Research Article

On the Cauchy Problem of a Quasilinear Degenerate Parabolic Equation

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By Oleinik's line method, we study the existence and the uniqueness of the classical solution of the Cauchy problem for the following equation in $[0, T] \times \mathbb{R}^2$: $\partial_{xx}u + u\partial_yu - \partial_tu = f(\cdot, u)$, provided that T is suitable small. Results of numerical experiments are reported to demonstrate that the strong solutions of the above equation may blow up in finite time.

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1. Introduction

We consider the following Cauchy problem:

$$\partial_{xx}u + u\partial_yu - \partial_tu = f(\cdot, u), \quad (t, x, y) \in (0, T] \times \mathbb{R}^2, \quad (1.1)$$

$$u(0, \cdot) = u_0(x, y), \quad (x, y) \in \mathbb{R}^2. \quad (1.2)$$

This problem arises in financial mathematics recently; more and more mathematicians have been interested in it. In [1], Antonelli et al. introduced a new model for agents' decision under risk, in which the utility function is the solution to (1.1)-(1.2); they also proved, by means of probability methods, the existence of a continuous viscosity solution of (1.1)-(1.2), which satisfies

$$|u(x, y, t) - u(\xi, \eta, \tau)| \leq C_T(|x - \xi| + |y - \eta|) \quad (1.3)$$

for every $(x, y), (\xi, \eta) \in \mathbb{R}^2$, $t \in [0, T]$, under the assumption that f is uniformly Lipschitz continuous function. In [2], Citti et al. studied the interior regularity properties of this problem; they proved that the viscosity solutions are indeed classical solutions. On the

other hand, Antonelli and Pascucci [3] showed that the solution u found in [1] can be also considered as a distributional solution.

However, all the above results are obtained when T is suitably small; say, the solution is local. The global weak solutions of the Cauchy problem for a more general class of equations, that contains (1.1), are obtained in [4–7], and so forth. This kind of solutions, however, is few regular and does not satisfy condition (1.3) in general.

In this paper, we will solve the Cauchy problem (1.1)-(1.2) in another simpler way and get the result as [2] again. Moreover, some examples are provided by numerical computation. The results of computation show that the strong solutions of the above equation may blow-up in finite time, though there exist the global weak solutions.

2. Line Method

In order to describe our method, we have to quote the well-known Prandtl system for a nonstationary boundary layer arising in an axially symmetric incompressible flow past a solid body, it has the form

$$\begin{aligned}\partial_t u + u\partial_x u + v\partial_y u &= \partial_t U + U\partial_x U + \partial_y^2 u, \\ \partial_x(ru) + \partial_y(rv) &= 0\end{aligned}\tag{2.1}$$

in a domain $D = \{0 < t < T, 0 < x < X, 0 < y < \infty\}$, where $U(t, x)$ and $r(x)$ are given functions. If we introduce the Crocco variables:

$$\tau = t, \quad \xi = x, \quad \eta = \frac{u(t, x)}{U(t, x)},\tag{2.2}$$

we obtain the following equation for $w(\tau, \xi, \eta) = \partial_y u / U$:

$$w^2 w_{\eta\eta} - w_\tau - \eta U w_\xi + A w_\eta + B w = 0.\tag{2.3}$$

Oleinik and Samokhin [8] had done excellent work in the boundary theory by the line method. Comparing this equation with (1.1), it is natural to conjecture that we are able to solve problem (1.1)-(1.2) by Oleinik's method.

Consider the following initial boundary problem:

$$w_{\eta\eta} - w_\tau + w w_\xi = f(\tau, \xi, \eta, w),\tag{2.4}$$

$$w(0, \xi, \eta) = u_0(\xi, \eta),\tag{2.5}$$

where $u_0 \in C^2(R^2)$; its first-order derivatives and $u_{0\eta\eta}$ are all bounded.

Definition 2.1. A function $w(\tau, \xi, \eta)$ is said to be a solution of problem (2.4)-(2.5) if w has first-order derivatives in (2.4) which is continuous in $(0, T] \times R^2$, and its derivative $w_{\eta\eta}$ is continuous; w satisfies (2.4) in $(0, T] \times R^2$, together with condition (2.5).

The solution of problem (2.4)-(2.5) will be constructed as the limit of a sequence w^n , $n \rightarrow \infty$, which consists of solutions of the equations

$$L_n(w^n) = w_{\eta\eta}^n - w_\tau^n + w^{n-1}w_\xi^n - f(\cdot, w^n) = 0, \quad (2.6)$$

$$w^n(0, \xi, \eta) = w_0(\xi, \eta). \quad (2.7)$$

As $w^0(\tau, \xi, \eta)$ we take a function which is smooth in $[0, T] \times R^2$.

Suppose that for some nonnegative number p

$$|f(\cdot, v)| \leq c(1 + |v|^p), \quad (2.8)$$

and when $v_1 - v_2 \geq 0$,

$$c_1(v_1 - v_2) \geq f(\cdot, v_1) - f(\cdot, v_2) \geq c_2(v_1 - v_2),$$

$$\max \left\{ \left| \frac{\partial f}{\partial \tau} \right|, \left| \frac{\partial f}{\partial \xi} \right|, \left| \frac{\partial f}{\partial \eta} \right|, \left| \frac{\partial^2 f}{\partial v^2} \right| \right\} \leq c. \quad (2.9)$$

Lemma 2.2. *Let V be a smooth function such that $L_n(V) \geq 0$ in $(0, T) \times R^2$, $V \leq w^n$ for $\tau = 0$. Then $V \leq w^n$ everywhere in $(0, T) \times R^2$. Let V_1 be a smooth function such that $L_n(V_1) \leq 0$ in $(0, T) \times R^2$, $V \geq w^n$ for $\tau = 0$. Then $V_1 \geq w^n$ everywhere in $(0, T) \times R^2$.*

Proof. Let us prove the first statement of Lemma 2.2. The difference $z^n = w^n - V$ satisfies the inequality

$$0 \geq L_n(z^n) = L_n(w^n) - L_n(V) = z_{\eta\eta}^n - z_\tau^n + w^{n-1}z_\xi^n - (f(\cdot, w^n) - f(\cdot, V)). \quad (2.10)$$

Let $z_1^n = e^{-\alpha\tau} z^n$. Then

$$\begin{aligned} 0 &\geq z_{1\eta\eta}^n - z_{1\tau}^n + \alpha z_1^n + w^{n-1}z_{1\xi}^n - e^{-\alpha\tau}(f(\cdot, w^n) - f(\cdot, V)) \\ &\geq z_{1\eta\eta}^n - z_{1\tau}^n - \alpha z_1^n + w^{n-1}z_{1\xi}^n - c_1 z_1^n. \end{aligned} \quad (2.11)$$

If we choose α large enough, by the maximal principle, we know $V \leq w^n$ everywhere in $(0, T) \times R^2$.

Let us construct functions satisfying the conditions of Lemma 2.2. To this end, we define a twice continuously differentiable even function such that $V_1 = (1 - e^{-\beta|\eta|})e^{\beta\tau}$ for $|\eta| > 1$, $V_1 = \varphi(\eta)e^{\beta\tau}$ for $|\eta| \leq 1$, where $\varphi(\eta)$ is a C^2 function, $|\varphi_{\eta\eta}| \leq c$.

When $|\eta| > 1$,

$$\begin{aligned} L_n(V_1) &= V_{1\eta\eta}^n - V_{1\tau}^n - w^{n-1}V_{1\xi}^n - f(\cdot, V_1) \\ &= -\beta_1^2 e^{-\beta_1|\eta|} e^{\beta\tau} - \beta \left(1 - e^{-\beta_1|\eta|}\right) e^{\beta\tau} - f(\cdot, V_1) \\ &\leq -\beta_1^2 e^{-\beta_1|\eta|} e^{\beta\tau} - \beta \left(1 - e^{-\beta_1|\eta|}\right) e^{\beta\tau} + c \left(1 - e^{-\beta_1|\eta|}\right)^p e^{p\beta\tau} + c < 0 \end{aligned} \quad (2.12)$$

if we chose β large enough and $\beta\tau \leq T_0$ small enough.

When $|\eta| \leq 1$,

$$\begin{aligned} L_n(V_1) &= \varphi_{\eta\eta} e^{\beta\tau} - \beta\varphi e^{\beta\tau} - f(\cdot, V_1) \\ &\leq \varphi_{\eta\eta} e^{\beta\tau} - \beta\varphi e^{\beta\tau} + c(1 + \varphi^p e^{\beta\tau p}) < 0 \end{aligned} \quad (2.13)$$

by the same reason.

Let $V = \psi(\eta)e^{-\alpha\tau}$, $\alpha_1 > \psi(\eta) \geq \alpha_0 > 0$, $|\psi_{\eta\eta}| \leq c$. Then

$$\begin{aligned} L_n(V) &= \psi_{\eta\eta} e^{-\alpha\tau} + \alpha\psi e^{\alpha\tau} - f(\cdot, V) \\ &\geq \psi_{\eta\eta} e^{-\alpha\tau} + \alpha\psi e^{\alpha\tau} - c(1 + \psi^p e^{\alpha\tau p}) \geq 0 \end{aligned} \quad (2.14)$$

if we chose α large enough and $\alpha\tau \leq T_0$ small enough.

Similarly, we are able to prove the second statement of Lemma 2.2. $\square$

Thus we have the following.

Lemma 2.3. *Suppose that f satisfies (2.9) and $V(0, \xi, \eta) \leq w_0 \leq V_1(0, \xi, \eta)$, then*

$$V \leq w^n \leq V_1. \quad (2.15)$$

The smooth functions V, V_1 can be constructed as in [8], and we omit details here.

Let

$$\Phi_n = \Phi = (u_\tau^n)^2 + (u_\xi^n)^2 + (u_\eta^n)^2, \quad (2.16)$$

where $u^n = w^n$. We will show that there exist positive constants M and T such that the conditions $\Phi_\mu \leq M$ for $\tau \leq T$, $\mu \leq n-1$, imply that $\Phi_n \leq M$ for $\tau \leq T$.

First, we rewrite (2.6) as

$$u_{\eta\eta}^n - u_\tau + u^{n-1}u_\xi^n - f(\cdot, u^n) = 0, \quad (\tau, \xi, \eta) \in (0, T] \times \mathbb{R}^2. \quad (2.17)$$

Applying the operator $2u_\tau^n(\partial/\partial\tau) + 2u_\xi^n(\partial/\partial\xi) + 2u_\eta^n(\partial/\partial\eta)$ to (2.17),

$$\begin{aligned} &2u_\tau^n u_{\tau\eta\eta}^n + 2u_\tau^n (u_\tau^{n-1} u_\xi^n + u^{n-1} u_{\xi\tau}^n) - 2u_\tau^n u_{\tau\tau}^n - 2 \frac{\partial f}{\partial u} (u_\tau^n)^2 - 2u_\tau^n \frac{\partial f}{\partial \tau}, \\ &2u_\xi^n u_{\xi\eta\eta}^n + 2u_\xi^n (u_\xi^{n-1} u_\xi^n + u^{n-1} u_{\xi\xi}^n) - 2u_\xi^n u_{\tau\xi}^n - 2 \frac{\partial f}{\partial u} (u_\xi^n)^2 - 2u_\xi^n \frac{\partial f}{\partial \xi}, \\ &2u_\eta^n u_{\eta\eta\eta}^n + 2u_\eta^n (w_\eta^{n-1} u_\xi^n + w^{n-1} u_{\xi\eta}^n) - 2u_\eta^n u_{\tau\eta}^n - 2 \frac{\partial f}{\partial u} (u_\eta^n)^2 - 2u_\eta^n \frac{\partial f}{\partial \xi} = 0, \end{aligned} \quad (2.18)$$

then

$$\begin{aligned}
u^{n-1}\Phi_\xi &= \left(2u_\tau^n u_{\tau\xi}^n + 2u_\xi^n u_{\xi\xi}^n + 2u_\eta^n u_{\eta\xi}^n\right)u^{n-1}, \\
-\Phi_\tau &= -2u_\tau^n u_{\tau\tau}^n - 2u_\xi^n u_{\xi\tau}^n - 2u_\eta^n u_{\eta\tau}^n, \\
\Phi_{\eta\eta} &= 2(u_{\tau\eta})^2 + 2u_\tau^n u_{\tau\eta\eta}^n + 2(u_{\xi\eta})^2 + 2u_\xi^n u_{\xi\eta\eta}^n + 2(u_{\eta\eta})^2 + 2u_\eta^n u_{\eta\eta\eta}^n, \\
\Phi_{\eta\eta} + u^{n-1}\Phi_\xi - \Phi_\tau - 2\frac{\partial f}{\partial u}\Phi + 2u_\tau^n u_\xi^n u_\xi^{n-1} + 2(u_\xi^n)^2 u_\xi^{n-1} + 2u_\eta^n u_\xi^n u_\eta^{n-1} - 2u_\tau^n \frac{\partial f}{\partial \tau} - 2u_\xi^n \frac{\partial f}{\partial \xi} - 2u_\eta^n \frac{\partial f}{\partial \xi} &= 0.
\end{aligned} \tag{2.19}$$

By (2.9), (2.15), and Cauchy inequality, we are able to get

$$\Phi_{\eta\eta} + u^{n-1}\Phi_\xi - \Phi_\tau + R^n\Phi \geq 0, \tag{2.20}$$

where R^n depends on u^{n-1} and its derivatives are up to the second. Let $\Phi_1 = \Phi e^{-\gamma\tau}$ with a positive constant γ to be chosen later. Then

$$\Phi_{1\eta\eta} + u^{n-1}\Phi_{1\xi} - \Phi_{1\tau} + (R^n - \gamma)\Phi \geq 0 \tag{2.21}$$

if we choose γ according to M such that $R^n - \gamma \leq -1$. If Φ_1 attains its positive maximum at $\tau = 0$, then

$$\Phi_1|_{\tau=0} = \Phi e^{-\gamma\tau}|_{\tau=0} = \Phi|_{\tau=0} \leq c, \tag{2.22}$$

where the constant c does not depend on n . At the same time, the positive maximum of Φ_1 in $(0, T] \times R^2$ cannot be attained by maximal principle. Thus we have

$$\Phi_1 \leq c. \tag{2.23}$$

So, if we let $T_1 \leq T$ small enough such that $e^{\gamma T_1} = 2$ and set $M = 2c$, then

$$\Phi \leq ce^{\gamma T_1} = M. \tag{2.24}$$

In order to estimate the second derivatives of u^n in $[0, T_1] \times R^2$, consider the function

$$F = (u_{\tau\tau}^n)^2 + (u_{\xi\xi}^n)^2 + (u_{\eta\eta}^n)^2 + (u_{\tau\xi}^n)^2 + (u_{\xi\eta}^n)^2 + (u_{\tau\eta}^n)^2. \tag{2.25}$$

Applying the operator

$$P = 2u_{\tau\tau}^n \frac{\partial^2}{\partial \tau^2} + 2u_{\xi\xi}^n \frac{\partial^2}{\partial \xi^2} + 2u_{\eta\eta}^n \frac{\partial^2}{\partial \eta^2} + 2u_{\tau\xi}^n \frac{\partial^2}{\partial \tau \partial \xi} + 2u_{\tau\eta}^n \frac{\partial^2}{\partial \tau \partial \eta} + 2u_{\xi\eta}^n \frac{\partial^2}{\partial \xi \partial \eta} \tag{2.26}$$

to both sides of (2.17), we find that

$$\begin{aligned}
0 = & 2u_{\tau\tau}^n u_{\eta\eta\tau\tau}^n + 2u_{\tau\tau}^n \left(u_{\tau\tau}^{n-1} u_{\xi}^n + 2u_{\tau}^{n-1} u_{\xi\tau}^n + u_{\xi\tau\tau}^{n-1} u_{\xi}^n \right) \\
& - 2u_{\tau\tau}^n u_{\tau\tau\tau\tau}^n - 2u_{\tau\tau}^n \left(\frac{\partial^2 f}{\partial u^2} (u_{\tau}^n)^2 + \frac{\partial f}{\partial u} u_{\tau\tau}^n \right) \\
& + 2u_{\xi\xi}^n u_{\eta\eta\xi\xi}^n + 2u_{\xi\xi}^n \left(u_{\xi\xi}^{n-1} u_{\xi}^n + 2u_{\xi}^{n-1} u_{\xi\xi}^n + u_{\xi\xi\xi}^{n-1} u_{\xi}^n \right) \\
& - 2u_{\xi\xi}^n u_{\tau\xi\xi}^n - 2u_{\xi\xi}^n \left(\frac{\partial^2 f}{\partial u^2} (u_{\xi}^n)^2 + \frac{\partial f}{\partial u} u_{\xi\xi}^n \right) \\
& + 2u_{\eta\eta}^n u_{\eta\eta\eta\eta}^n + 2u_{\eta\eta}^n \left(u_{\eta\eta}^{n-1} u_{\xi}^n + 2u_{\eta}^{n-1} u_{\xi\eta}^n + u_{\xi\eta\eta}^{n-1} u_{\xi}^n \right) \\
& - 2u_{\eta\eta}^n u_{\tau\xi\xi}^n - 2u_{\eta\eta}^n \left(\frac{\partial^2 f}{\partial u^2} (u_{\eta}^n)^2 - \frac{\partial f}{\partial u} u_{\eta\eta}^n \right) \\
& + 2u_{\tau\xi}^n u_{\eta\eta\tau\xi}^n + 2u_{\tau\xi}^n \left(u_{\tau\xi}^{n-1} u_{\xi}^n + u_{\tau}^{n-1} u_{\xi\xi}^n + u_{\xi}^{n-1} u_{\xi\tau}^n + u_{\xi\xi\tau}^{n-1} u_{\xi}^n \right) \\
& - 2u_{\tau\xi}^n u_{\tau\tau\xi}^n - 2u_{\tau\xi}^n \left(\frac{\partial^2 f}{\partial u^2} u_{\tau}^n u_{\xi}^n + \frac{\partial f}{\partial u} u_{\tau\xi}^n \right) \\
& + 2u_{\xi\eta}^n u_{\eta\eta\xi\eta}^n + 2u_{\xi\eta}^n \left(u_{\xi\eta}^{n-1} u_{\xi}^n + u_{\xi}^{n-1} u_{\xi\eta}^n + u_{\eta}^{n-1} u_{\xi\xi}^n + u_{\xi\xi\eta}^{n-1} u_{\xi}^n \right) \\
& - 2u_{\xi\eta}^n u_{\tau\tau\xi}^n - 2u_{\xi\eta}^n \left(\frac{\partial^2 f}{\partial u^2} u_{\eta}^n u_{\xi}^n + \frac{\partial f}{\partial u} u_{\xi\eta}^n \right) \\
& + 2u_{\tau\eta}^n u_{\eta\eta\tau\eta}^n + 2u_{\tau\eta}^n \left(u_{\tau\eta}^{n-1} u_{\xi}^n + u_{\tau}^{n-1} u_{\xi\eta}^n + u_{\eta}^{n-1} u_{\xi\tau}^n + u_{\xi\tau\eta}^{n-1} u_{\xi}^n \right) - 2u_{\tau\eta}^n u_{\tau\tau\eta}^n \\
& - 2u_{\tau\eta}^n \left(\frac{\partial^2 f}{\partial u^2} u_{\eta}^n u_{\tau}^n + \frac{\partial f}{\partial u} u_{\tau\eta}^n \right).
\end{aligned} \tag{2.27}$$

At the same time, we can calculate that

$$\begin{aligned}
F_{\eta} &= 2u_{\tau\tau}^n u_{\tau\tau\eta}^n + 2u_{\xi\xi}^n u_{\xi\xi\eta}^n + 2u_{\eta\eta}^n u_{\eta\eta\eta}^n + 2u_{\tau\xi}^n u_{\tau\xi\eta}^n + 2u_{\xi\eta}^n u_{\xi\eta\eta}^n + 2u_{\tau\eta}^n u_{\tau\eta\eta}^n, \\
F_{\eta\eta} &= 2 \left(u_{\tau\tau\eta}^n \right)^2 + 2u_{\tau\tau}^n u_{\tau\tau\eta\eta}^n + 2 \left(u_{\xi\xi\eta}^n \right)^2 + 2u_{\xi\xi}^n u_{\eta\eta\xi\xi}^n + 2 \left(u_{\eta\eta\eta}^n \right)^2 + 2u_{\eta\eta}^n u_{\eta\eta\eta\eta}^n \\
&\quad + 2 \left(u_{\tau\xi\eta}^n \right)^2 + 2u_{\tau\xi}^n u_{\tau\xi\eta\eta}^n + 2 \left(u_{\eta\xi\xi}^n \right)^2 + 2u_{\xi\eta}^n u_{\xi\eta\eta\eta}^n + 2 \left(u_{\tau\eta\eta}^n \right)^2 + 2u_{\tau\eta}^n u_{\tau\eta\eta\eta}^n, \\
u^{n-1} F_{\xi} &= u^{n-1} \left(2u_{\tau\tau}^n u_{\tau\tau\xi}^n + 2u_{\xi\xi}^n u_{\xi\xi\xi}^n + 2u_{\eta\eta}^n u_{\eta\eta\xi}^n + 2u_{\tau\xi}^n u_{\tau\xi\xi}^n + 2u_{\xi\eta}^n u_{\xi\eta\xi}^n + 2u_{\tau\eta}^n u_{\tau\eta\xi}^n \right), \\
-F_{\tau} &= - \left(2u_{\tau\tau}^n u_{\tau\tau\tau}^n + 2u_{\xi\xi}^n u_{\xi\xi\tau}^n + 2u_{\eta\eta}^n u_{\eta\eta\tau}^n + 2u_{\tau\xi}^n u_{\tau\xi\tau}^n + 2u_{\xi\eta}^n u_{\xi\eta\tau}^n + 2u_{\tau\eta}^n u_{\tau\eta\tau}^n \right),
\end{aligned} \tag{2.28}$$

and so we have

$$\begin{aligned}
& F_{\eta\eta} + u^{n-1}F_{\xi} - F_{\tau} - 2\frac{\partial f}{\partial u}F - 2u_{\tau\tau}^n(w_{\tau\tau}^{n-1}u_{\xi}^n + 2w_{\tau}^{n-1}u_{\xi\tau}^n) \\
& - 2u_{\xi\xi}^n(u_{\xi\xi}^{n-1}u_{\xi}^n + 2u_{\xi}^{n-1}u_{\xi\xi}^n) - 2u_{\eta\eta}^n(u_{\eta\eta}^{n-1}u_{\xi}^n + 2w_{\eta}^{n-1}u_{\xi\eta}^n) \\
& - 2u_{\tau\xi}^n(u_{\tau\xi}^{n-1}u_{\xi}^n + u_{\tau}^{n-1}u_{\xi\xi}^n + u_{\xi}^{n-1}u_{\xi\tau}^n) - 2u_{\xi\eta}^n(u_{\xi\eta}^{n-1}u_{\xi}^n + u_{\xi}^{n-1}u_{\xi\eta}^n + u_{\eta}^{n-1}u_{\xi\xi}^n) \\
& - 2u_{\tau\eta}^n(u_{\tau\eta}^{n-1}u_{\xi}^n + u_{\tau}^{n-1}u_{\xi\eta}^n + u_{\eta}^{n-1}u_{\xi\tau}^n) - 2u_{\tau\tau}^n\frac{\partial^2 f}{\partial u^2}(u_{\tau}^n)^2 - 2u_{\xi\xi}^n\frac{\partial^2 f}{\partial u^2}(u_{\xi}^n)^2 - 2u_{\eta\eta}^n\frac{\partial^2 f}{\partial u^2}(u_{\eta}^n)^2 \\
& - 2u_{\tau\xi}^n\frac{\partial^2 f}{\partial u^2}u_{\tau}^nu_{\xi}^n - 2u_{\xi\eta}^n\frac{\partial^2 f}{\partial u^2}u_{\eta}^nu_{\xi}^n - 2u_{\tau\eta}^n\frac{\partial^2 f}{\partial u^2}u_{\tau}^nu_{\eta}^n \\
& - 2\frac{\partial f}{\partial u}\left[(u_{\tau}^n)^2 + (u_{\xi}^n)^2 + (u_{\eta}^n)^2\right] = 0.
\end{aligned} \tag{2.29}$$

By the introduced assumption that the first-order and second-order derivatives of u^{n-1} , $\partial f/\partial u$, and $\partial^2 f/\partial u^2$ are all bounded and using Cauchy inequality, we can get from (2.29) that

$$F_{\eta\eta} - 2\alpha F_{\eta} - u^{n-1}F_{\xi} - F_{\tau} + R_1^n F \geq 0. \tag{2.30}$$

By the transformation $F_1 = Fe^{-\gamma\tau}$, if we chose γ large enough, we are able to show that there exist positive constants M and T such that the conditions $F_{\mu} \leq M$ for $\tau \leq T$, $\mu \leq n-1$, imply that $F_n \leq M$ for $\tau \leq T$. Thus we have the following.

Theorem 2.4. *Let w^n be the solutions of problems (2.4)-(2.5), then the derivatives of w^n up to the second-order are uniformly bounded with respect to n in the domain $(0, T] \times \mathbb{R}^2$ with a small positive number T .*

Now let us establish uniform convergence of $w^n = u^n$ in $[0, T] \times \mathbb{R}^2$. For $v^n = w^n - w^{n-1}$ we obtain the following equation from (2.6):

$$\begin{aligned}
& v_{\eta\eta}^n - v_{\tau}^n + w^{n-1}v_{\xi}^n - v^{n-1}w_{\xi}^{n-1} - \left(f(\cdot, w^n) - f(\cdot, w^{n-1})\right) = 0, \\
& v^n(0, \xi, \eta) = 0.
\end{aligned} \tag{2.31}$$

Let $v^n = e^{\alpha\tau}v_1^n$. Then

$$\begin{aligned}
& v_{1\eta\eta}^n - v_{1\tau}^n + w^{n-1}v_{1\xi}^n - v_1^{n-1}w_{\xi}^{n-1} - \alpha v_1^n - e^{-\alpha\tau}\left(f(\cdot, w^n) - f(\cdot, w^{n-1})\right) = 0, \\
& v_{1\eta\eta}^n - v_{1\tau}^n + w^{n-1}v_{1\xi}^n - v_1^{n-1}w_{\xi}^{n-1} \\
& = \alpha v_1^n + e^{-\alpha\tau}\left(f(\cdot, w^n) - f(\cdot, w^{n-1})\right) = \alpha v_1^n + e^{-\alpha\tau}\frac{\partial f}{\partial w}v_1^n \geq (\alpha - c)v_1^n,
\end{aligned} \tag{2.32}$$

where we have chosen $\tau \leq T$ small enough such that $e^{-\alpha\tau} = 2$, and $2(\partial f/\partial w) \geq -c$.

If v_1 attains its positive maximal value in $(0, T] \times R^2$, we can choose α large enough such that

$$\left| \frac{w_\xi^{n-1}}{\alpha - c} \right| < 1, \quad (2.33)$$

and then at the maximal point we have

$$(\alpha - c)v_1^n \leq -v_1^{n-1}w_\xi^{n-1}. \quad (2.34)$$

If v_1^n attains its negative minimal value in $(0, T] \times R^2$, we have

$$(\alpha - c)(-v_1^n) \leq -v_1^{n-1}w_\xi^{n-1}. \quad (2.35)$$

Notice that at $\tau = 0$, $v_1^n = v^n = 0$. By (2.34) and (2.35),

$$\max |v_1^n| \leq q \max |v_1^{n-1}|, \quad q < 1, \quad (2.36)$$

which means that the series $v_1^1 + v_1^2 + \dots + v_1^n + \dots$, whose sum has the form $w^n e^{-\alpha\tau}$, is majorized by a geometrical progression and, therefore, is uniformly convergent. The fact that w^n and its derivatives up to the second-order are bounded implies that the first derivatives of w^n are uniformly convergent as $n \rightarrow \infty$.

It follows from (2.6) that $w_{\eta\eta}^n$ are also uniformly convergent as $n \rightarrow \infty$.

Now, we can take $w^{-1} = w^0 = w_0$; then by the above discussion, we have the following theorem.

Theorem 2.5. *Suppose that $V(0, \xi, \eta) \leq w_0 \leq V_1(0, \xi, \eta)$ and f satisfies (2.9) and is suitable smooth, then there exists a small positive number T such that the Cauchy problem (2.4) has a classical solution.*

By the way, it is easy to prove the uniqueness of the solution for the Cauchy problem (2.4), and we omit the details here.

3. Computational Examples

In this section, a numerical simulate is made for the equations by differential method. Numerical computation of these examples shows that the strong solutions for the corresponding Cauchy problem of (1.1)-(1.2) will blow-up in finite time.

Let $\Omega = [0, L_x] \times [0, L_y]$ and $u(x, y, 0) = u_0(x, y)$, $(x, y) \in \Omega$, but $u(x, y, 0) = 0$, $(x, y) \in R^2/\Omega$. Then instead of studying the Cauchy Problem (1.1)-(1.2), we can study the following

Figure 1: $f(\cdot, u) = u$. Figure 2: $f(\cdot, u) = \sin u$.

initial boundary problem:

$$\begin{aligned}
 \partial_{xx}u + u\partial_y u - \partial_t u &= f(\cdot, u), \quad (x, y, t) \in \Omega \times (0, T], \\
 u(x, y, 0) &= u_0(x, y), \quad (x, y) \in \Omega, \\
 u|_{\partial\Omega} &= 0.
 \end{aligned} \tag{3.1}$$

(a)

(b)

Figure 3: The numerical results in (a) at $t = 0$ and in (b) at $t = 0.046$ when $f(\cdot, u) = u^2$.

If $f(\cdot, 0) = 0$, it is clear that if $u(x, y, t)$ is a classical solution of (3.1), then $u(x, y, t)$ is a strong solution of the Cauchy problem (1.1)-(1.2).

To dissect domain Ω , suppose that $L_x = L_y = 2\pi$ and $h_x = 2\pi/N$, $h_y = 2\pi/M$ stands for the space step-length in the axis x and axis y , and $k = T/J$ stands for the time step-length. Let $\Omega_h = \{(ih_x, jh_y) \mid 0 \leq i \leq N; 0 \leq j \leq M\}$ and define $u_{ij}^n = u(ih_x, jh_y, nk)$. The differential scheme of the original equation is (to ensure numerical stability, here we apply arithmetic averages in order to avoid "oscillation" and "shifting" of the numerical solution)

$$\begin{aligned}
 & \frac{u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n}{h_x^2} + \frac{u_{i+1,j}^n + u_{i,j}^n + u_{i-1,j}^n + u_{i,j+1}^n + u_{i,j}^n + u_{i,j-1}^n}{6} \frac{u_{i,j+1}^n - u_{i,j-1}^n}{2h_y} \\
 & \quad - \frac{u_{i,j}^{n+1} - (1/4)(u_{i-1,j}^n + u_{i+1,j}^n + u_{i,j+1}^n + u_{i,j-1}^n)}{k} \\
 & = f\left(ih_x, jh_y, nk, \frac{u_{i+1,j}^n + u_{i,j}^n + u_{i-1,j}^n + u_{i,j+1}^n + u_{i,j}^n + u_{i,j-1}^n}{6}\right), \\
 & u^n|_{\partial\Omega_h} = 0, \quad (n = 1, 2, \dots), \quad u_{i,j}^0 = u_0(ih_x, jh_y).
 \end{aligned} \tag{3.2}$$

Figure 4: $f(\cdot, u) = u$.

So we get the following explicit formula:

$$\begin{aligned}
 u_{i,j}^{n+1} = & \frac{1}{4} \left(u_{i-1,j}^n + u_{i+1,j}^n + u_{i,j+1}^n + u_{i,j-1}^n \right) + \frac{k}{h_x^2} \left(u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n \right) \\
 & + \frac{k}{12h_y} \left(u_{i+1,j}^n + 2u_{i,j}^n + u_{i-1,j}^n + u_{i,j+1}^n + u_{i,j-1}^n \right) \left(u_{i,j+1}^n - u_{i,j-1}^n \right) \\
 & - kf \left(ih_x, jh_y, nk, \frac{u_{i+1,j}^n + 2u_{i,j}^n + u_{i-1,j}^n + u_{i,j+1}^n + u_{i,j-1}^n}{6} \right).
 \end{aligned} \tag{3.3}$$

Experiment 1. Suppose $\Omega = [0, 2\pi] \times [0, 2\pi]$, $h_x = h_y = 2\pi/40$, $k = 0.001$, $u_0(x, y) = \sin x \sin y$ which itself does not satisfy (1.1); we get the graphs (see Figures 1–3) where $u(x, y, t)$ changes according to the changes of t when different functions are given to $f(\cdot, u)$.

Figure 1 shows that when $f(\cdot, u) = u$, at $t = 0.04$, the numerical solutions become oscillatory; at $t = 0.042$, the bifurcation of solutions occurs; when $t > 0.042$, the solutions will blow-up. Similarly Figure 2 shows that when $f(\cdot, u) = \sin u$, at $t = 0.6$, the bifurcation of solutions occurs; when $t > 0.6$, the solutions will blow-up. Figure 3 is the spatiotemporal graphs of solutions when $f(\cdot, u) = u^2$ at $t = 0$ (initial value) and $t = 0.0046$. When $t > 0.0046$, the solutions will blow-up.

Experiment 2. The initial value is unknown in the general situation; so we use random numbers $([-0.01, 0.01])$ as the initial value and draw graphs (see Figures 4 and 5) where $u(x, y, t)$ changes as t changes when different functions are given to $f(\cdot, u)$.

Figures 4 and 5 show that even though the initial value is sufficiently small, the blow-up will appear in finite time for the different functions.

Figure 5: $f(\cdot, u) = 1 - \sin u$.

The numerical result shows that there is a locality solution of the equation. When t becomes larger, the bifurcation of solutions occurs in finite time and blow-up appears. For this problem, it is essential to have a further research.

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