. This situation can be represented with payoff functions as those depicted in Figure 2(a). Schelling [1] gives a qualitative analysis of this scenario and classifies it as being characterized by global stability of the unique equilibrium point. Using the words of Schelling [1, page 401] "If we suppose any kind of damped adjustment, we have a *stable equilibrium* at the intersection". His argument is based on the fact that $A(x) > B(x)$ on the left of x^* (hence increasing x whenever $x < x^*$) and $A(x) < B(x)$ on the right of x^* (hence decreasing x whenever $x > x^*$). While this statement of global stability is true when assuming a continuous time scale, in our discrete-time model we can observe oscillations of x_t.

This is shown in Figure 2(b), obtained with $B(x) = 1.5x$, $A(x) = 0.25 + 0.5x$, $g(\cdot) = (2/\pi)\arctan(\cdot)$, $\delta_B = \delta_A = 0.5$, $\lambda = 35$. In this case, x increases in the right neighborhood of 0, and decreases in the left neighborhood of 1, nevertheless, the unique equilibrium x^* is unstable, and persistent oscillations, periodic or chaotic, are observed around it. The wide spectrum of asymptotic dynamic behaviors that characterize this model is summarized in the bifurcation diagram depicted in Figure 2(c), which is obtained with the same values of parameters δ and payoff functions as in Figure 2(b) and by considering the parameter λ that varies in the range $(0, 70)$. However, Figure 2(c) also shows that for high values of the parameter λ the asymptotic dynamics settle on a given periodic cycle (of period 3 in this case) according to the values of the parameters δ_B and δ_A. This can be easily forecasted from the study of the limiting map (3.1), as shown in Section 4. The occurrence of oscillations is typical of a discrete-time process, and is caused when individual players overshoot, or overreact. For example, in the model of binary choice in car usage described above, overshooting occurs for sufficiently large values of λ (high speed of reaction). This means that whenever traffic congestion is reported, on the following day many people will stay at home; vice versa when no traffic congestion is reported most all of the people will use their car. This kind of reactions generates a typical oscillatory time pattern which is a common situation observed in everyday life (for a discussion about the chaos and complexity

in sociology the reader may refer to [8]). This sort of realistic situation would be completely ruled out when adopting a continuous time one-dimensional dynamic model. However, several of the examples proposed in literature are characterized by decisions that cannot be continuously revised, and lags between observations and decisions are often finite. As a consequence decision processes typically occur in a discrete-time setting. The reader may also refer to Schelling [1, chapter 3] for several qualitative descriptions of overshooting and cyclic phenomena in social systems.

Concerning the condition for stability of the unique equilibrium x^*, when considering the discrete-time dynamic model (2.1) we realize that the slope of the function f at the steady state may be positive or negative. More precisely, assuming that $A(x)$ and $B(x)$ are differentiable functions, since $B'(x^*) \geq A'(x^*)$, both the left and right tangents $f'_-(x^*)$ and $f'_+(x^*)$ are less than 1:

$$\begin{aligned} f'_-(x^*) &= 1 + \delta_A \{ g'[\lambda(A(x^*) - B(x^*))]\lambda(A'_-(x^*) - B'_-(x^*)) \}(1 - x^*) \\ &= 1 + \delta_A \lambda g'[0](A'_-(x^*) - B'_-(x^*))(1 - x^*) < 1, \\ f'_+(x^*) &= 1 - \delta_B \{ g'[\lambda(A(x^*) - B(x^*))]\lambda(A'_-(x^*) - B'_-(x^*)) \} x^* \\ &= 1 - \delta_B [g'(0)\lambda(B'_+(x^*) - A'_+(x^*))] x^* < 1. \end{aligned} \tag{2.2}$$

Hence x^* is stable (indeed, globally stable) as far as $f'_-(x^*) > -1$ and $f'_+(x^*) > -1$, and it may become unstable when at least one of these slopes decrease below -1 (see, e.g., [9, 10]). However, if $f'_-(x^*) < -1$ and $f'_+(x^*) > 0$, then the fixed point is still globally stable, because in this case any initial condition taken on the right of x^* generates a decreasing trajectory that converges to x^*, whereas an initial condition x_0 taken on the left of x^* has the rank-1 image $f(x_0) > x^*$, after which convergence to x^* follows (see, Figure 3(a)). The same argument holds, just reversing left and right, if $0 < f'_-(x^*) < 1$ and $f'_+(x^*) < -1$. Instead, if $f'_-(x^*) < -1$ and $f'_+(x^*) < 0$ then the stability of x^* depends on the product $f'_-(x^*)f'_+(x^*)$, being it stable if $f'_-(x^*)f'_+(x^*) \leq 1$, otherwise it is unstable with a stable cycle (or a chaotic attractor) around it (see, Figure 3(b)).

The results of our discussion can be summarized as follows.

Proposition 2.2. *If $A : [0,1] \to A$ and $B : [0,1] \to A$ are differentiable functions such that*

(i) $A(0) > B(0)$,

(ii) $A(1) < B(1)$,

(iii) *there exists a unique $x^* \in (0,1)$ such that $A(x^*) = B(x^*)$,*

then the dynamical system (2.1) *has only one fixed point at $x = x^*$, which is stable if $f'_-(x^*)f'_+(x^*) \leq 1$.*

It is worth to note that both the slopes $f'_-(x^*)$ and $f'_+(x^*)$ decrease as λ or δ_B or δ_A increase, that is, if the impulsivity of the agents and/or their propensity to switch to the opposite choice increase.

3. Impulsivity in Agents' Reaction

In this section we examine what happens when the parameter λ increases, because this corresponds to the case in which agents are impulsive. Indeed, impulsivity is an important

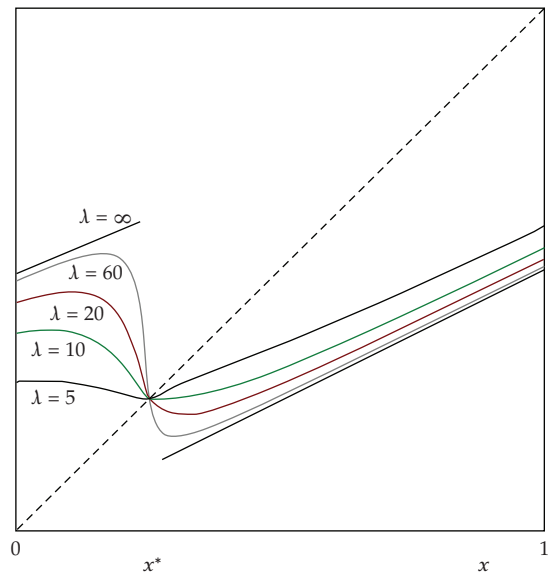

Figure 4: The map f for different values of parameter λ and in the limiting case $\lambda = \infty$.

construct in the psychological and psychiatric literature (see, [6]) and, as a matter of fact, is directly mentioned in the DSM_IV diagnostic criteria. (The Fourth edition of the Diagnostic and Statistical Manual of Mental Disorders presents the descriptions of diagnostic categories of mental disorders coded on different axes. It is used in the United States and around the world by clinicians researchers health insurance companies and policy markers). In order to obtain an insight on the effect of increasing values of λ, in the following we study the limiting case obtained as $\lambda \to +\infty$. This is equivalent to consider $g(x) = 1$ if $x \neq 0$ and $g(x) = 0$ if $x = 0$, as a consequence the switching rate only depends on the sign of the difference between payoffs, no matter how much they differ. In this case the dynamical system assumes the following form

$$x_{t+1} = f_\infty(x_t) = \begin{cases} (1-\delta_A)x_t + \delta_A, & \text{if } B(x_t) < A(x_t), \\ x_t, & \text{if } B(x_t) = A(x_t), \\ (1-\delta_B)x_t, & \text{if } B(x_t) > A(x_t). \end{cases} \tag{3.1}$$

Such a limiting situation may appear as rather extreme, because the map f_∞ becomes discontinuous at the internal equilibria defined by the equation $B(x) = A(x)$. However, a study of the global properties of $f_\infty(x_t)$ gives some insight into the asymptotic properties of the continuos map (2.1) for high values of λ and emphasizes the role of the parameters δ_A and δ_B.

For example, when the payoff functions satisfy the assumptions of Proposition 2.2, increasing values of λ cause the loss of stability of the equilibrium via a flip bifurcation that opens the usual route to chaos through a period doubling cascade. However, as shown in the bifurcation diagram of Figure 2(c), such a chaotic behavior can only be observed for intermediate values of the parameter λ, as the asymptotic dynamics settles on a stable

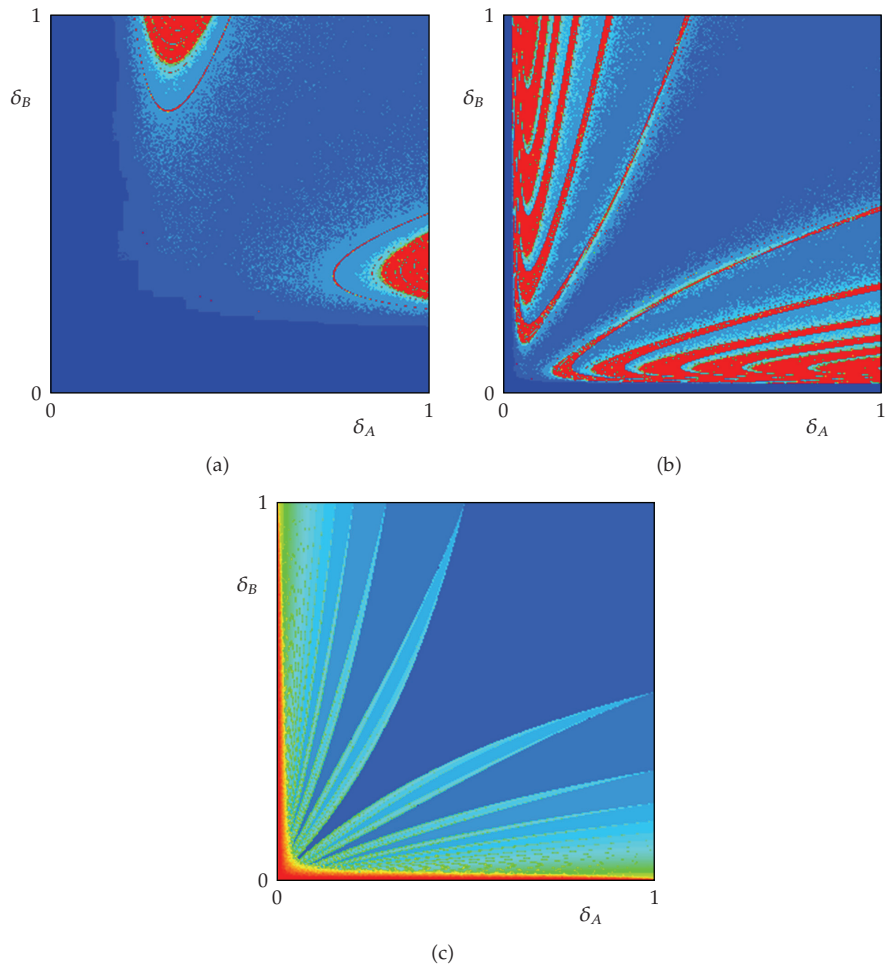

Figure 5: Bifurcation diagrams in the parameters' plane (δ_A, δ_B), with (a) $\lambda = 20$ (b) $\lambda = 60$ (c) $\lambda = 500$. Initial condition: $x = .28$; transient = 4000, iterations = 2000.

periodic cycle for sufficiently high values of λ. This can be numerically observed for many different values of the parameters δ_A and δ_B, the only difference being the period of the stable cycle that prevails at high values of λ. In order to have a complete understanding of the dependence on δ_A and δ_B of the periodicity that characterizes the asymptotic dynamics of impulsive agents, we will study the discontinuous map (3.1), to which the continuous map (2.1) gradually approaches for increasing values of λ, see Figure 4. It is also interesting to observe how the corresponding two-dimensional bifurcation diagram in the parameters' plane (δ_A, δ_B) evolves as λ increases. The different colors shown in the three pictures of Figure 5, obtained with $\lambda = 20$, $\lambda = 60$, $\lambda = 500$ respectively, represent the kind of asymptotic behavior numerically observed: convergence to the stable fixed point or a stable periodic cycle of low period when the parameters are chosen in the blue regions (with different blue shades representing different periods) or the convergence to a chaotic attractor, or periodic cycle of very high period, when the parameters are chosen in the red regions. It can be seen that chaotic behavior becomes quite common at intermediate values of λ whereas periodic

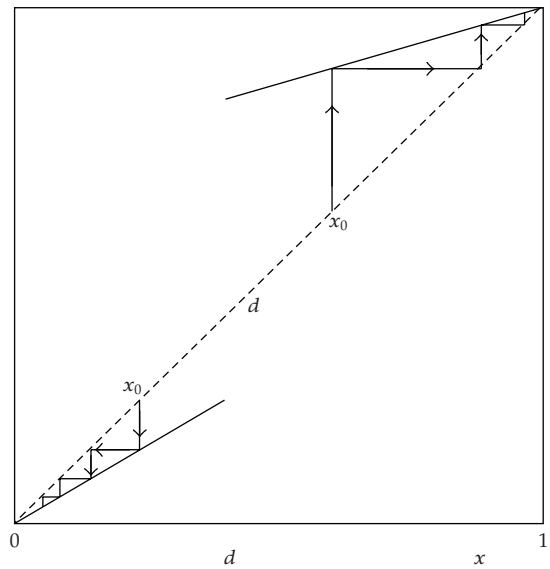

Figure 6: Discontinuous limiting map T_1 with $m_L = 1 - \delta_A = 0.6$, $m_R = 1 - \delta_B = 0.3$, $d = 0.4$.

cycles of low period prevail for very high values of λ. However, it is worth to remark that the set of parameters' values corresponding to chaotic attractors is given by the union of one-dimensional subsets in the two-dimensional space of parameters shown in Figure 5, whereas the regions related to attracting cycles of different periods are open two-dimensional subsets.

3.1. The Analysis of the Impulsive Agents Limit Case

Let us consider the discontinuous limiting map (3.1) in the two cases of payoff curves that intersect in a unique interior point, as described in Propositions 2.1 and 2.2, given by

$$x' = T_1(x) = \begin{cases} (1-\delta_A)x, & \text{if } x < d, \\ x, & \text{if } x = d, \\ (1-\delta_B)x + \delta_B, & \text{if } x > d, \end{cases}$$
$$x' = T_2(x) = \begin{cases} (1-\delta_A)x + \delta_A, & \text{if } x < d \\ x, & \text{if } x = d, \\ (1-\delta_B)x, & \text{if } x > d, \end{cases} \tag{3.2}$$

respectively, where the parameter $d \in (0,1)$ represents the discontinuity point located at the interior equilibrium, that is, $d = x^*$, and the parameters δ_A, δ_B are subject to the constraints $0 \le \delta_A \le 1$, $0 \le \delta_B \le 1$. It is worth noticing that the value of the map in the discontinuity point, $x = d$, is not important for the analysis which follows, therefore it will often be omitted.

Let us first consider the map $T_1(x)$. It has a discontinuity in the point d with an "increasing" jump, that is, $T_1(d^-) < T_1(d^+)$, see Figure 6. In this case, whichever is the position

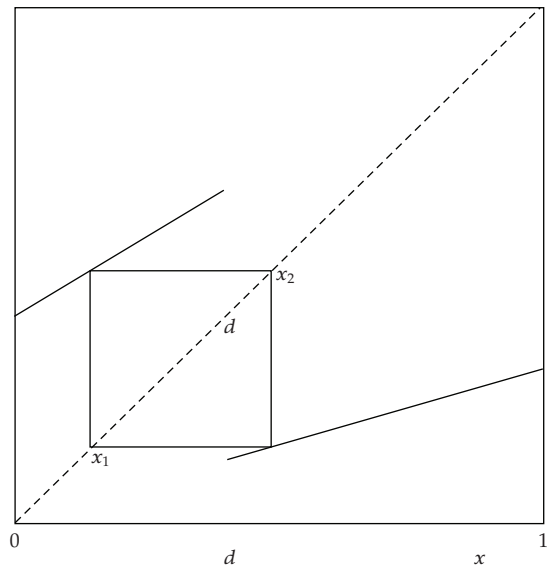

Figure 7: Discontinuous limiting map T_2 with $m_L = 1 - \delta_A = 0.6$, $m_R = 1 - \delta_B = 0.3$, $d = 0.4$.

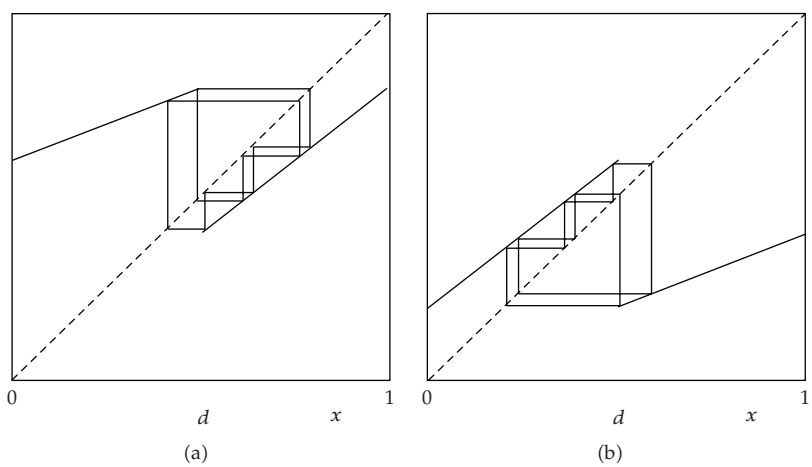

Figure 8: Discontinuous limiting map T_2 with (a) $m_L = 1 - \delta_A = 0.4$, $m_R = 1 - \delta_B = 0.8$, $d = 0.5$; (b) $m_L = 1 - \delta_A = 0.8$, $m_R = 1 - \delta_B = 0.4$, $d = 0.5$.

of the discontinuity point d, the dynamics are very simple: there are two stable fixed points, the boundary steady states $x = 0$ and $x = 1$, with basins of attraction separated by the discontinuity point: any initial condition $x_0 \in (0, d)$ will converge to the fixed point $x = 0$ while any initial condition $x_0 \in (d, 1)$ will generate a trajectory that converges to the fixed point $x = 1$.

Quite different is the situation for the map $T_2(x)$, where the discontinuity point has a "decreasing" jump, that is, $T_2(d^-) > T_2(d^+)$. In this case we will see that periodic cycles of any period may occur. A first example is shown in Figure 7, where a stable cycle of period 2 is shown.

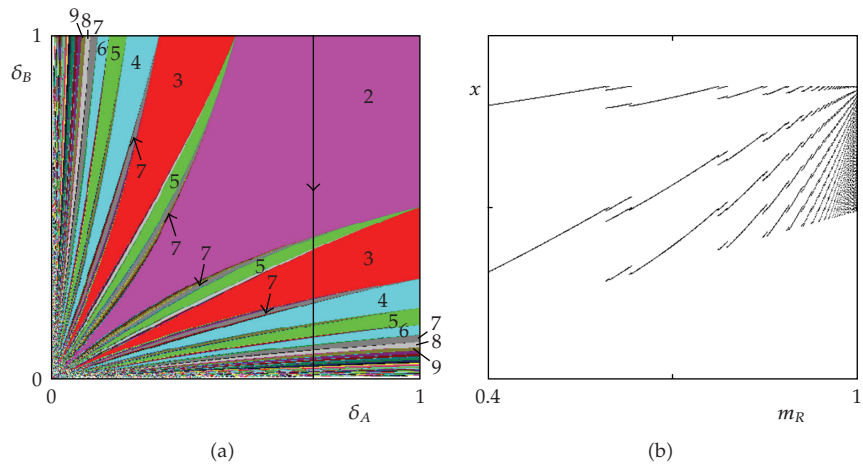

Figure 9: (a) Bifurcation diagrams in the parameters' plane (δ_A, δ_B) with $d = 0.5$. The regions of periodicity are represented by different colors; (b) Bifurcation diagram obtained with a fixed value of $\delta_A = 0.7$ and $m_R = 1 - \delta_B$ anging from 0.4 up to 1.

In Figure 8 we show another example to illustrate that several different cycle of the same period may exist: in Figure 8(a), a 7-cycle (cycle of period 7) has 2 points in the left branch and 5 in the right one, while in Figure 8(b) (obtained with different values of δ_A and δ_B) another 7-cycle has 5 periodic points in the left branch and 2 in the right branch. We will see that it is even easy to find 7-cycles having 3 points on the left branch and 4 in the right one, or 4 points on the left and 3 in the right as well. Indeed, for any given period, all the possible combinations may occur depending on the values of the parameters δ_A and δ_B.

We will also prove that, given δ_A and δ_B, and consequently the slopes of the left branch $m_L = (1 - \delta_A)$ and the right branch $m_R = (1 - \delta_B)$ respectively, the map has only one attractor, a stable cycle of some period k, and any initial condition $x_0 \in [0, 1]$ gives a trajectory converging to such k-cycle. Before giving a proof of this statement, we prefer to show first a numerical computation of a two-dimensional bifurcation diagram, in the plane of the parameters δ_A and δ_B, by using different colors to denote the regions where stable cycles of different periods characterize the asymptotic dynamics. The analytic computation of the bifurcation curves that bound these regions will be given later.

In Figure 9(a) we show the parameter plane (δ_A, δ_B) covered by regions of different colors, often called "periodicity tongues" (due to their particular shape), each characterized by a different period (indicated by a number in some of the tongues, only the larger ones). Figure 9(a) has been obtained fixing the discontinuity point at $d = 0.5$ and changing the parameter δ_A and δ_B between 0 and 1. Figure 9(b) shows a bifurcation diagram which gives the asymptotic behavior of the state variable x as the parameter δ_A is fixed at the value $\delta_A = 0.7$, whereas the parameter δ_B decreases from 0.6 to 0, that is, the parameter m_R increases from 0.4 up to 1. It can be seen that between a cycle of period 2 and a cycle of period 3 there is a region where a cycle of period 5 exists. Moreover, zooming in the scale of the horizontal axis it is possible to see that between the regions of the 2-cycle and the 3-cycle, there exist infinitely many other intervals of existence of cycles of period $2n + 3m$ for any integer $n \geq 1$ and any integer $m \geq 1$. The reason for this will be clarified later. The analytic study of the different regions of periodicity in the parameters' plane (δ_A, δ_B) is the goal of the next sections. However, let us first remark that the existence of the cycles of any period

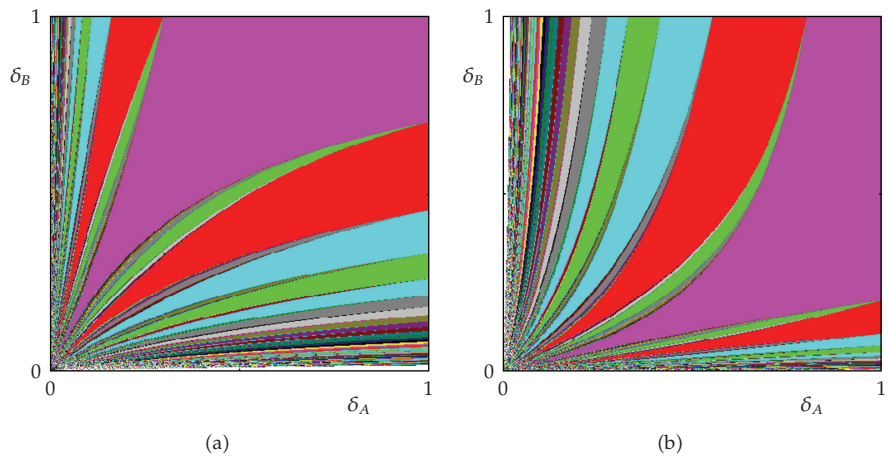

Figure 10: Bifurcation diagrams in the parameters' plane (δ_A, δ_B) obtained with different values of the discontinuity point (a) $d = 0.3$; (b) $d = 0.8$.

is not substantially influenced by the position of the discontinuity point, in the sense that all the bifurcation curves continue to exist for different values of the discontinuity as well, only showing slight modifications of their shape. For example, in Figure 10 we show the periodicity tongues numerically obtained for $d = 0.3$ and for $d = 0.8$ respectively. Notice that the colors of the periodicity tongues are practically the same, and are also similar to those shown in Figure 9, obtained with $d = 0.5$.

3.2. Analytic Expressions of the Boundaries of the Periodicity Tongues

The study of the dynamic properties of iterated piecewise linear maps with one or more discontinuity points has been rising increasing interest in recent years, as witnessed by the high number of papers and books devoted to this topic, both in the mathematical literature (see e.g., [11–16]) and in applications to electrical and mechanical engineering ([17–26]) or to social sciences ([27–31]).

The bifurcations involved in discontinuous maps are often described in terms of the so called *border-collision bifurcations*,that can be defined as due to contacts between an invariant set of a map with the border of its region of definition. The term *border-collision bifurcation* was introduced for the first time by Nusse and Yorke [32] (see also [33]) and it is now widely used in this context. However the study and description of such bifurcations was started several years before by Leonov [34, 35], who described several bifurcations of that kind and gave a recursive relation to find the analytic expression of the sequence of bifurcations occurring in a one-dimensional piecewise linear map with one discontinuity point. His results are also described and used by Mira [36, 37]. Analogously, important results in this field have been obtained by Feigen in 1978, as reported in di Bernardo et al. [14].

We now apply the methods suggested by Leonov [34, 35], see also Mira [36, 37] to the map T_2, in order to show that it is possible to give the analytical equation of the bifurcation curves that we have seen in Figures 9 and 10. As we will see, the boundaries that separate two adjacent periodicity tongues are characterized by the occurrence of a *border-collision*, involving the contact between a periodic point of the cycles existing inside the regions and

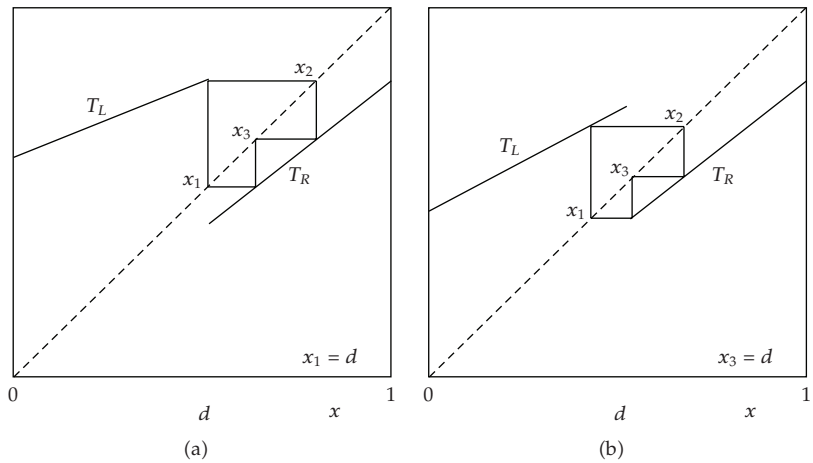

Figure 11: (a) Starting condition for a cycle of period 3 (b) closing condition for the same cycle, both related to border-collision boundaries of the corresponding periodicity tongue in the space of parameters.

the discontinuity point. To better formalize and explain our results it is suitable to label the two components of our map $x' = T_2(x)$ as follows:

$$x' = T_2(x) \begin{cases} T_L(x) = m_L x + (1 - m_L), & \text{if } x < d, \\ x, & \text{if } x = d, \\ T_R(x) = m_R x, & \text{if } x > d, \end{cases} \tag{3.3}$$

where $m_L = (1 - \delta_A)$ and $m_R = (1 - \delta_B)$ are the slopes of the two linear branch on the left and on the right of the discontinuity point $x = d$ respectively.

First of all, notice that all the possible cycles of the map T_2 of period $k > 1$ are always stable. In fact, the stability of a k-cycle is given by the slope (or eigenvalue) of the function $T_2^k = T_2 \circ \cdots \circ T_2$ (k times) in the periodic points of the cycle, which are fixed points for the map T_2^k, so that, considering a cycle with p points on the left side of the discontinuity and $(k - p)$ on the right side, the eigenvalue is given by $m_L^p m_R^{(k-p)}$ which, in our assumptions, is always positive and less than 1.

To study the conditions for the existence of the periodic cycles we limit our analysis to the bifurcation curves of the so-called "principal tongues", or "main tongues" ([14, 17–19, 21, 22]) or "tongues of first degree" ([34–37]), which are the cycles of period k having one point on one side of the discontinuity point and $(k - 1)$ points on the other side (for any integer $k > 1$). Let us begin with the conditions to determine the existence of a cycle of period k having one point on the left side L and $(k - 1)$ points on the right side R. The condition (i.e., the bifurcation) that marks its creation is that *the discontinuity point $x = d$ is a periodic point to which we apply, in the sequence, the maps $T_L, T_R, \ldots, T_R$.* In the qualitative picture shown in Figure 11(a) we show the condition for the creation of a 3-cycle, that is, $k = 3$, given by, $T_R \circ T_R \circ T_L(d) = d$. Then the k-cycle with periodic points $x_1, \ldots, x_k$, numbered with the first point on the left side, satisfies $x_2 = T_L(x_1)$, $x_3 = T_R(x_2), \ldots, x_1 = T_R(x_k)$, and this cycle ends to exist when the last point (x_k) merges with the discontinuity point, that is, $x_k = d$ which may be stated as *the point $x = d$ is a periodic point to which we apply, in the*

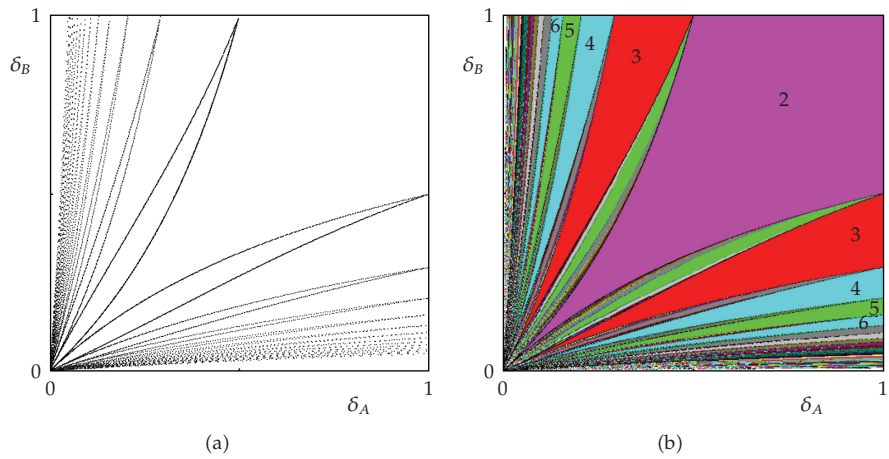

Figure 12: (a) For $d = 0.5$, bifurcation curves of the principal tongues of period k, with $k = 2, \ldots, 15$ in the plane (δ_A, δ_B), $\delta_A = (1 - m_L)$, $\delta_B = (1 - m_R)$; (b) corresponding bifurcation diagrams in the plane (δ_A, δ_B) with colors obtained numerically according to the different periods observed.

sequence, the maps $T_R, T_L, T_R, \ldots, T_R$. In the qualitative picture in Figure 11(b) we show the closing condition related with the 3-cycle, that is, $T_R \circ T_L \circ T_R(d) = d$. Notice that both these conditions express the occurrence of a border collision bifurcation, being related to a contact between a periodic point and the boundary (or border) of the region of differentiability of the corresponding branch of the map. Of course, at the bifurcation the discontinuity point, which is a fixed point according to the definition of the map, represents a stable equilibrium as any trajectory converging to the stable cycle is definitely captured by the fixed point $x = d$ as it coincides with a periodic point at the bifurcation. However, this only happens at the bifurcation points, that is, it represents a structurally unstable situation, as any slight changes of a parameter with respect to the bifurcation value, that is, just before or just after a bifurcation situation, the fixed point $x = d$ is unstable. So, as previously stated, we can neglect such nongeneric stability conditions of the fixed point that only represent bifurcation situations. In general, for a cycle of period $k > 1$, the equation of one boundary of the corresponding region of periodicity is

$$m_L = m_{Li} = \frac{m_R^{(k-1)} - d}{(1-d)m_R^{(k-1)}}, \tag{3.4}$$

while the other boundary, that is, the closure of the periodicity tongue of the same cycle, is given by

$$m_L = m_{Lf} = \frac{m_R^{(k-2)} - d}{(1 - m_R d)m_R^{(k-2)}}. \tag{3.5}$$

The proof of these two equations is reported in the appendix. Thus the k-cycle exists for $m_R^{k-2} > d$ and m_L in the range

$$m_{Li} \leq m_L \leq m_{Lf}, \tag{3.6}$$

and the periodic points of the k-cycle, say $(x_1^*, x_2^*, \ldots, x_k^*)$ where $x_1^* < d$ and $x_i^* > d$ for $i > 1$, can be obtained explicitly as:

$$\begin{aligned}
x_1^* &= \frac{m_R^{(k-1)}(1-m_L)}{1-m_L m_R^{(k-1)}}, \\
x_2^* &= T_L(x_1^*) = m_L x_1^* + 1 - m_L, \\
x_3^* &= T_R(x_2^*) = m_R(m_L x_1^* + 1 - m_L), \\
x_4^* &= T_R(x_3^*) = m_R^2(m_L x_1^* + 1 - m_L), \\
&\ldots \\
x_k^* &= T_R(x_{k-1}^*) = m_R^{(k-2)}(m_L x_1^* + 1 - m_L).
\end{aligned} \tag{3.7}$$

It is plain that we can reason symmetrically for the other kind of cycles (with 1 periodic point in the R branch and $k-1$ in the L branch) just swapping L and R. So, we can easily get the following expressions for the bifurcation curves that mark the creation of a k-cycle:

$$m_R = m_{Ri} = \frac{d - 1 + m_L^{(k-1)}}{d m_L^{(k-1)}}, \tag{3.8}$$

while the closure of the same periodicity tongue is given by the following expression:

$$m_R = m_{Rf} = \frac{d - 1 + m_L^{(k-2)}}{m_L^{(k-2)}(m_L d + 1 - m_L)}. \tag{3.9}$$

So, the k-cycle exists for $m_L^{k-1} > (1-d)$ and m_R in the range

$$m_{Ri} \leq m_R \leq m_{Rf}. \tag{3.10}$$

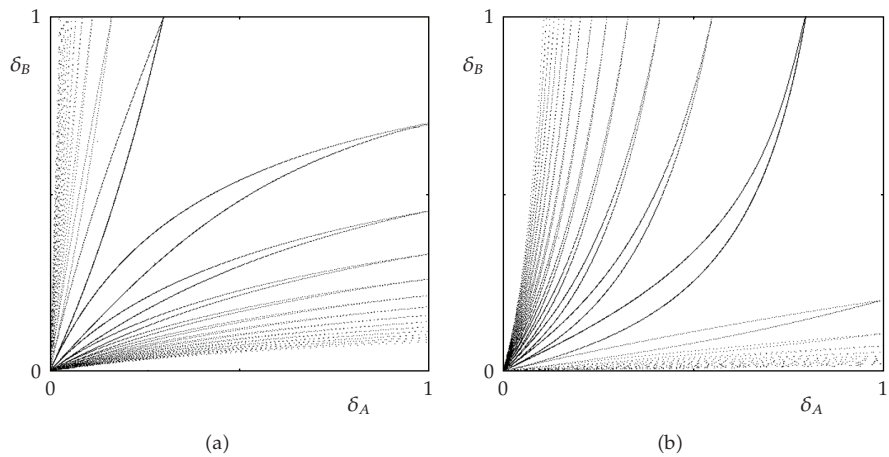

Figure 13: Bifurcation curves of the principal tongues of period k, with $k = 2, \ldots, 15$ (a) $d = 0.3$; (b) $d = 0.8$.

Moreover, the periodic points of the k-cycle, $(x_1^*, x_2^*, \ldots, x_k^*)$ where $x_1^* > d$ and $x_i^* < d$ for $i > 1$, are obtained from the existence condition, so that we have:

$$\begin{aligned}
x_1^* &= \frac{1 - m_L^{(k-1)}}{1 - m_R m_L^{(k-1)}}, \\
x_2^* &= T_R(x_1^*) = m_R x_1^*, \\
x_3^* &= T_L(x_2^*) = m_L m_R x_1^* + 1 - m_L, \\
x_4^* &= T_L(x_3^*) = m_L^2 m_R x_1^* + m_L(1 - m_L) + (1 - m_L), \\
&\ldots \\
x_k^* &= T_L(x_{k-1}^*) = m_L^{(k-2)} m_R x_1^* + \left(1 - m_L^{(k-2)}\right).
\end{aligned} \tag{3.11}$$

It is easy to see that for $k = 2$ the formulas in (3.4) and in (3.8) give the same bifurcation curves, and similarly for $k = 2$ the formulas in (3.5) and in (3.9) give the same equations. Instead, for $k = 3, \ldots, 15$ with the formulas in (3.4) and in (3.5) we get all the bifurcation curves in Figure 12(a), *below the main diagonal*, and with those in (3.8) and (3.9) we get all the bifurcation curves in Figure 12(a), *above the main diagonal*.

Note that the formulas given in (3.4) and in (3.8) are generic, and hold whichever is the position of the discontinuity point $x = d$. In Figure 13 we show the bifurcation curves for $k = 2, \ldots, 15$ in the case $d = 0.3$ and $d = 0.8$ respectively. It is worth noticing that following the same arguments it is possible to find the boundaries of the other bifurcation curves as well. In fact, besides the regions associated with the "tongues of first degree" ([34–37]) there are infinitely many other periodicity tongues, with periods that can be obtained from the property that between any two tongues having periods k_1 and k_2 there exists also a tongue having period $k_1 + k_2$ (see e.g., the periods indicated in Figure 9).

To be more specific, in the description of the periodicity tongues we can associate a number to each region, which may be called "*rotation number*", in order to classify all the periodicity tongues. In this notation a periodic orbit of period k is characterized not only by the period but also by the number of points in the two branches separated by the discontinuity point (denoted by T_L and T_R resp.,). So, we can say that a cycle has a rotation number p/k if a k-cycle has p points on the L side and the other $(k-p)$ on the R side. Then between any pair of periodicity regions associated with the "rotation number" p_1/k_1 and p_2/k_2 there exists also the periodicity tongue associated with the "rotation number" $p_1/k_1 \oplus p_2/k_2 = (p_1+p_2)/(k_1+k_2)$ (also called Farey composition rule $\oplus$, see e.g., [38]).

Then, following Leonov [34, 35] (see also [36, 37]), between any pair of contiguous "*tongues of first degree*", say $1/k_1$ and $1/(k_1+1)$, we can construct two infinite families of periodicity tongues, called "*tongues of second degree*" by the sequence obtained by adding with the Farey composition rule $\oplus$ iteratively the first one or the second one, that is, $1/k_1 \oplus 1/(k_1+1) = 2/(2k_1+1)$, $2/(2k_1+1) \oplus 1/k_1 = 3/(3k_1+1), \ldots$ and so on, that is:

$$\frac{n}{nk_1+1} \quad \text{for any } n > 1, \tag{3.12}$$

and $1/k_1 \oplus 1/(k_1+1) = 2/(2k_1+1)$, $2/(2k_1+1) \oplus 1/(k_1+1) = 3/(3k_1+2)$, $3/(3k_1+2) \oplus 1/(k_1+1) = 4/(4k_1+3) \ldots$, that is:

$$\frac{n}{nk_1+n-1} \quad \text{for any } n > 1 \tag{3.13}$$

which give two sequences of tongues accumulating on the boundary of the two starting tongues.

Clearly, this mechanisms can be repeated: between any pair of contiguous "*tongues of second degree*", for example $n/(nk_1+1)$ and $(n+1)/((n+1)k_1+1)$, we can construct two infinite families of periodicity tongues, called "*tongues of third degree*" by the sequence obtained by adding with the composition rule $\oplus$ iteratively the first one or the second one. And so on. All the rational numbers are obtained in this way, giving all the infinitely many periodicity tongues.

Besides the notation used above, called method of the *rotation numbers,* we may also follow a different approach, related with the symbolic sequence associated to a cycle. In this notation, considering the principal tongue of a periodic orbit of period k constituted by one point on the L side and $(k-1)$ on the R side, we associate to the cycle the *symbolic sequence* $LR\cdots(k-1\,times)\cdot R$. Then the composition of two consecutive cycles is given by:

$$LR\cdots(k-1\,times)\cdot R \oplus LR\cdots(k\,times)\cdot R = LR\cdots(k-1\,times)\cdot RLR\cdots(k\,times)\cdot R \tag{3.14}$$

that is, the two sequences are just put together in file (and indeed this sequence of bifurcations is also called "*boxes in files*" in [37]), and the sequence of maps to apply in order to get the

cycle are listed from left to right. More generally, it is true that given a periodicity tongue associated with a symbolic sequence σ (consisting of letters L and R, giving the cycle from left to right) and a second one with a symbolic sequence τ, then also the composition of the two sequences exists, associated with a periodicity tongue with symbolic sequence $\sigma\tau$:

$$\sigma \oplus \tau = \sigma\tau. \tag{3.15}$$

Finally, we notice that all the tongues are disjoint, that is, they never overlap, and this implies that coexistence of different periodic cycles is not possible. In other words, we can have only a single attractor for each pair of parameters δ_A and δ_B.

4. Conclusions

In this paper we have considered an adaptive discrete-time dynamic model of a binary game with externalities proposed by Bischi and Merlone [4] and based on the qualitative description of binary choice processes given by Schelling [1]. We focused on the case of a switching intensity that tends to infinity, a limiting that may be interpreted as agents' impulsivity, that is, actors that decide to switch the strategy choice even when the discrepancy between the payoffs observed in the previous period is extremely small. This may even be interpreted as the automatic change of an electrical or mechanical device that changes its state according to a measured difference between two indexes of performance.

In this limiting case the dynamical system is represented by the iteration of a one-dimensional piecewise linear discontinuous map that depends on three parameters, and whose dynamic properties and the analytically computed border collision bifurcations allowed us to give a quite complete description of existence, uniqueness and stability of periodic cycles of any period. In fact, for this piecewise continuous map with only one discontinuity point we could combine and usefully apply some geometric and analytic methods taken from the recent literature, as well as some results proposed several years ago but not sufficiently known in our opinion. How close the bifurcation curves of the limiting case are to those of the original continuous model, with a high value of the parameter λ, can be deduced comparing Figure 5(c) with Figure 9(a).

We have obtained the analytic expression of the border collision bifurcation curves that bound the periodicity tongues of first degree in the parameters' plane.

The results obtained show that the limiting case of impulsive agents is always characterized by convergence to a fixed point or to a periodic cycle, whereas for intermediate values of the switching parameters chaotic motion can be easily observed as well.

The methods followed to obtain such analytic expressions are quite general and can be easily generalized to cases with several discontinuities and with slopes different from the ones considered in the model studied in this paper. Indeed, in Schelling [1] also the case with two intersections between the payoff curves has been discussed, see also Granovetter, [3]. This gives rise to a model with two discontinuity points in the limiting case of impulsive agents. The bifurcation diagrams obtained in this case are studied in the paper Bischi et al. [39], and the same arguments can be applied in order to extend the discussion to the case where even more intersections (hence more discontinuities) occur.

Appendix

The condition of existence of a k-cycle with periodic points $x_1, \ldots, x_k$, having one point on the L side and $(k-1)$ points in the R side (also called principal orbits), is obtained by considering the fact that a periodic cycle is created in the form of a critical orbit, that is with a periodic point in the discontinuity point, for example, $x_1 = d$ and then $x_2 = T_L(x_1)$, $x_3 = T_R(x_2), \ldots, x_1 = T_R(x_k)$. From this condition we get:

$$\begin{aligned} x_1 &= d, \\ x_2 &= T_L(x_1) = m_L d + 1 - m_L, \\ x_3 &= T_R(x_2) = m_R(m_L d + 1 - m_L), \\ &\ldots \\ x_{k+1} &= T_R(x_k) = m_R^{(k-1)}(m_L d + 1 - m_L), \end{aligned} \tag{A.1}$$

and the condition for a k-cycle is given by:

$$x_{k+1} = d = m_R^{(k-1)}(m_L d + 1 - m_L). \tag{A.2}$$

Rearranging we obtain (for any $k > 1$):

$$m_{Li} = \frac{m_R^{(k-1)} - d}{(1-d)m_R^{(k-1)}}. \tag{A.3}$$

The cycle exists until a periodic point has a contact with the discontinuity point, $x = d$, at which we apply, in the sequence, the maps $T_R, T_L, T_R, \ldots, T_R$. Thus we obtain the following expressions:

$$\begin{aligned} x_1 &= d, \\ x_2 &= T_R(x_1) = m_R d, \\ x_3 &= T_L(x_2) = m_L m_R d + 1 - m_L, \\ x_4 &= T_R(x_3) = m_R(m_L m_R d + 1 - m_L), \\ &\ldots \\ x_{k+1} &= T_R(x_k) = m_R^{(k-2)}(m_L m_R d + 1 - m_L) \end{aligned} \tag{A.4}$$

and the condition for a k-cycle is:

$$x_{k+1} = d = m_R^{(k-2)}(m_L m_R d + 1 - m_L) \tag{A.5}$$

which, rearranged, gives (for any $k > 1$)

$$m_{Lf} = \frac{m_R^{(k-2)} - d}{(1 - m_R d) m_R^{(k-2)}}. \tag{A.6}$$

Acknowledgment

The authors thank the participants to the Workshop MDEF-2008 in Urbino for their interesting comments and suggestions, in particular Fabio Tramontana for useful discussions during the preparation of this paper. We also thank the two referees for their useful suggestions and remarks. The usual disclaimers apply.

References

[1] T. C. Schelling, "Hockey helmets, concealed weapons and daylight saving," *Journal of Conflict Resolution*, vol. 17, no. 3, pp. 381–428, 1973.
[2] D. Challet and Y. C. Zhang, "Emergence of cooperation and organization in an evolutionary game," *Physica A*, pp. 246–407, 1997.
[3] M. Granovetter, "Threshold models of collective behavior," *The American Journal of Sociology*, vol. 83, no. 6, pp. 1420–1443, 1978.
[4] G. I. Bischi and U. Merlone, "Global dynamics in binary choice models with social influence," *Journal of Mathematical Sociology*, vol. 33, pp. 1–26, 2009.
[5] J. B. Rosser, "On the complexities of complex economic dynamics," *Journal of Economic Perspectives*, vol. 13, pp. 169–172, 1999.
[6] F. G. Moeller, E. S. Barratt, D. M. Dougherty, J. M. Schmitz, and A. C. Swann, "Psychiatric aspects of impulsivity," *American Journal of Psychiatry*, vol. 158, pp. 1783–1793, 2001.
[7] J. H. Patton, M. S. Stanford, and E. S. Barratt, "Factor structure of the Barratt impulsiveness scale," *Journal of Clinical Psychology*, vol. 51, pp. 768–774, 1995.
[8] R. A. Eve, S. Horsfall, and M. E. Lee, Eds., *Chaos, Complexity, and Sociology: Myths, Models, and Theories*, Sage Publications, London, UK, 1997.
[9] R. L. Devaney, *Chaotic Dynamical Systems*, Perseus Books, Reading, Mass, USA, 2nd edition, 1989.
[10] C. Robinson, *Dynamical Systems: Stability, Symbolic Dynamics, and Chaos*, Studies in Advanced Mathematics, CRC Press, Boca Raton, Fla, USA, 2nd edition, 1999.
[11] V. Avrutin and M. Schanz, "On multi-parametric bifurcations in a scalar piecewise-linear map," *Nonlinearity*, vol. 19, no. 3, pp. 531–552, 2006.
[12] V. Avrutin and M. Schanz, "On the fully developed bandcount adding scenario," *Nonlinearity*, vol. 21, no. 5, pp. 1077–1103, 2008.
[13] V. Avrutin, M. Schanz, and S. Banerjee, "Multi-parametric bifurcations in a piecewise-linear discontinuous map," *Nonlinearity*, vol. 19, no. 8, pp. 1875–1906, 2006.
[14] M. di Bernardo, M. I. Feigin, S. J. Hogan, and M. E. Homer, "Local analysis of C-bifurcations in n-dimensional piecewise-smooth dynamical systems," *Chaos, Solitons & Fractals*, vol. 10, no. 11, pp. 1881–1908, 1999.
[15] I. Sushko, L. Gardini, and T. Puu, "Tongues of periodicity in a family of two-dimensional discontinuous maps of areal Möbius type," *Chaos, Solitons & Fractals*, vol. 21, no. 2, pp. 403–412, 2004.
[16] I. Sushko, A. Agliari, and L. Gardini, "Bifurcation structure of parameter plane for a family of unimodal piecewise smooth maps: border-collision bifurcation curves," *Chaos, Solitons & Fractals*, vol. 29, no. 3, pp. 756–770, 2006.
[17] Yu. L. Maĭstrenko, V. L. Maĭstrenko, and L. O. Chua, "Cycles of chaotic intervals in a time-delayed Chua's circuit," *International Journal of Bifurcation and Chaos in Applied Sciences and Engineering*, vol. 3, no. 6, pp. 1557–1572, 1993.
[18] Yu. L. Maĭstrenko, V. L. Maĭstrenko, S. I. Vikul, and L. O. Chua, "Bifurcations of attracting cycles from time-delayed Chua's circuit," *International Journal of Bifurcation and Chaos in Applied Sciences and Engineering*, vol. 5, no. 3, pp. 653–671, 1995.

[19] Yu. L. Maĭstrenko, V. L. Maĭstrenko, and S. I. Vikul, "On period-adding sequences of attracting cycles in piecewise linear maps," *Chaos, Solitons & Fractals*, vol. 9, no. 1-2, pp. 67–75, 1998.
[20] S. Banerjee and C. Grebogi, "Border-collision bifurcations in two-dimensional piecewise smooth maps," *Physical Review E*, vol. 59, no. 4, pp. 4052–4061, 1999.
[21] S. Banerjee, M. S. Karthik, G. Yuan, and J. A. Yorke, "Bifurcations in one-dimensional piecewise smooth maps—theory and applications in switching circuits," *IEEE Transactions on Circuits and Systems I*, vol. 47, no. 3, pp. 389–394, 2000.
[22] S. Banerjee, P. Ranjan, and C. Grebogi, "Bifurcations in two-dimensional piecewise smooth maps—theory and applications in switching circuits," *IEEE Transactions on Circuits and Systems I*, vol. 47, no. 5, pp. 633–643, 2000.
[23] O. Feely, D. Fournier-Prunaret, I. Taralova-Roux, and D. Fitzgerald, "Nonlinear dynamics of bandpass sigma-delta modulation. An investigation by means of the critical lines tool," *International Journal of Bifurcation and Chaos*, vol. 10, no. 2, pp. 303–327, 2000.
[24] Z. T. Zhusubaliyev and E. Mosekilde, *Bifurcations and Chaos in Piecewise-Smooth Dynamical Systems*, vol. 44 of *World Scientific Series on Nonlinear Science. Series A: Monographs and Treatises*, World Scientific, River Edge, NJ, USA, 2003.
[25] Z. T. Zhusubaliyev, E. Mosekilde, S. Maity, S. Mohanan, and S. Banerjee, "Border collision route to quasiperiodicity: numerical investigation and experimental confirmation," *Chaos*, vol. 16, no. 2, Article ID 023122, 11 pages, 2006.
[26] Z. T. Zhusubaliyev, E. Soukhoterin, and E. Mosekilde, "Quasiperiodicity and torus breakdown in a power electronic dc/dc converter," *Mathematics and Computers in Simulation*, vol. 73, no. 6, pp. 364–377, 2007.
[27] L. Gardini, T. Puu, and I. Sushko, "A goodwin-type model with a piecewise linear investment function," in *Business Cycles Dynamics. Models and Tools*, T. Puu and I. Sushko, Eds., Springe, New York, NY, USA, 2006.
[28] L. Gardini, I. Sushko, and A. Naimzada, "Growing through chaotic intervals," *Journal of Economic Theory*, vol. 143, pp. 541–557, 2008.
[29] T. Puu, L. Gardini, and I. Sushko, "A Hicksian multiplier-accelerator model with floor determined by capital stock," *Journal of Economic Behavior and Organization*, vol. 56, pp. 331–348, 2005.
[30] T. Puu, "The Hicksian trade cycle with floor and ceiling dependent on capital stock," *Journal of Economic Dynamics & Control*, vol. 31, no. 2, pp. 575–592, 2007.
[31] F. Tramontana, L. Gardini, and T. Puu, "Duopoly games with alternative technologies," *Journal of Economic Dynamic and Control*, vol. 33, pp. 250–265, 2008.
[32] H. E. Nusse and J. A. Yorke, "Border-collision bifurcations including "period two to period three" for piecewise smooth systems," *Physica D*, vol. 57, no. 1-2, pp. 39–57, 1992.
[33] H. E. Nusse and J. A. Yorke, "Border-collision bifurcations for piecewise smooth one-dimensional maps," *International Journal of Bifurcation and Chaos in Applied Sciences and Engineering*, vol. 5, no. 1, pp. 189–207, 1995.
[34] N. N. Leonov, "Map of the line onto itself," *Radiofisica*, vol. 3, no. 3, pp. 942–956, 1959.
[35] N. N. Leonov, "On a discontinuous point transformation of the line into the line," *Doklady Akademii Nauk SSSR*, vol. 143, no. 5, pp. 1038–1041, 1962.
[36] C. Mira, "Sur la structure des bifurcations des difféomorphismes du cercle," *Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences. Series A*, vol. 287, no. 13, pp. 883–886, 1978.
[37] C. Mira, *Chaotic Dynamics: From the One-Dimensional Endomorphism to the Two-Dimensional Diffeomorphism*, World Scientific, Singapore, 1987.
[38] B. L. Hao, *Elementary Symbolic Dynamics and Chaos in Dissipative Systems*, World Scientific, Teaneck, NJ, USA, 1989.
[39] G. I. Bischi, L. Gardini, and U. Merlone, "Periodic cycles and bifurcation curves for one-dimensional maps with two discontinuities," *Journal of Dynamical Systems and Geometric Theory*. In press.