Research Article
Periodic Solutions for a System of Difference Equations

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This paper deals with the second-order nonlinear systems of difference equations, we obtain the existence theorems of periodic solutions. The theorems are proved by using critical point theory.

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1. Introduction

Let \(\mathbb{N}, \mathbb{Z}, \mathbb{R}\) be the set of all natural numbers, integers, and real numbers, respectively. For \(a, b \in \mathbb{Z}\), note that \(\mathbb{Z}[a, b] = \{a, a + 1, \ldots, b\}\), where \(a \leq b\).

In this paper, we consider the existence of periodic solutions for the system of difference equations of the form

\[
\Delta (p_{n1} (\Delta x_{(n-1)1})^\delta) + q_{n1} (x_{n1})^\delta = f_1(n, X_n),
\]
\[
\Delta (p_{n2} (\Delta x_{(n-1)2})^\delta) + q_{n2} (x_{n2})^\delta = f_2(n, X_n),
\]
\[
\vdots
\]
\[
\Delta (p_{nk} (\Delta x_{(n-1)k})^\delta) + q_{nk} (x_{nk})^\delta = f_k(n, X_n),
\]

which can be recorded as

\[
\Delta (\widetilde{P}_n (\Delta X_{n-1})^\delta) + \widetilde{Q}_n (X_n)^\delta = f(n, X_n), \quad n \in \mathbb{Z},
\]
where $k$ is a positive integer,

\[
\bar{P}_n = \begin{pmatrix} p_{n1} & 0 & \cdots & 0 \\ 0 & p_{n2} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & p_{nk} \end{pmatrix}, \quad \bar{Q}_n = \begin{pmatrix} q_{n1} & 0 & \cdots & 0 \\ 0 & q_{n2} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & q_{nk} \end{pmatrix},
\]

and $\bar{P}_{n+\omega} = \bar{P}_n > 0$ (i.e., $p_{n1} > 0, p_{n2} > 0, \ldots, p_{nk} > 0$), $\bar{Q}_{n+\omega} = \bar{Q}_n$, $f = (f_1, f_2, \ldots, f_k)^T$, $f_i = f_i(n, X_n) = f_i(n, x_{n1}, x_{n2}, \ldots, x_{nk})$, $f(n + \omega, U) = f(n, U)$ for any $(n, U) \in \mathbb{Z} \times \mathbb{R}^k$, $\omega > 0$ is a positive integer, $(-1)^\delta = -1$, $\delta$ is the ratio of odd positive integers, $\Delta X_n^T = X_{n+1} - X_n = (x_{n+1} - x_n, x_{n+2} - x_{n+1}, \ldots, x_{n+k} - x_{n+k})^T$, $\Delta^2 X_{n-1} = \Delta X_n^T = \Delta X_{n-1} - \Delta X_{n-2}$. For $U = (u_1, u_2, \ldots, u_k) \in \mathbb{R}^k$, define $U^\delta = (u_1^\delta, u_2^\delta, \ldots, u_k^\delta)$. A sequence $X = \{X_n\}_{n \in \mathbb{Z}}$ is an $\omega$-periodic solution of (1.2) if substitution of it into (1.2) yields an identity for all $n \in \mathbb{Z}$.

In [1, 2], the qualitative behavior of linear difference equations

\[
\Delta (p_n \Delta x_n) + q_n x_n = 0
\]

has been investigated. In [3], the nonlinear difference equation

\[
\Delta (p_n \Delta x_{n-1}) + q_n x_n = f(n, x_n)
\]

has been considered. In [4], by critical point method, the existence of periodic and subharmonic solutions of equation

\[
\Delta^2 x_{n-1} + f(n, x_n) = 0, \quad n \in \mathbb{Z}
\]

has been studied. Other interesting results can be found in [5–8]. In [9], the authors consider the existence of periodic solutions for second-order nonlinear difference equation

\[
\Delta (p_n (\Delta x_{n-1})^\delta) + q_n x_n^\delta = f(n, x_n), \quad n \in \mathbb{Z},
\]

using critical point theory, obtaining some new results. It is a discrete analogues of differential equation

\[
(p(t)\phi(u'))' + f(t, u) = 0.
\]

They do have physical applications in the study of nuclear physics, gas aerodynamics, and so on (see [10, 11]). In this paper, we obtain some new results of existence of periodic solution for the second-order nonlinear system of difference equations by using critical point theory. We remark, however, the result in [9] is only good for (1.7) which is much less general than our results in what follows.
2. Some Basic Lemmas

Let $E$ be a real Hilbert space, $I \in C^1(E, \mathbb{R})$ mean that $I$ is continuously Fréchet differentiable functional defined on $E$. $I$ is said to be satisfying Palais-Smale condition (P-S condition) if any bounded sequence $\{I(u_n)\}$ and $I'(u_n) \to 0 \ (n \to \infty)$ possess a convergent subsequence in $E$. Let $B_\rho$ be the open ball in $E$ with radius $\rho$ and centered at $\theta$, and let $\partial B_\rho$ denote its boundary, $\theta$ is null element of $E$.

Lemma 2.1 (see [12]). Let $E$ be a real Hilbert space, and assume that $I \in C^1(E, \mathbb{R})$ satisfies the P-S condition and the following conditions:

(I) $\exists \rho > 0$ and $a > 0$ such that $I(x) \geq a$ for all $x \in \partial B_\rho$, where $B_\rho = \{x \in E : \|x\| < \rho\}$;

(II) $I(0) \leq 0$ and there exists $x_0 \notin B_\rho$ such that $I(x_0) \leq 0$.

Then $c = \inf_{x \in \Gamma} \sup_{s \in [0,1]} I(h(s))$ is a positive critical value of $I$, where

$$\Gamma = \{h \in C([0,1], X) : h(0) = \theta, h(1) = x_0\}. \quad (2.1)$$

Let $\Omega_\omega$ be the set of sequences

$$X = \{X_n\}_{n \in \mathbb{Z}} = \{\ldots, X_{-n}, \ldots, X_{-1}, X_0, X_1, \ldots, X_n, \ldots\}, \quad (2.2)$$

where $X_n = (x_{n1}, x_{n2}, \ldots, x_{nk}) \in \mathbb{R}^k$, that is,

$$\Omega_\omega = \{X = \{X_n\}_{n \in \mathbb{Z}} : X_n \in \mathbb{R}^k, n \in \mathbb{Z}\}. \quad (2.3)$$

For any $X, Y \in \Omega_\omega$, $a, b \in \mathbb{R}$, $aX + bY$ is defined by

$$aX + bY = \{aX_n + bY_n\}_{n \in \mathbb{Z}}, \quad (2.4)$$

then $\Omega_\omega$ is a vector space. For given positive integer $\omega$, $E_\omega$ is defined as a subspace of $\Omega_\omega$ by

$$E_\omega = \{X = \{X_n\} \in \Omega_\omega : X_{n+\omega} = X_n, n \in \mathbb{Z}\}. \quad (2.5)$$

Obviously, $E_\omega$ is isomorphic to $\mathbb{R}^{k\omega}$, for any $X, Y \in E_\omega$, defined inner product

$$(X,Y) = \sum_{i=1}^{\omega} \langle X_i, Y_i \rangle, \quad (2.6)$$

by which the norm $\|\cdot\|$ can be induced by

$$\|X\| = \left(\sum_{i=1}^{\omega} \|X_i\|^2\right)^{1/2}, \quad X \in E_\omega. \quad (2.7)$$
where \( \|X_i\| = \left( \sum_{j=1}^{k} |x_{ij}|^2 \right)^{1/2} \). It is obvious that \( E_\omega \) with the inner product defined by (2.6) is a finite-dimensional Hilbert space and linearly homeomorphic to \( \mathbb{R}^{k\omega} \). Define the functional \( J \) on \( E_\omega \) as follows:

\[
J(X) = \frac{1}{\delta + 1} \sum_{n=1}^{\omega} \langle P_{n\ell} (\Delta X_{n-1})^{\delta+1} \rangle - \frac{1}{\delta + 1} \sum_{n=1}^{\omega} \langle Q_n, X_n^{\delta+1} \rangle + \sum_{n=1}^{\omega} F(n, X_n), \quad X \in E_\omega, \tag{2.8}
\]

where \( F(n, X_n) \) such that \( \nabla X F(n, \Omega) = f(n, U) \), that is,

\[
f_i(n, U) = f_i(n, u_1, u_2, \ldots, u_k) = \frac{\partial}{\partial u_i} F(n, u_1, u_2, \ldots, u_k) \quad \tag{2.9}
\]

for any \( (n, U) \in \mathbb{Z}[1, \omega] \times \mathbb{R}^k, \ P_n = (p_{n1}, p_{n2}, \ldots, p_{nk}), \ Q_n = (q_{n1}, q_{n2}, \ldots, q_{nk}). \) Clearly \( J \in C^1(E_\omega, \mathbb{R}) \), and for any \( X = \{X_n\}_{n \in \mathbb{Z}} \in E_\omega \) by \( X_0 = X_\omega \) and \( X_1 = X_{\omega+1} \), we have

\[
\frac{\partial J(X)}{\partial x_{ni}} = -\Delta(p_{ni}(\Delta x_{n-1})^{\delta}) - q_{ni}(x_n)^{\delta} + f_i(n, X_n), \quad l \in [1, k], \ n \in \mathbb{Z}[1, \omega]. \tag{2.10}
\]

Thus \( X = \{X_n\}_{n \in \mathbb{Z}} \) is a critical point of \( J \) on \( E_\omega \) \( (J(X) = 0) \) if and only if

\[
\Delta(p_{ni}(\Delta x_{n-1})^{\delta}) + q_{ni}(x_n)^{\delta} = f_i(n, X_n), \quad l \in [1, k], \ n \in \mathbb{Z}[1, \omega]. \tag{2.11}
\]

That is,

\[
\Delta(p_{ni}^{\top}(\Delta X_{n-1}^{\top})^\delta) + Q_{ni}^{\top}(X_n^{\top})^\delta = f(n, X_n), \quad n \in \mathbb{Z}. \tag{2.12}
\]

By the periodicity of \( X_n \) and \( f(n, X_n) \) in the first variable \( n \), we know that if \( X = \{X_n\}_{n \in \mathbb{Z}} \in E_\omega \) is a critical point of the real functional \( J \) defined by (2.8), then it is a periodic solution of (1.2).

For \( X = \{X_n\}_{n \in \mathbb{Z}} \in E_\omega \), \( X_n = (x_{n1}, x_{n2}, \ldots, x_{nk}) \in \mathbb{R}^k, k > 1, \) denote

\[
\|X\|_r = \left( \sum_{i=1}^{k} \|X_i\|^r \right)^{1/r}, \quad \|X_n\|_r = \left( \sum_{i=1}^{k} \|x_{ni}\|^r \right)^{1/r}. \tag{2.13}
\]

Clearly, \( \|X\|_2 = \|X\|, \|X_n\|_2 = \|X_n\|. \) Because of \( \|\cdot\|_2 \) and \( \|\cdot\|_r \) being equivalent when \( r_1, r_2 > 1, \) so there exist constants \( c_1, c_2, c_3, c_4, h_1, h_2, h_3 \), and \( h_4 \) such that \( c_2 \geq c_1 > 0, c_4 \geq c_3 > 0, h_2 \geq h_1 > 0, \) and \( h_4 \geq h_3 > 0, \)

\[
c_1\|X\| \leq \|X\|_{\delta+1} \leq c_2\|X\|, \quad c_3\|X\| \leq \|X\|_\beta \leq c_4\|X\|, \quad h_1\|X_n\| \leq \|X_n\|_{\delta+1} \leq h_2\|X_n\|, \quad h_3\|X_n\| \leq \|X_n\|_\beta \leq h_4\|X_n\|, \tag{2.14}
\]

for all \( X \in E_\omega, \delta > 0, \) and \( \beta > 1. \)
Lemma 2.2. Suppose that

\((F_1)\) there exist constants \(a_1 > 0, a_2 > 0, \beta > \delta + 1\) such that

\[
F(n, U) \leq -a_1 \|U\|^\beta + a_2
\]  
(2.15)

for any \((n, U) \in \mathbb{Z}[1, \omega] \times R^k;\)

\((F_2)\)

\[
q_{ni} \leq 0, \quad n \in \mathbb{Z}, \; i \in \mathbb{Z}[1, k].
\]  
(2.16)

Then

\[
J(X) = \frac{1}{\delta + 1} \sum_{n=1}^{\omega} \left( P_n \left( \Delta X_{n-1} \right)^{\delta+1} \right) - \frac{1}{\delta + 1} \sum_{n=1}^{\omega} \left( Q_n \left( X_n \right)^{\delta+1} \right) + \sum_{n=1}^{\omega} F(n, X_n)
\]  
(2.17)

satisfies P-S condition.

Proof. For any sequence \(\{X^{(0)}\} = \{X_0^{(0)}, \ldots, X_1^{(0)}, X_2^{(0)}, \ldots, X_{\omega}^{(0)}\} \in E_\omega,\) \(J(X^{(0)})\) is bounded and \(J'(X^{(0)}) \to 0 \; (l \to \infty).\) Then there exists a positive constant \(M > 0,\) such that \(|J(X^{(0)})| \leq M.\) From \((F_1),\) we have

\[
-M \leq J(X^{(0)})
\]

\[
= \frac{1}{\delta + 1} \sum_{n=1}^{\omega} \left[ \left( P_n \left( X_n^{(0)} - X_{n-1}^{(0)} \right)^{\delta+1} \right) - \left( Q_n \left( X_n^{(0)} \right)^{\delta+1} \right) \right] + \sum_{n=1}^{\omega} F(n, X_n^{(0)})
\]

\[
\leq \frac{1}{\delta + 1} \sum_{n=1}^{\omega} 2^{\delta+1} \left( P_n \left( |X_n^{(0)}|^{\delta+1} + |X_{n-1}^{(0)}|^{\delta+1} \right) \right) - \frac{1}{\delta + 1} \sum_{n=1}^{\omega} \left( Q_n \left( X_n^{(0)} \right)^{\delta+1} \right) + \sum_{n=1}^{\omega} F(n, X_n^{(0)})
\]

\[
\leq \frac{2^{\delta+1}}{\delta + 1} \sum_{n=1}^{\omega} \left( P_n + P_{n+1} \right) \left( X_n^{(0)} \right)^{\delta+1} \) - \frac{1}{\delta + 1} \sum_{n=1}^{\omega} \left( Q_n \left( X_n^{(0)} \right)^{\delta+1} \right) + \sum_{n=1}^{\omega} F(n, X_n^{(0)})
\]

\[
\leq \frac{2^{\delta+1}}{\delta + 1} \sum_{n=1}^{\omega} \left( P_n + P_{n+1} \right) \left( X_n^{(0)} \right)^{\delta+1} \) - \frac{1}{\delta + 1} \sum_{n=1}^{\omega} \left( Q_n \left( X_n^{(0)} \right)^{\delta+1} \right) - a_1 \sum_{n=1}^{\omega} \|X_n^{(0)}\|^\beta + a_2 \omega
\]

\[
= \frac{1}{\delta + 1} \sum_{n=1}^{\omega} \left( 2^{\delta+1} \left( P_n + P_{n+1} \right) - Q_n \left( X_n^{(0)} \right)^{\delta+1} \right) - a_1 \|X_n^{(0)}\|^\beta + a_2 \omega.
\]
Set

\[ A_0 = \max_{n \in \mathbb{Z}[1, \omega], l \in \mathbb{Z}[1, k]} \left\{ 2^{\delta + 1} (p_m + p_{(n+1)i}) - q_{nl} \right\}. \]  \hspace{0.5cm} (2.19)

Then \( A_0 > 0, \) and

\[-M \leq J(X^{(l)}) \]
\[ \leq \frac{A_0}{\delta + 1} \sum_{n=1}^{\infty} \|X_n^{(l)}\|^{\delta + 1} \|X^{(l)}\|^{\beta} + a_1 \omega \]
\[ \leq \frac{A_0 \delta^{\delta + 1}}{\delta + 1} \sum_{n=1}^{\infty} \|X_n^{(l)}\|^{\delta + 1} \|X^{(l)}\|^{\beta} + a_1 \omega \]
\[ = \frac{A_0 \delta^{\delta + 1}}{\delta + 1} \|X^{(l)}\|^{\delta + 1} \|X^{(l)}\|^{\beta} + a_1 \omega. \]  \hspace{0.5cm} (2.20)

Because of \( \beta > \delta + 1, \) and \( (\beta - \delta - 1)/\beta + (\delta + 1)/\beta = 1, \) in view of Hölder inequality, we have

\[ \sum_{n=1}^{\infty} \|X_n^{(l)}\|^{\delta + 1} \leq \omega^{(\beta-\delta-1)/\beta} \left( \sum_{n=1}^{\infty} \|X_n^{(l)}\|^{\beta} \right)^{(\delta + 1)/\beta}. \]  \hspace{0.5cm} (2.21)

Thus

\[ \|X^{(l)}\|^{\beta} \geq \omega^{(\delta+1-\beta)/(\delta+1)} \|X^{(l)}\|^{\beta}_{\delta+1}. \]  \hspace{0.5cm} (2.22)

Then we have

\[-M \leq J(X^{(l)}) \]
\[ \leq \frac{A_0 \delta^{\delta + 1}}{\delta + 1} \|X^{(l)}\|^{\delta + 1} \|X^{(l)}\|^{\beta} + a_1 \omega \]
\[ \leq \frac{A_0 \delta^{\delta + 1}}{\delta + 1} \|X^{(l)}\|^{\delta + 1} \|X^{(l)}\|^{\beta} + a_1 \omega. \]  \hspace{0.5cm} (2.23)

Thus, for any \( l \in \mathbb{N}, \)

\[ a_1 \omega^{(\delta+1-\beta)/(\delta+1)} \|X^{(l)}\|^{\beta}_{\delta+1} - \frac{A_0 \delta^{\delta + 1}}{\delta + 1} \|X^{(l)}\|^{\delta + 1}_{\delta+1} \leq M + a_2 \omega. \]  \hspace{0.5cm} (2.24)

Because of \( \beta > \delta + 1, \) it is easily seen that the inequality (2.24) implies that \( \{X^{(l)}\} \) is a bounded sequence in \( E_{\omega}. \) Thus \( \{X^{(l)}\} \) possesses convergent subsequences. The proof is complete. \( \square \)
3. Main Result

Theorem 3.1. Suppose that condition \((F_1)\) holds, and

\((F_3)\) for each \(n \in \mathbb{Z},\)

\[
\lim_{\|U\| \to 0} \frac{F(n, U)}{\|U\|^{\delta+1}} = 0; \tag{3.1}
\]

\((F_4)\) for any \(i \in \mathbb{Z}[1, k], n \in \mathbb{Z}[1, \omega],\)

\[q_{ni} < 0; \tag{3.2}\]

\((F_5)\) \(F(n, \theta) = 0.\)

Then \((1.2)\) has at least two nontrivial \(\omega\)-periodic solutions.

Proof. By Lemma 2.2, \(J\) satisfies P-S condition. Next, we will verify the conditions \((I_1)\) and \((I_2)\) of Lemma 2.1. By \((F_3),\) there exists \(\rho > 0,\) such that

\[|F(n, U)| \leq -\frac{q_{\text{max}} h_1^{\delta+1}}{2(\delta + 1)} \|U\|^{\delta+1} \tag{3.3}\]

for any \(\|U\| < \rho\) and \(n \in \mathbb{Z}[1, \omega],\) where \(q_{\text{max}} = \max_{n \in \mathbb{Z}[1, \omega], i \in \mathbb{Z}[1, k]} q_{ni} < 0.\) Thus

\[
J(X) \geq -\frac{1}{\delta + 1} \sum_{n=1}^{\omega} \langle Q_n, X_n^{\delta+1} \rangle + \sum_{n=1}^{\omega} F(n, X_n)
\]

\[
\geq -\frac{q_{\text{max}} h_1^{\delta+1}}{\delta + 1} \sum_{n=1}^{\omega} \|X_n\|^{\delta+1} + \frac{q_{\text{max}} h_1^{\delta+1}}{2(\delta + 1)} \sum_{n=1}^{\omega} \|X_n\|^{\delta+1}
\]

\[
\geq -\frac{q_{\text{max}} h_1^{\delta+1}}{2(\delta + 1)} \|X\|^{\delta+1}
\]

\[
\geq -\frac{q_{\text{max}} h_1^{\delta+1} c_1^{\delta+1}}{2(\delta + 1)} \|X\|^{\delta+1}
\]

for any \(X \in E_{\omega},\) with \(\|X\| \leq \rho.\) We choose \(a = -h_1^{\delta+1} c_1^{\delta+1} (q_{\text{max}} / 2(\delta + 1)) \rho^{\delta+1},\) then we have

\[
J(X)|_{\partial B_{\rho}} \geq a > 0, \tag{3.5}
\]

that is, the condition \((I_1)\) of Lemma 2.1 holds.
Obviously, $J(0) = 0$. For any given $V \in E_\omega$ with $\|V\| = 1$ and constant $\alpha > 0$,

$$J(\alpha V) = \frac{1}{\delta + 1} \sum_{n=1}^{\omega} \left( P_n (\alpha V_n - \alpha V_{n-1})^\delta + \sum_{n=1}^{\omega} (Q_n (\alpha V_n)^\delta) + F(n, \alpha V_n) \right)$$

$$= \frac{1}{\delta + 1} \sum_{n=1}^{\omega} \left( p_n (\alpha v_{n1} - \alpha v_{n-1,1})^\delta + p_n (\alpha v_{n2} - \alpha v_{n-1,2})^\delta + \cdots + p_n (\alpha v_{nk} - \alpha v_{n-1,k})^\delta \right)$$

$$- \frac{1}{\delta + 1} \sum_{n=1}^{\omega} \left( q_n (\alpha v_{n1})^\delta + q_n (\alpha v_{n2})^\delta + \cdots + q_n (\alpha v_{nk})^\delta \right) + \sum_{n=1}^{\omega} F(n, \alpha V_n)$$

$$\leq \frac{1}{\delta + 1} \sum_{n=1}^{\omega} \left( 2^\delta + 1 \right) \left( \|P_n\| + \|Q_n\| \right) \alpha^\delta + a_1 \alpha^\beta \sum_{n=1}^{\omega} \|V_n\|^\beta + a_2 \omega$$

$$\leq \frac{1}{\delta + 1} \sum_{n=1}^{\omega} \left( 2^\delta + 1 \right) \left( \|P_n\| + \|Q_n\| \right) \alpha^\delta + a_1 \alpha^\beta \|V\|^\beta + a_2 \omega$$

$$\leq \frac{1}{\delta + 1} \sum_{n=1}^{\omega} \left( 2^\delta + 1 \right) \left( \|P_n\| + \|Q_n\| \right) \alpha^\delta + a_1 \alpha^\beta \|V\|^\beta + a_2 \omega$$

$$\rightarrow -\infty, \quad (\alpha \rightarrow +\infty).$$

Thus we can choose a sufficiently large $\alpha$ such that $\alpha > \rho$, and $\overline{X} = \alpha V \in E_\omega$, $J(\overline{X}) < 0$. According to Lemma 2.1, there exists at least one critical value $c \geq \alpha > 0$. We suppose that $X^*$ is a critical point corresponding to $c$, then $J(X^*) = c$ and $J'(X^*) = 0$.

By similar argument of Lemma 2.2, we know that $J(X)$ is bounded from above, so there exists $X^{**} \in E_\omega$ such that $J(X) \leq J(X^{**}) = c_{\text{max}}$ for any $X \in E_\omega$. Obviously, $X^{**} \neq 0$. If $X^{**} \neq X^*$, then the proof is complete. Otherwise, $X^{**} = X^*$, $c = c_{\text{max}}$. In view of Lemma 2.1,

$$c = \inf_{h \in \Gamma} \sup_{s \in [0,1]} J(h(s)), \quad (3.7)$$

where $\Gamma = \{h \in C([0,1], E_\omega) : h(0) = \theta, h(1) = \overline{X}\}$. Then $c_{\text{max}} = \max_{s \in [0,1]} J(h(s))$ for any $h \in \Gamma$ holds. In view of the continuity of $J(h(s))$ in $s$, $J(\theta) \leq 0$, and $J(\overline{X}) < 0$, we know that there
exists some \( s_0 \in (0, 1) \) such that \( J(h(s_0)) = c_{\text{max}} \). If we choose \( h_1, h_2 \in \Gamma \) such that
\[
\{h_1(s) : s \in (0, 1)\} \cap \{h_2(s) : s \in (0, 1)\} = \emptyset,
\]
then there exist \( s_1, s_2 \in (0, 1) \) such that \( J(h_1(s_1)) = J(h_2(s_2)) = c_{\text{max}} \). Then \( J \) possesses two different critical points \( \hat{Y} = h_1(s_1) \) and \( \hat{Z} = h_2(s_2) \) in \( E_\omega \), hence, we obtain at least two nontrivial critical points which correspond to the critical value \( c_{\text{max}} \). Thus (1.2) possesses at least two nontrivial \( \omega \)-periodic solutions. The proof is complete.

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