Research Article

A Recurrence Formula for D Numbers $D_{2n}^{(2n-1)}$

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we establish a recurrence formula for D numbers $D_{2n}^{(2n-1)}$. A generating function for D numbers $D_{2n}^{(2n-1)}$ is also presented.

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1. Introduction and Results

The Bernoulli polynomials $B_n^{(k)}(x)$ of order k , for any integer k , may be defined by (see [1–4])

$$\left(\frac{t}{e^t - 1}\right)^k e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x) \frac{t^n}{n!}, \quad |t| < 2\pi. \quad (1.1)$$

The numbers $B_n^{(k)} = B_n^{(k)}(0)$ are the Bernoulli numbers of order k , $B_n^{(1)} = B_n$ are the ordinary Bernoulli numbers (see [2, 5]). By (1.1), we can get (see [4, page 145])

$$\frac{d}{dx} B_n^{(k)}(x) = n B_{n-1}^{(k)}(x), \quad (1.2)$$

$$B_n^{(k+1)}(x) = \frac{k-n}{k} B_n^{(k)}(x) + (x-k) \frac{n}{k} B_{n-1}^{(k)}(x), \quad (1.3)$$

$$B_n^{(k+1)}(x+1) = \frac{nx}{k} B_{n-1}^{(k)}(x) - \frac{n-k}{k} B_n^{(k)}(x), \quad (1.4)$$

where $n \in \mathbb{N}$, with $\mathbb{N}$ being the set of positive integers.

The numbers $B_n^{(n)}$ are called the Nörlund numbers (see [2, 4, 6]). A generating function for the Nörlund numbers $B_n^{(n)}$ is (see [4, page 150])

$$\frac{t}{(1+t)\log(1+t)} = \sum_{n=0}^{\infty} B_n^{(n)} \frac{t^n}{n!}. \quad (1.5)$$

The D numbers $D_{2n}^{(k)}$ may be defined by (see [4, 7, 8])

$$(t \csc t)^k = \sum_{n=0}^{\infty} (-1)^n D_{2n}^{(k)} \frac{t^{2n}}{(2n)!}, \quad |t| < \pi. \quad (1.6)$$

By (1.1), (1.6), and note that $\csc t = 2i/(e^{it} - e^{-it})$ (where $i^2 = -1$), we can get

$$D_{2n}^{(k)} = 4^n B_{2n}^{(k)} \left(\frac{k}{2}\right). \quad (1.7)$$

Taking $k = 1, 2$ in (1.7), and note that $B_{2n}^{(1)}(1/2) = (2^{1-2n} - 1)B_{2n}$, $B_{2n}^{(2)}(1) = (1 - 2n)B_{2n}$ (see [4, page 22, page 145]), we have

$$D_{2n}^{(1)} = (2 - 2^{2n})B_{2n}, \quad D_{2n}^{(2)} = 4^n(1 - 2n)B_{2n}. \quad (1.8)$$

The D numbers $D_{2n}^{(k)}$ satisfy the recurrence relation (see [7])

$$D_{2n}^{(k)} = \frac{(2n - k + 2)(2n - k + 1)}{(k - 2)(k - 1)} D_{2n}^{(k-2)} - \frac{2n(2n - 1)(k - 2)}{k - 1} D_{2n-2}^{(k-2)}. \quad (1.9)$$

By (1.9), we may immediately deduce the following (see [4, page 147]):

$$D_{2n}^{(2n+1)} = \frac{(-1)^n (2n)!}{4^n} \binom{2n}{n}, \quad D_{2n}^{(2n+2)} = \frac{(-1)^n 4^n}{2n+1} (n!)^2, \quad (1.10)$$

$$D_{2n}^{(2n+3)} = \frac{(-1)^n (2n)!}{2 \cdot 4^{2n}} \binom{2n+2}{n+1} \left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots + \frac{1}{(2n+1)^2}\right). \quad (1.11)$$

The numbers $D_{2n}^{(2n)}$ are called the D -Nörlund numbers that satisfy the recurrence relation (see [7])

$$\sum_{j=0}^n \frac{(-1)^j}{4^j (2j+1)} \binom{2j}{j} \frac{D_{2n-2j}^{(2n-2j)}}{(2n-2j)!} = \frac{(-1)^n}{4^n} \binom{2n}{n}, \quad (1.12)$$

so we find $D_0^{(0)} = 1, D_2^{(2)} = -2/3, D_4^{(4)} = 88/15, D_6^{(6)} = -3056/21, D_8^{(8)} = 319616/45, D_{10}^{(10)} = -18940160/33, \dots$

A generating function for the D -Nörlund numbers $D_{2n}^{(2n)}$ is (see [7])

$$\frac{t}{\sqrt{1+t^2} \log(t + \sqrt{1+t^2})} = \sum_{n=0}^{\infty} D_{2n}^{(2n)} \frac{t^{2n}}{(2n)!}, \quad |t| < 1. \quad (1.13)$$

These numbers $D_{2n}^{(2n)}$ and $D_{2n}^{(2n-1)}$ have many important applications. For example (see [4, page 246])

$$\begin{aligned} \int_0^{\pi/2} \frac{\sin t}{t} dt &= \sum_{n=0}^{\infty} \frac{(-1)^n D_{2n}^{(2n)}}{(2n+1)!}, & \int_0^{\pi/2} \frac{\sin t}{t} dt &= \frac{\pi}{2} \sum_{n=0}^{\infty} \frac{(-1)^{n+1} D_{2n}^{(2n-1)}}{2^{2n} (2n-1)(n!)^2}, \\ \frac{2}{\pi} &= \sum_{n=0}^{\infty} \frac{(-1)^{n+1} D_{2n}^{(2n-1)}}{(2n-1)(2n)!}. \end{aligned} \quad (1.14)$$

The main purpose of this paper is to prove a recurrence formula for D numbers $D_{2n}^{(2n-1)}$ and to obtain a generating function for D numbers $D_{2n}^{(2n-1)}$. That is, we will prove the following main conclusion.

Theorem 1.1. *Let $n \in \mathbb{N}$. Then*

$$\sum_{j=1}^n \binom{2n}{2j} (-1)^{j-1} 4^{j-1} ((j-1)!)^2 D_{2n-2j}^{(2n-1-2j)} = \frac{(-1)^{n-1} 2(2n)!}{4^n} \binom{2n-2}{n-1}, \quad (1.15)$$

so one finds $D_2^{(1)} = -1/3$, $D_4^{(3)} = 17/5$, $D_6^{(5)} = -1835/21$, $D_8^{(7)} = 195013/45$, $D_{10}^{(9)} = -3887409/11, \dots$

Theorem 1.2. *Let t be a complex number with $|t| < 1$. Then*

$$\sum_{n=0}^{\infty} D_{2n}^{(2n-1)} \frac{t^{2n}}{(2n)!} = \frac{1}{\sqrt{1+t^2}} \left(\frac{t}{\log(t + \sqrt{1+t^2})} \right)^2. \quad (1.16)$$

2. Proof of the Theorems

Proof of Theorem 1.1. Note the identity (see [4, page 203])

$$B_{2n}^{(k)} \left(x + \frac{k}{2} \right) = \sum_{j=0}^n \binom{2n}{2j} \frac{D_{2n-2j}^{(k-2j)}}{2^{2n-2j}} x^2 (x^2 - 1^2) (x^2 - 2^2) \cdots (x^2 - (j-1)^2), \quad (2.1)$$

we have

$$\frac{B_{2n}^{(k)}(x + k/2) - B_{2n}^{(k)}(k/2)}{x^2} = \sum_{j=1}^n \binom{2n}{2j} \frac{D_{2n-2j}^{(k-2j)}}{2^{2n-2j}} (x^2 - 1^2) (x^2 - 2^2) \cdots (x^2 - (j-1)^2). \quad (2.2)$$

Therefore,

$$\lim_{x \rightarrow 0} \frac{B_{2n}^{(k)}(x + k/2) - B_{2n}^{(k)}(k/2)}{x^2} = \sum_{j=1}^n \binom{2n}{2j} \frac{D_{2n-2j}^{(k-2j)}}{2^{2n-2j}} (-1)^{j-1} ((j-1)!)^2. \quad (2.3)$$

By (2.3) and (1.2), we have

$$\lim_{x \rightarrow 0} \frac{2n(2n-1)B_{2n-2}^{(k)}(x + k/2)}{2} = \sum_{j=1}^n \binom{2n}{2j} \frac{D_{2n-2j}^{(k-2j)}}{2^{2n-2j}} (-1)^{j-1} ((j-1)!)^2. \quad (2.4)$$

That is,

$$n(2n-1)B_{2n-2}^{(k)}\left(\frac{k}{2}\right) = \sum_{j=1}^n \binom{2n}{2j} \frac{D_{2n-2j}^{(k-2j)}}{2^{2n-2j}} (-1)^{j-1} ((j-1)!)^2. \quad (2.5)$$

By (2.5) and (1.7), we have

$$D_{2n-2}^{(k)} = \frac{1}{n(2n-1)} \sum_{j=1}^n \binom{2n}{2j} (-1)^{j-1} 4^{j-1} ((j-1)!)^2 D_{2n-2j}^{(k-2j)}. \quad (2.6)$$

Setting $k = 2n-1$ in (2.6), and note (1.10), we immediately obtain Theorem 1.1. This completes the proof of Theorem 1.1. $\square$

Remark 2.1. Setting $k = 2n$ in (2.6), and note (1.10), we may immediately deduce the following recurrence formula for D -Nörlund numbers $D_{2n}^{(2n)}$:

$$\sum_{j=1}^n \binom{2n}{2j} (-1)^j 4^j ((j-1)!)^2 D_{2n-2j}^{(2n-2j)} = (-1)^n n 4^n ((n-1)!)^2 \quad (n \in \mathbb{N}). \quad (2.7)$$

Proof of Theorem 1.2. Note the identity (see [9])

$$\sum_{n=0}^{\infty} (-1)^n 4^n (n!)^2 \frac{t^{2n}}{(2n)!} = \frac{1}{1+t^2} \left(1 - \frac{t}{\sqrt{1+t^2}} \log(t + \sqrt{1+t^2}) \right), \quad (2.8)$$

where $|t| < 1$. We have

$$\sum_{n=0}^{\infty} (-1)^n 4^n (n!)^2 \frac{t^{2n+2}}{(2n+2)!} = \frac{1}{2} \left(\log(t + \sqrt{1+t^2}) \right)^2, \quad (2.9)$$

That is,

$$\sum_{n=1}^{\infty} (-1)^{n-1} 4^{n-1} ((n-1)!)^2 \frac{t^{2n}}{(2n)!} = \frac{1}{2} \left(\log(t + \sqrt{1+t^2}) \right)^2. \quad (2.10)$$

On the other hand,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} 2(2n)!}{4^n} \binom{2n-2}{n-1} \frac{t^{2n}}{(2n)!} = \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{4^n} \binom{2n}{n} t^{2n+2} = \frac{t^2}{2\sqrt{1+t^2}}. \quad (2.11)$$

Thus, by (2.10), (2.11), and Theorem 1.1, we have

$$\sum_{n=1}^{\infty} (-1)^{n-1} 4^{n-1} ((n-1)!)^2 \frac{t^{2n}}{(2n)!} \sum_{n=1}^{\infty} D_{2n}^{(2n-1)} \frac{t^{2n}}{(2n)!} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 2(2n)!}{4^n} \binom{2n-2}{n-1} \frac{t^{2n}}{(2n)!}. \quad (2.12)$$

That is,

$$\frac{1}{2} \left(\log(t + \sqrt{1+t^2}) \right)^2 \sum_{n=1}^{\infty} D_{2n}^{(2n-1)} \frac{t^{2n}}{(2n)!} = \frac{t^2}{2\sqrt{1+t^2}}. \quad (2.13)$$

By (2.13), and note that

$$\lim_{t \rightarrow 0} \frac{t}{\log(t + \sqrt{1+t^2})} = 1, \quad D_0^{(-1)} = 1, \quad (2.14)$$

we immediately obtain Theorem 1.2. This completes the proof of Theorem 1.2. $\square$

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