Research Article

On the Global Attractivity of a Max-Type Difference Equation

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We investigate asymptotic behavior and periodic nature of positive solutions of the difference equation $x_n = \max\{A/x_{n-1}, 1/x_{n-3}^\alpha\}$, $n = 0, 1, \dots$, where $A > 0$ and $0 < \alpha < 1$. We prove that every positive solution of this difference equation approaches $\bar{x} = 1$ or is eventually periodic with period 2.

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1. Introduction

In the recent years, there has been a lot of interest in studying the global attractivity and the periodic nature of, so-called, max-type difference equations (see, e.g., [1–17] and references therein).

In [10], the following difference equation was proposed by Ladas:

$$x_n = \max \left\{ \frac{A_1}{x_{n-1}}, \frac{A_2}{x_{n-2}}, \dots, \frac{A_p}{x_{n-p}} \right\}, \quad n = 0, 1, \dots, \quad (1.1)$$

where $A_1, A_2, \dots, A_p$ are real numbers and initial conditions are nonzero real numbers.

In [17], asymptotic behavior of positive solutions of the difference equation was investigated

$$x_n = \max \left\{ \frac{1}{x_{n-1}^\alpha}, \frac{A}{x_{n-2}} \right\}, \quad n = 0, 1, \dots, \quad (1.2)$$

where $0 < \alpha < 1$ and $0 < A$. It was showed that every positive solution of this difference equation approaches $\bar{x} = 1$ or is eventually periodic with period 4.

In [14], it was proved that every positive solution of the difference equation,

$$x_n = \max \left\{ \frac{A}{x_{n-1}^\alpha}, \frac{B}{x_{n-2}^\beta} \right\}, \quad n = 0, 1, \dots, \quad (1.3)$$

where $0 < \alpha, \beta < 1, 0 < A$, and $0 < B$, converges to $\bar{x} = \max\{A^{1/(\alpha+1)}, B^{1/(\beta+1)}\}$.

In this paper, we investigate the difference equation

$$x_n = \max \left\{ \frac{A}{x_{n-1}}, \frac{1}{x_{n-3}^\alpha} \right\}, \quad n = 0, 1, \dots, \quad (1.4)$$

where $0 < A, 0 < \alpha < 1$ and initial conditions are positive real numbers. We prove that every positive solution of this difference equation approaches $\bar{x} = 1$ or is eventually periodic with period 2.

2. The Case $A = 1$

In this section, we consider the difference equation

$$x_n = \max \left\{ \frac{1}{x_{n-1}}, \frac{1}{x_{n-3}^\alpha} \right\}, \quad n = 0, 1, \dots, \quad (2.1)$$

where $0 < \alpha < 1$.

Theorem 2.1. *Let x_n be a solution of (2.1). Then x_n approaches $\bar{x} = 1$.*

Proof. Choose a number B such that $0 < B < 1$, let $x_n = B^{y_n}$ for $n \geq -3$. Then (2.1) implies the difference equation

$$y_n = \min \{ -y_{n-1}, -\alpha y_{n-3} \}, \quad n = 0, 1, \dots, \quad (2.2)$$

where $0 < \alpha < 1$ and initial conditions are real numbers.

Let y_n be a solution of (2.2). Then it suffices to prove $y_n \rightarrow 0$. Observe that there exists a positive integer N such that

$$y_N = -y_{N-1}, \quad y_{N+1} = -\alpha y_{N-2} \quad \text{for } N \geq 0. \quad (2.3)$$

By computation, we get that

$$y_{N+2} = -y_{N+1}, \quad y_{N+3} = -\alpha y_N, \quad y_{N+4} = -y_{N+3} \quad (2.4)$$

and then

$$\begin{aligned} y_{4n+N} &= \alpha^n y_N, & y_{4n+N+1} &= \alpha^n y_{N+1}, & \forall n \geq 0. \\ y_{4n+N+2} &= -y_{4n+N+1}, & y_{4n+N+3} &= -\alpha y_{4n+N} \end{aligned} \quad (2.5)$$

So, $y_{4n+N} \rightarrow 0, y_{4n+N+1} \rightarrow 0, y_{4n+N+2} \rightarrow 0, y_{4n+N+3} \rightarrow 0$. This implies $y_n \rightarrow 0$. $\square$

3. The Case $0 < A < 1$

In this section, we consider (1.4), where $0 < \alpha < 1$.

Let $x_n = A^{y_n}, n \geq -3$. Equation (1.4) implies the difference equation

$$y_n = \min \{1 - y_{n-1}, -\alpha y_{n-3}\}, \quad n = 0, 1, \dots, \quad (3.1)$$

where initial conditions are real numbers.

Lemma 3.1. *Let y_n be a solution of (3.1). Then for all $n \geq 0$,*

$$|y_n| \leq \max \{|y_{n-1}| - 1, \alpha |y_{n-3}|\}. \quad (3.2)$$

Proof. From (3.1), we have the following statements:

- if $y_{n-1} \geq 0$ and $y_{n-3} \geq 0$, then $|y_n| \leq \max\{|y_{n-1}| - 1, \alpha |y_{n-3}|\}$;
- if $y_{n-1} \leq 0$ and $y_{n-3} \leq 0$, then $|y_n| \leq \alpha |y_{n-3}|$;
- if $y_{n-1} \geq 0$ and $y_{n-3} \leq 0$, then $|y_n| \leq \max\{|y_{n-1}| - 1, \alpha |y_{n-3}|\}$;
- if $y_{n-1} \leq 0$ and $y_{n-3} \geq 0$, then $|y_n| = \alpha |y_{n-3}|$.

In general, we have $|y_n| \leq \max\{|y_{n-1}| - 1, \alpha |y_{n-3}|\}$ for all $n \geq 0$. $\square$

Theorem 3.2. *if x_n is a solution of (1.4), x_n approaches $\bar{x} = 1$.*

Proof. Let y_n be a solution of (3.1). To prove $x_n \rightarrow 1$, it suffices to prove $y_n \rightarrow 0$.

Choose a number β such that $0 < |y_{n-1}| - 1 \leq \beta |y_n|$. Then from inequality (3.2), we get that

$$|y_n| \leq \max \{\beta |y_{n-1}|, \alpha |y_{n-3}|\}, \quad n \geq 0. \quad (3.3)$$

Let $\gamma = \max\{\beta, \alpha\}$, then $0 < \gamma < 1$ and

$$|y_n| \leq \gamma \max\{|y_{n-1}|, |y_{n-3}|\}, \quad n \geq 0. \quad (3.4)$$

From (3.4) and by induction, we get that

$$|y_n| \leq \gamma^{\lfloor n/3 \rfloor + 1} \max\{|y_{-1}|, |y_{-2}|, |y_{-3}|\}, \quad n \geq 0. \quad (3.5)$$

This implies $y_n \rightarrow 0$. $\square$

4. The Case $A > 1$

In this section, we consider (1.4). Let $x_n = A^{y_n}$ for $n \geq -3$. Equation (1.4) implies the difference equation

$$y_n = \max \{1 - y_{n-1}, -\alpha y_{n-3}\}, \quad n = 0, 1, \dots, \quad (4.1)$$

where $0 < \alpha < 1$ and initial conditions are real numbers.

Theorem 4.1. *If x_n is a solution of (1.4), then the following statements are true:*

(a) x_n approaches $\bar{x} = 1$, if there is an integer N such that

$$x_N = \frac{A}{x_{N-1}}, \quad x_{N+1} = \frac{1}{x_{N-2}^\alpha} \quad \text{for } 0 \leq n \leq N. \quad (4.2)$$

(b) x_n is eventually periodic with period 2, if there is an integer N such that

$$x_N = \frac{A}{x_{N-1}}, \quad x_{N+1} = x_{N-1}, \quad A^{-\alpha/(1-\alpha)} \leq x_{N-1} \leq A^{1/(1-\alpha)} \quad \text{for } 0 \leq n \leq N. \quad (4.3)$$

Proof. (a) Change of variables $x_n = A^{y_n}$, $n \geq -3$. If $x_N = A/x_{N-1}$ and $x_{N+1} = 1/x_{N-2}^\alpha$ for $0 \leq n \leq N$, then $y_N = 1 - y_{N-1}$ and $y_{N+1} = -\alpha y_{N-2}$. Let y_n be a solution of (4.1). So, to prove $x_n \rightarrow 1$, it suffices to prove $y_n \rightarrow 0$. From (4.1), there is at least an integer N such that $y_N = 1 - y_{N-1} > 0$ and $y_{N+1} = -\alpha y_{N-2}$ for $0 \leq n \leq N$. By computation from (4.1), we get that $y_{N+1} < 0$, $y_{N-2} \geq y_N > 0$ and then

$$\begin{aligned} y_{N+2} &= 1 + \alpha y_{N-2}, & y_N &> y_{N+2} > 0, \\ y_{N+3} &= -\alpha y_N, & y_{N+1} &< y_{N+3} < 0, \\ y_{N+4} &= 1 + \alpha y_N, & y_N &> y_{N+2} \geq y_{N+4} > 0, \\ y_{N+5} &= -\alpha y_{N+2}, & y_{N+1} &\leq y_{N+3} < y_{N+5} < 0. \end{aligned} \quad (4.4)$$

So, we have

$$y_{4n+N} > y_{4n+N+2} \geq y_{4n+N+4} > 0, \quad y_{4n+N+1} \leq y_{4n+N+3} < y_{4n+N+5} < 0 \quad (4.5)$$

for all $n \geq 0$. This implies $y_n \rightarrow 0$.

(b) Change of variables $x_n = A^{y_n}$, $n \geq -3$. Let y_n be a solution of (4.1).

If $x_N = A/x_{N-1}$, $x_{N+1} = x_{N-1}$, and $A^{-\alpha/(1-\alpha)} \leq x_{N-1} \leq A^{1/(1-\alpha)}$ for $0 \leq n \leq N$, then

$$y_N = 1 - y_{N-1}, \quad y_{N+1} = y_{N-1}, \quad \frac{-\alpha}{1-\alpha} \leq y_{N-1} \leq \frac{1}{1-\alpha}. \quad (4.6)$$

Clearly, there is at least an integer N such that $y_N = 1 - y_{N-1}$ for $0 \leq n \leq N$.

Suppose that $y_{N+1} = y_{N-1}$ and $y_{N-1} \notin [-\alpha/(1-\alpha), 1/(1-\alpha)]$.

If $y_{N-1} > 1/(1-\alpha)$ then from (4.1), we have $y_{N+2} = -\alpha y_{N-1}$. So, from (a) we get immediately that $y_n \rightarrow 0$.

If $y_{N-1} < -\alpha/(1-\alpha)$, then we have $y_{N+2} = 1 - y_{N+1}$ and $y_{N+3} = -\alpha y_N$. Then we get that $y_n \rightarrow 0$, from (a).

We assume that $y_{N+1} = y_{N-1}$ and $-\alpha/(1-\alpha) \leq y_{N-1} \leq 1/(1-\alpha)$. To prove the desired result, it suffices to show that y_n is eventually periodic with period 2. By computation from (4.1), we get immediately $y_N = 1 - y_{N-1}$ for all $0 \leq n \leq N$. This is the desired result. $\square$

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