Research Article

On the Recursive Sequence $x_n = A + x_{n-k}^p / x_{n-1}^r$

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This paper studies the dynamic behavior of the positive solutions to the difference equation $x_n = A + x_{n-k}^p / x_{n-1}^r$, $n = 1, 2, \ldots$, where $A, p, r$ are positive real numbers, and the initial conditions are arbitrary positive numbers. We establish some results regarding the stability and oscillation character of this equation for $p \in (0, 1)$.

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1. Introduction

In recent years, there has been intense interest in the dynamic behavior of the positive solutions to a class of difference equations of the form

$$x_n = A + x_{n-k}^p / x_{n-1}^r, \quad n \in \mathbb{N},$$

where $A$ and $p$ are positive real numbers. Now, let us make a brief review on the advances in this class of difference equations.

In 1999, Amleh et al. [1] studied the second-order rational difference equation

$$x_n = A + x_{n-2} / x_{n-1}, \quad n \in \mathbb{N}.$$  (1.2)

Later, Berenhaut and Stević [2], Stević [3], and El-Owaidy et al. [4] extended this work to the following more general second-order difference equation:

$$x_n = A + x_{n-2}^p / x_{n-1}^r, \quad n \in \mathbb{N}.$$  (1.3)
On the other hand, DeVault et al. [5] investigated the following higher-order version of (1.2):

\[ x_n = A + \frac{x_{n-k}}{x_{n-1}}, \quad n \in \mathbb{N}. \tag{1.4} \]

By combining (1.3) and (1.4), Berenhaut and Stević [6] examined a larger class of difference equations, which are of the form

\[ x_n = A + \frac{x_{n-k}}{x_{n-1}}, \quad n \in \mathbb{N}. \tag{1.5} \]

Very recently, Berenhaut et al. [7] studied the following generalization of (1.5):

\[ x_n = A + \frac{x_{n-k}}{x_{n-m}}, \quad n \in \mathbb{N}. \tag{1.6} \]

For some related work, the interested reader is referred to [1, 3, 8–19].

Inspired by the previous work and by the work owing to Stević [15], this paper studies the behavior of the recursive equation

\[ x_n = A + \frac{x_{n-k}}{x_{n-1}}, \quad n \in \mathbb{N}. \tag{1.7} \]

We establish some interesting results regarding the stability and oscillation character of this equation for \( p \in (0, 1) \).

2. Stability Character

In this section we investigate the stability character of the positive solutions to (1.7).

A point \( \overline{x} \in \mathbb{R} \) is an equilibrium point of (1.7) if and only if it is a root for the function

\[ g(x) = x - x^{p-r} - A, \tag{2.1} \]

that is,

\[ \overline{x} = \overline{x}^{p-r} + A. \tag{2.2} \]

**Lemma 2.1.** Let \( 0 < p < r + 1 \), then (1.7) has a unique equilibrium point \( \overline{x} > 1 \).

**Proof**

**Case 1.** \( p = r \). Then \( \overline{x} = A + 1 > 1 \).
Case 2. \( r < p < r + 1 \). Then \( g \) defined by (2.1) is decreasing on \([0, (p-r)^{1/(r-p+1)}]\) and increasing on \([(p-r)^{1/(r-p+1)}, \infty)\). Since \( g(1) = -A \) and \( \lim_{x \to \infty} g(x) = \infty \), then \( g \) has a unique zero \( \bar{x} > 1 \).

Case 3. \( 0 < p < r \). Since \( g \) is increasing on \([0, \infty)\), \( g(1) = -A \) and \( \lim_{x \to \infty} g(x) = \infty \), then \( g \) has a unique zero \( \bar{x} > 1 \).

**Lemma 2.2.** Let \( 0 < p < r + 1 \). Assume that \( \bar{x} \) is the equilibrium point of (1.7). If \( (p+r)^{(p-r)/(r+1-p)}(p+r-1) < A \), then \( \bar{x} \) is locally asymptotically stable.

*Proof.* By the Linearized Stability Theorem \([11]\), \( \bar{x} \) is locally asymptotically stable if and only if \( \bar{x}^{r+1-p} > p + r \). A simple calculations shows that

\[
g\left((p+r)^{1/(r+1-p)}\right) = (p+r)^{(p-r)/(r+1-p)}(p+r-1) - A < 0, \tag{2.3}
\]

where \( g \) is defined by (2.1). Then since \( \lim_{x \to \infty} g(x) = \infty \), we have \( \bar{x} > (p+r)^{1/(r+1-p)} \) and \( \bar{x}^{r+1-p} > p + r \). The proof is complete. \( \square \)

**Lemma 2.3.** If \( p \in (0,1) \), then every positive solution to (1.7) is bounded.

*Proof.* Note that each \( n \in \mathbb{N} \) can be written in the form \( lk+i \) for some \( l \in \mathbb{N}_0 \) and \( i \in \{0,1,\ldots,k-1\} \). From (1.7) and since \( x_n > A \) for every \( n \geq 0 \), we have that

\[
x_{l+1} = A + \frac{x_{l-1}^p}{x_{l-1}^r} < A + \frac{x_{l-1}^p}{A^r}, \tag{2.4}
\]

for every \( l \in \mathbb{N}_0 \) and \( i \in \{0,1,\ldots,k-1\} \). Let \( (u_i^{(l)})_{l \in \mathbb{N}_0} \) be the solution to the difference equation

\[
u_i^{(l)} = A + \left(\frac{u_{l-1}^{(l)}}{A^r}\right)^p, \quad u_0^{(l)} = x_{l+i}.
\]

From (2.4) and by induction we see that \( x_{l+1} \leq u_i^{(l)}, l \in \mathbb{N}_0 \). Hence it is enough to prove that the sequences \( (u_i^{(l)})_{l \in \mathbb{N}_0}, i \in \{0,1,\ldots,k-1\} \) are bounded.

Since the function \( f(x) = A + x^p / A^r \), \( x \in (0, \infty) \) is increasing and concave for \( p \in (0,1) \), it follows that there is a unique fixed point \( \bar{x} \) of the equation \( f(x) = x \) and that the function \( f \) satisfies

\[
(f(x) - x)(x - \bar{x}) < 0, \quad x \in (0, \infty). \tag{2.6}
\]

Using this fact it is easy to see that if \( u_i^{(l)} \in (0, \bar{x}) \), the sequence is nondecreasing and bounded from above by \( \bar{x} \), and if \( u_i^{(l)} \geq \bar{x} \), it is nonincreasing and bounded from below by \( \bar{x} \).

Hence for every \( x_0^{(l)} \in (0, \infty) \), each of the sequences \( u_i^{(l)}, i \in \{0,1,\ldots,k-1\} \) is bounded. The claimed result follows. \( \square \)
Lemma 2.4 (see [18]). Let $s, t$ be distinct nonnegative integers. Consider the difference equation

$$x_n = f(x_{n-s}, x_{n-t}), \quad n = 1, 2, 3, \ldots,$$

$$x_1 = \max(s, t), x_2 = \max(s, t), \ldots, x_0 \in [a, b]. \quad (2.7)$$

Suppose $f$ satisfies the following conditions.

(H$_1$) $f : [a, b]^2 \to [a, b]$ is a continuous function that is nondecreasing in the first argument and is nonincreasing in the second argument.

(H$_2$) The system

$$x = f(x, y),$$

$$y = f(y, x) \quad (2.8)$$

has a unique solution $(x, x) \in [a, b] \times [a, b]$.

Then $x$ is the global attractor of all solutions to (2.7).

Theorem 2.5. Let $p + r \leq 1$, then the unique equilibrium $x$ to (1.7) is globally asymptotically stable.

Proof. By Lemma 2.3, there must exist positive constants $P$ and $Q$ such that $P \leq x_n \leq Q$. Let $f(u, v) = A + u^p / v^r$, $u, v \in [P, Q]$, it is easy to verify that (H$_1$) holds. In addition, if

$$x = A + \frac{x^p}{y^r},$$

$$y = A + \frac{y^p}{x^r}, \quad (2.9)$$

then

$$\frac{x - A}{y - A} = \frac{x^{p+r}}{y^{p+r}}. \quad (2.10)$$

Assume that $x \neq y$, then $x > y$ or $x < y$.

In case $x > y$, we have $(x - A) / (y - A) > x / y \geq x^{p+r} / y^{p+r}$, which contradicts with (2.10).

In case $x < y$, we have $(x - A) / (y - A) < x / y \leq x^{p+r} / y^{p+r}$, again a contradiction. Thus $x = y = \overline{x}$. By Lemma 2.4, the required result follows.

Theorem 2.6. Let $0 < p \leq r < 1$ and $A^{-p+1} \geq p / r$. Then every positive solution to (1.7) converges to the unique equilibrium $\overline{x}$. 
**Proof.** By Lemma 2.3, every positive solution \( \{x_n\} \) to (1.7) is bounded, which implies that there are finite \( \lim \inf x_n = I \) and \( \lim \sup x_n = S \). Assume that \( I \neq S \) (\( I < S \)). Taking the \( \lim \inf \) and \( \lim \sup \) in (1.7), it follows that

\[
A + \frac{lp}{Sr} \leq I < S \leq A + \frac{Sp}{Ir}.
\]  

(2.11)

From this and \( r \in (0, 1) \), it follows that

\[
AS' + Ip \leq IS' < SI' \leq AI' + Sp,
\]  

(2.12)
yielding

\[
AS' - Sp < AI' - Ip.
\]  

(2.13)

Define function \( f(x) = Ax' - x^p \), \( x \in (A, \infty) \). Since

\[
f'(x) = Arx'^{-1} - px'^{-1} = x'^{-1}(Arx'^{-p} - p) > x'^{-1}(rA^{-p+1} - p) \geq 0,
\]  

we deduce that \( f \) is increasing, and thus (2.13) cannot hold. Therefore we have \( I = S \), which implies the result. \( \square \)

**Theorem 2.7.** Let \( 0 < p < 1 \), \( r \geq 1 \), and \( A^{-p+1} \geq r + p - 1 \). Then every positive solution to (1.7) converges to the unique equilibrium \( \bar{x} \).

**Proof.** From (2.11) we have

\[
AI'^{-1}Sr' + Ip'^{-1} \leq IS' \leq AI'S'^{-1} + Sp'^{-1}.
\]  

(2.15)

Consequently, we obtain \((AI'^{-1}S'^{-1})(S - I) \leq (S'^{p-1} - I'^{p-1})\). Suppose that \( I \neq S \), we get

\[
AI'^{-1}S'^{-1} \leq \frac{S'^{p-1} - I'^{p-1}}{S - I} = (r + p - 1)\gamma^{p+2},
\]  

(2.16)

where \( \gamma \in (I, S) \), leading to

\[
A' S'^{-1} \leq AI'^{-1}S'^{-1} \leq (r + p - 1)p^{p+2} < (r + p - 1)A^{-1}S'^{-1}.
\]  

(2.17)

This implies that \( A^{-p+1} < r + p - 1 \), which is a contradiction. Hence, \( I = S = \bar{x} \). \( \square \)
3. Oscillation Character

In this section we investigate the oscillation character of the positive solutions to (1.7).

**Theorem 3.1.** Let \( \{x_n\}_{n=k}^{\infty} \) be a positive solution to (1.7). Then either \( \{x_n\}_{n=k}^{\infty} \) consists of a single semicycle or \( \{x_n\}_{n=k}^{\infty} \) oscillates about the equilibrium \( \bar{x} \) with semicycles having at most \( k-1 \) terms.

**Proof.** Suppose that \( \{x_n\}_{n=k}^{\infty} \) has at least two semicycles. Then there exists \( N \geq -k \) such that either \( x_N < \bar{x} \leq x_{N+1} \) or \( x_{N+1} < \bar{x} \leq x_N \). Assume that \( x_N < \bar{x} \leq x_{N+1} \). (The argument for the case \( x_{N+1} < \bar{x} \leq x_N \) is similar and is omitted.) Now suppose that the positive semicycle beginning with the term \( x_{N+1} \) has \( k-1 \) terms. Then \( x_N < \bar{x} \leq x_{N+k-1} \) and so

\[
x_N^{k+1} = A + \frac{x_N^p}{x_{N+k-1}^r} < A + \frac{\bar{x}_p}{\bar{x}_r} = A + \bar{x}^{p-r} = \bar{x}.
\]

This completes the proof. \( \square \)

**Theorem 3.2.** Suppose that \( k \) is even and let \( \{x_n\}_{n=k}^{\infty} \) be a solution to (1.7), which has \( k-1 \) consecutive semicycles of length one, then every semicycle after this point is of length one.

**Proof.** There exists \( N \geq -k \) such that either

\[
x_N, x_{N+2}, \ldots, x_{N+k-2} < \bar{x} \leq x_{N+1}, x_{N+3}, \ldots, x_{N+k-1}
\]

or

\[
x_{N+1}, x_{N+3}, \ldots, x_{N+k-1} < \bar{x} \leq x_N, x_{N+2}, \ldots, x_{N+k-2}. \]

We prove the former case. The proof for the latter is similar and is omitted. Now, we have

\[
x_N^{k+1} = A + \frac{x_N^p}{x_{N+k-1}^r} < A + \frac{\bar{x}_p}{\bar{x}_r} = A + \bar{x}^{p-r} = \bar{x},
\]

\[
x_{N+k+1} = A + \frac{x_{N+1}^p}{x_{N+k}^r} > A + \frac{\bar{x}_p}{\bar{x}_r} = A + \bar{x}^{p-r} = \bar{x}.
\]

The result then follows by induction. \( \square \)

**Lemma 3.3.** Let \( 0 < p < r + 1 \). Then (1.7) has no nontrivial periodic solutions of (not necessarily prime) period \( k-1 \).

**Proof.** Suppose that \( \{x_n\}_{n=k}^{\infty} \) is a positive solution to (1.7) satisfying \( x_{n-1} = x_{n-k} \) for all \( n \geq 1 \), then \( x_n = A + x_{n-k}^p / x_{n-1}^r = A + x_{n-1}^{p-r} \) implies that \( x_{n-1} = x_n = \bar{x} \) for all \( n > -k \). The proof is complete. \( \square \)

**Theorem 3.4.** Assume that \( p \leq r \). Let \( \{x_n\}_{n=k}^{\infty} \) be a positive solution to (1.7), which consists of a single semicycle, then \( \{x_n\}_{n=k}^{\infty} \) converges to the equilibrium \( \bar{x} \).
Proof. Suppose $x_n \geq \overline{x}$ (the case for $x_n \leq \underline{x}$ is similar and is omitted) for all $n \geq -k$, then

$$x_{n+1} = A + \frac{x_n^p}{x_{n-k}^p} \geq \overline{x} = A + \overline{x}^{p-r},$$

(3.5)

implying that

$$x_{n-(k-1)} \geq \overline{x}^{(p-r)/p} x_n^{r/p} \geq x_n^{(p-r)/p} x_n^{r/p} = x_n,$$

(3.6)

and so

$$x_{n-(k-1)} \geq x_n \geq \overline{x} \quad \text{for } n = 1, 2, \ldots.$$  

(3.7)

From here it is clear that for $i = 0, \ldots, k-2$ there exists $\alpha_i$ such that

$$\lim_{n \to \infty} x_{n(k-1)+i} = \alpha_i.$$  

(3.8)

But then $\alpha_0, \alpha_1, \ldots, \alpha_{k-2}$ is a periodic solution of (not necessarily prime) period $k - 1$. By Lemma 3.3 the result holds.

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