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Research Article

Weighted Composition Operators and Integral-Type Operators between Weighted Hardy Spaces on the Unit Ball

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We study the boundedness and compactness of the weighted composition operators as well as integral-type operators between weighted Hardy spaces on the unit ball.

1. Introduction

Let $\mathbb{B}$ denote the open unit ball of the n-dimensional complex vector space $\mathbb{C}^n$, $\partial\mathbb{B}$ its boundary, and let $H(\mathbb{B})$ denote the space of all holomorphic functions on $\mathbb{B}$. For $0 < p < \infty$ and $\alpha \geq 0$ we define the *weighted Hardy space* $H_\alpha^p(\mathbb{B})$ as follows:

$$H_\alpha^p(\mathbb{B}) = \left\{ f \in H(\mathbb{B}) \ : \ \sup_{0<r<1} (1-r)^\alpha \int_{\partial\mathbb{B}} |f(r\zeta)|^p \, d\sigma(\zeta) < \infty \right\}, \tag{1.1}$$

where $d\sigma$ is the normalized Lebesgue measure on $\partial\mathbb{B}$ (see, also [1], as well as [2], for an equivalent definition of the space). Note that for $\alpha = 0$ the weighted Hardy space becomes the Hardy space $H^p(\mathbb{B})$. We define the norm $\|\cdot\|_{H_\alpha^p}$ on this space as follows:

$$\|f\|_{H_\alpha^p}^p = \sup_{0<r<1} (1-r)^\alpha \int_{\partial\mathbb{B}} |f(r\zeta)|^p d\sigma(\zeta). \tag{1.2}$$

With this norm $H_\alpha^p(\mathbb{B})$ is a Banach space when $1 \le p < \infty$. For a related space on the unit polydisk; see [3]. In this paper, we investigate two types of operators acting between weighted Hardy spaces.

Let φ be a holomorphic self-map of $\mathbb{B}$ and $u \in H(\mathbb{B})$. Then φ and u induce a *weighted composition operator* uC_φ on $H(\mathbb{B})$ which is defined by $uC_\varphi f = u(f \circ \varphi)$. This type of operators has been studied on various spaces of holomorphic functions in $\mathbb{C}^n$, by many authors; see, for example, [4], recent papers [5–17], and the references therein.

Let $g \in H(\mathbb{D})$ and φ be a holomorphic self-map of the open unit disk $\mathbb{D}$ in the complex plane. Products of integral and composition operators on $H(\mathbb{D})$ were introduced by S. Li and S. Stević in a private communication (see [18–21], as well as papers [22] and [23] for closely related operators) as follows:

$$C_\varphi J_g f(z) = \int_0^{\varphi(z)} f(\zeta) g'(\zeta) d\zeta, \tag{1.3}$$

$$J_g C_\varphi f(z) = \int_0^{z} f(\varphi(\zeta)) g'(\zeta) d\zeta. \tag{1.4}$$

In [24] the first author of this paper has extended the operator in (1.4) in the unit ball settings as follows (see also [25, 26]). Assume $g \in H(\mathbb{B})$, $g(0) = 0$, and φ is a holomorphic self-map of $\mathbb{B}$, then we define an operator on the unit ball as follows:

$$P_\varphi^g(f)(z) = \int_0^1 f(\varphi(tz)) g(tz) \frac{dt}{t}, \quad f \in H(\mathbb{B}), \quad z \in \mathbb{B}. \tag{1.5}$$

If $n = 1$, then $g \in H(\mathbb{D})$ and $g(0) = 0$, so that $g(z) = z g_0(z)$, for some $g_0 \in H(\mathbb{D})$. By the change of variable $\zeta = tz$, it follows that

$$P_\varphi^g f(z) = \int_0^1 f(\varphi(tz)) tz g_0(tz) \frac{dt}{t} = \int_0^z f(\varphi(\zeta)) g_0(\zeta) d\zeta. \tag{1.6}$$

Thus the operator (1.5) is a natural extension of operator $J_g C_\varphi$ in (1.4). For related operators see [27–33] as well as the references therein.

In this paper we study the boundedness and compactness of the weighted composition operators as well as the integral-type operator P_φ^g, between different weighted Hardy spaces on the unit ball.

Throughout this paper, constants are denoted by C, they are positive and may differ from one occurrence to the other. The notation $a \preceq b$ means that there is a positive constant C such that $a \le Cb$. Moreover, if both $a \preceq b$ and $b \preceq a$ hold, then one says that $a \asymp b$.

2. Weighted Composition Operators

This section is devoted to studying weighted composition operators between weighted Hardy spaces. Weighted composition operators between different Hardy spaces on the unit ball were previously studied in [15, 34], while the composition operators on the unit ball were studied in [35, 36]. For the case of the unit disk see also [37].

Before we formulate the main results in this section we quote several auxiliary results which will be used in the proofs of these ones.

Lemma 2.1. *Let $0 < p < \infty$ and $\alpha \geq 0$. Suppose that $u \in H(\mathbb{B})$ and φ is a holomorphic self-map of $\mathbb{B}$. Then for each $f \in H(\mathbb{B})$*

$$\left\| uC_\varphi f \right\|_{H_\alpha^p} \leq \liminf_{R \to 1^-} \left\| uC_\varphi f_R \right\|_{H_\alpha^p}, \tag{2.1}$$

where $uC_\varphi f_R(z) = u(z) f(R\varphi(z))$.

Proof. Fix $r \in (0,1)$. Fatou's lemma shows that

$$\begin{aligned}
(1-r)^\alpha \int_{\partial\mathbb{B}} |u(r\zeta) f(\varphi(r\zeta))|^p d\sigma(\zeta) &\leq (1-r)^\alpha \liminf_{R \to 1^-} \int_{\partial\mathbb{B}} |u(r\zeta) f(R\varphi(r\zeta))|^p d\sigma(\zeta) \\
&= \liminf_{R \to 1^-} (1-r)^\alpha \int_{\partial\mathbb{B}} \left| u(r\zeta) f(R\varphi(r\zeta)) \right|^p d\sigma(\zeta) \\
&\leq \liminf_{R \to 1^-} \| uC_\varphi f_R \|_{H_\alpha^p}.
\end{aligned} \tag{2.2}$$

Hence we have the desired inequality. □

Recall that an $f \in H(\mathbb{B})$ has the homogeneous expansion

$$f(z) = \sum_{k=0}^{\infty} \sum_{|\gamma|=k} c(\gamma) z^\gamma, \tag{2.3}$$

where $\gamma = (\gamma_1, \ldots, \gamma_n)$ is a multi-index, $|\gamma| = \gamma_1 + \cdots + \gamma_n$ and $z^\gamma = z_1{}^{\gamma_1} \cdots z_n{}^{\gamma_n}$. For the homogeneous expansion of f and any integer $j \geq 1$, let

$$R_j f(z) = \sum_{k=0}^{\infty} \sum_{|\gamma|=k} c(\gamma) z^\gamma, \tag{2.4}$$

and $K_j = I - R_j$ where $If = f$ is the identity operator. Note that K_j is compact operator on $H_\alpha^p(\mathbb{B})$ for each $j \in \mathbb{N}$.

Lemma 2.2. *If $1 < p < \infty$, then R_j converges to 0 pointwise in the Hardy space $H^p(\mathbb{B})$ as $j \to \infty$.*

Proof. See [34, Corollary 3.4] . □

Lemma 2.2 and the uniform boundedness principle show that $\{R_j\}$ is an uniformly bounded sequence in $H^p(\mathbb{B})$.

The following lemma is proved similar to [4, Lemma 3.16]. We omit its proof.

Lemma 2.3. *If uC_φ is bounded from $H^p_\alpha(\mathbb{B})$ into $H^q_\beta(\mathbb{B})$, then*

$$\|uC_\varphi\|_{e,H^p_\alpha(\mathbb{B})\to H^q_\beta(\mathbb{B})} \le \liminf_{j\to\infty}\|uC_\varphi R_j\|_{H^p_\alpha(\mathbb{B})\to H^q_\beta(\mathbb{B})}, \tag{2.5}$$

where $\|\cdot\|_{e,H^p_\alpha(\mathbb{B})\to H^q_\beta(\mathbb{B})}$ and $\|\cdot\|_{H^p_\alpha(\mathbb{B})\to H^q_\beta(\mathbb{B})}$ denote the essential norm and the operator norm, respectively.

Lemma 2.4. *Let $0 < p \le q < \infty$. Suppose that μ is a positive Borel measure on $\mathbb{B}$ which satisfies*

$$\mu(B(\zeta,t)) \le C_1 t^{qn/p} \quad (\zeta \in \partial\mathbb{B}, t > 0), \tag{2.6}$$

for some positive constant C_1. Then there exists a positive constant C_2 which depends only on p, q, and the dimension n such that

$$\int_{\mathbb{B}} |f|^q d\mu \le C_1 C_2 \|f\|^q_{H^p}, \tag{2.7}$$

for any $f \in H^p(\mathbb{B})$. Here $B(\zeta,t) = \{z \in \mathbb{B} : |1-\langle z,\zeta\rangle| < t\}$.

Proof. See [38, page 13, Theorem] or [34, Lemma 2.1] . □

Let $0 < q < \infty$. For each $r \in (0,1)$, a holomorphic self-map φ of $\mathbb{B}$ and $u \in H(\mathbb{B})$, we define a positive Borel measure $\mu^r_{u,\varphi}$ on $\mathbb{B}$ by

$$\mu^r_{u,\varphi}(E) = \int_{(\varphi_r)^{-1}(E)} |u_r|^q d\sigma, \tag{2.8}$$

for all Borel sets E of $\mathbb{B}$. By the change of variables formula from measure theory, we can verify

$$\int_{\mathbb{B}} g\, d\mu^r_{u,\varphi} = \int_{\partial\mathbb{B}} |u(r\zeta)|^q (g\circ\varphi)(r\zeta)\, d\sigma(\zeta), \tag{2.9}$$

for each nonnegative measurable function g in $\mathbb{B}$.

Theorem 2.5. *Let $0 < p \le q < \infty$ and $\alpha, \beta \ge 0$. Suppose that $u \in H(\mathbb{B})$ and φ is a holomorphic self-map of $\mathbb{B}$. Then $uC_\varphi : H^p_\alpha(\mathbb{B}) \to H^q_\beta(\mathbb{B})$ is bounded if and only if*

$$\sup_{w\in\mathbb{B}}\sup_{0<r<1}(1-r)^\beta \int_{\partial\mathbb{B}} |u(r\zeta)|^q \left\{\frac{1-|w|^2}{|1-\langle\varphi(r\zeta),w\rangle|^2}\right\}^{q(\alpha+n)/p} d\sigma(\zeta) < \infty. \tag{2.10}$$

Proof. For $w \in \mathbb{B}$ we put

$$f_w(z) = \left\{ \frac{1-|w|^2}{(1-\langle z,w\rangle)^2} \right\}^{(\alpha+n)/p}. \tag{2.11}$$

Then we see that $f_w \in H^p_\alpha(\mathbb{B})$ and moreover $\sup_{w\in\mathbb{B}}\|f_w\|_{H^p_\alpha} \le C$. By a straightforward calculation, we have

$$\|uC_\varphi f_w\|^q_{H^q_\beta} = \sup_{0<r<1}(1-r)^\beta \int_{\partial\mathbb{B}} |u(r\zeta)|^q \left\{ \frac{1-|w|^2}{\left|1-\langle\varphi(r\zeta),w\rangle\right|^2} \right\}^{q(\alpha+n)/p} d\sigma(\zeta), \tag{2.12}$$

for all $w \in \mathbb{B}$. Hence if $uC_\varphi : H^p_\alpha(\mathbb{B}) \to H^q_\beta(\mathbb{B})$ is bounded, then u and φ satisfy the condition

$$\sup_{w\in\mathbb{B}}\sup_{0<r<1}(1-r)^\beta \int_{\partial\mathbb{B}} |u(r\zeta)|^q \left\{ \frac{1-|w|^2}{\left|1-\langle\varphi(r\zeta),w\rangle\right|^2} \right\}^{q(\alpha+n)/p} d\sigma(\zeta) \le C\|uC_\varphi\|^q_{H^p_\alpha(\mathbb{B})\to H^q_\beta(\mathbb{B})} < \infty. \tag{2.13}$$

Next we assume

$$M := \sup_{w\in\mathbb{B}}\sup_{0<r<1}(1-r)^\beta \int_{\partial\mathbb{B}} |u(r\zeta)|^q \left\{ \frac{1-|w|^2}{\left|1-\langle\varphi(r\zeta),w\rangle\right|^2} \right\}^{q(\alpha+n)/p} d\sigma(\zeta) < \infty. \tag{2.14}$$

Fix $r \in (0,1)$ and $R \in (0,1)$, respectively. For $\zeta \in \partial\mathbb{B}$ and $t, 0 < t \le t_R = 1-R$, we put $w = (1-t)\zeta$ and $w_R = (1-t_R)\zeta$. Since the function $f_w(z)$, which is defined by (2.11) for this w, satisfies

$$|f_w(z)|^q > 4^{-q(\alpha+n)/p}\, t^{-qn/p}(1-R)^{-q\alpha/p} \tag{2.15}$$

for all $z \in B(\zeta,t)$, we have

$$\begin{aligned}
\frac{\mu^r_{u,\varphi}(B(\zeta,t))}{t^{qn/p}} &\le 4^{q(\alpha+n)/p}(1-R)^{q\alpha/p}\int_{B(\zeta,t)} |f_w(z)|^q d\mu^r_{u,\varphi}(z) \\
&\le 4^{q(\alpha+n)/p}(1-R)^{q\alpha/p}\int_{\partial\mathbb{B}} |u(r\zeta)|^q \left\{ \frac{1-|w|^2}{\left|1-\langle\varphi(r\zeta),w\rangle\right|^2} \right\}^{q(\alpha+n)/p} d\sigma(\zeta) \\
&\le 4^{q(\alpha+n)/p}\frac{(1-R)^{q\alpha/p}}{(1-r)^\beta} M.
\end{aligned} \tag{2.16}$$

By the same argument, the function $f_{w_R}(z)$ gives the following estimate:

$$\mu^r_{u,\varphi}(B(\zeta,2t_R)) \le 9^{q(\alpha+n)/p}\frac{(1-R)^{q\alpha/p}}{(1-r)^\beta} M {t_R}^{qn/p}. \tag{2.17}$$

Now we need to prove that there exists a positive constant C such that

$$\mu_{u,\varphi}^{r}(B(\zeta,t)) \leq C\frac{(1-R)^{q\alpha/p}}{(1-r)^{\beta}}Mt^{qn/p}, \tag{2.18}$$

for all $\zeta \in \partial\mathbb{B}$ and $t > 0$. By the estimate (2.16), we see that the inequality (2.18) is true for all $t \in (0, t_R]$. Thus we assume $t > t_R$. By the same argument as in [36, pages 241-242, proof of Theorem 1.1] , we see that the inequality (2.17) shows that there exists a positive constant C_n which depends only on the dimension n such that

$$\begin{aligned}\mu_{u,\varphi}^{r}(B(\zeta,t)) &\leq C_n\left(\frac{t}{t_R}\right)^{n} 9^{q(\alpha+n)/p}\frac{(1-R)^{q\alpha/p}}{(1-r)^{\beta}}Mt_R{}^{qn/p}\\ &= C_n 9^{q(\alpha+n)/p}\frac{(1-R)^{q\alpha/p}}{(1-r)^{\beta}}Mt^{n}t_R{}^{(q/p-1)n}\\ &\leq C_n 9^{q(\alpha+n)/p}\frac{(1-R)^{q\alpha/p}}{(1-r)^{\beta}}Mt^{qn/p}.\end{aligned} \tag{2.19}$$

Hence for $C = \max\{4^{q(\alpha+n)/p}, C_n 9^{q(\alpha+n)/p}\}$, we have the inequality in (2.18).

For $f \in H_{\alpha}^{p}(\mathbb{B})$ the dilate function f_R belongs to the ball algebra, and so f_R is in the Hardy space $H^p(\mathbb{B})$. Hence Lemma 2.4 gives

$$\int_{\mathbb{B}} |f_R(z)|^q d\mu_{u,\varphi}^{r}(z) \leq C'C\frac{(1-R)^{q\alpha/p}}{(1-r)^{\beta}}M\|f_R\|_{H^p}^{q}, \tag{2.20}$$

for some positive constant C' and all $R \in (0,1)$. This implies that

$$(1-r)^{\beta}\int_{\partial\mathbb{B}} \left|uC_{\varphi}f_R(r\zeta)\right|^q d\sigma(\zeta) \leq C'CM\left[(1-R)^{\alpha}\int_{\partial\mathbb{B}} \left|f(R\zeta)\right|^p d\sigma(\zeta)\right]^{q/p}, \tag{2.21}$$

and so we have

$$\|uC_{\varphi}f_R\|_{H_{\beta}^{q}}^{q} \leq C'CM\|f\|_{H_{\alpha}^{p}}^{q}, \tag{2.22}$$

for all $R \in (0,1)$. By Lemma 2.1 we have

$$\begin{aligned}\|uC_{\varphi}f\|_{H_{\beta}^{q}}^{q} &\leq C'C\|f\|_{H_{\alpha}^{p}}^{q}\\ &\quad\times \sup_{w\in\mathbb{B}}\sup_{0<r<1}(1-r)^{\beta}\int_{\partial\mathbb{B}} |u(r\zeta)|^q\left\{\frac{1-|w|^2}{\left|1-\langle\varphi(r\zeta),w\rangle\right|^2}\right\}^{q(\alpha+n)/p} d\sigma(\zeta).\end{aligned} \tag{2.23}$$

This completes the proof. □

The following proposition is proved in a standard way; see, for example, the proofs of the corresponding results in [4, 32, 33, 39]. Hence we omit its proof.

Proposition 2.6. *Let $0 < p, q < \infty$ and $\alpha, \beta \geq 0$. Suppose that $u \in H(\mathbb{B})$ and φ is a holomorphic self-map of $\mathbb{B}$ which induce the bounded operator $uC_\varphi : H^p_\alpha(\mathbb{B}) \to H^q_\beta(\mathbb{B})$. Then $uC_\varphi : H^p_\alpha(\mathbb{B}) \to H^q_\beta(\mathbb{B})$ is compact if and only if for every bounded sequence $\{f_j\}_{j\in\mathbb{N}}$ in $H^p_\alpha(\mathbb{B})$ which converges to 0 uniformly on compact subsets of $\mathbb{B}$, $\{uC_\varphi f_j\}_{j\in\mathbb{N}}$ converges to 0 in $H^q_\beta(\mathbb{B})$.*

In the proof of Theorem 2.8, we need the following lemma.

Lemma 2.7. *Let $1 < p < \infty$, $\alpha \geq 0$, and f_w be the family of test functions defined in* (2.11). *Then $f_w \to 0$ weakly in $H^p_\alpha(\mathbb{B})$ as $|w| \to 1^-$.*

Proof. The family $\{f_w\}_{w\in\mathbb{B}}$ is bounded in $H^p_\alpha(\mathbb{B})$ and $f_w \to 0$ uniformly on compact subsets of $\mathbb{B}$ as $|w| \to 1^-$. By the definitions of the space $H^p_\alpha(\mathbb{B})$ and the norm $\|\cdot\|_{H^p_\alpha}$, we see that $H^p_\alpha(\mathbb{B})$ is a subspace of the weighted Bergman space $A^p_\alpha(\mathbb{B})$ and

$$\|f\|_{A^p_\alpha} \leq C(\alpha, p, n)\|f\|_{H^p_\alpha} \quad \left(f \in H^p_\alpha(\mathbb{B})\right), \tag{2.24}$$

for some positive constant $C(\alpha, p, n)$ which depends on α, p, and n. This inequality implies that the family $\{f_w\}_{w\in\mathbb{B}}$ is also bounded in $A^p_\alpha(\mathbb{B})$. Note also that the family converges to 0 uniformly on compact subsets of $\mathbb{B}$ as $|w| \to 1^-$. Hence $f_w \to 0$ weakly in $A^p_\alpha(\mathbb{B})$ as $|w| \to 1^-$.

In order to prove that $f_w \to 0$ weakly in $H^p_\alpha(\mathbb{B})$ as $|w| \to 1^-$, we take an arbitrary bounded linear functional Λ on $H^p_\alpha(\mathbb{B})$. By the Hahn-Banach theorem, Λ can be extended to a bounded linear functional $\widetilde{\Lambda}$ on $A^p_\alpha(\mathbb{B})$ so that $\widetilde{\Lambda}(f_w) = \Lambda(f_w)$ for all $w \in \mathbb{B}$. Since $f_w \to 0$ weakly in $A^p_\alpha(\mathbb{B})$ as $|w| \to 1^-$, we have $\Lambda(f_w) = \widetilde{\Lambda}(f_w) \to 0$ as $|w| \to 1^-$, and so $f_w \to 0$ weakly in $H^p_\alpha(\mathbb{B})$ as $|w| \to 1^-$. □

Theorem 2.8. *Let $1 < p \leq q < \infty$ and $\alpha, \beta \geq 0$. Suppose that $u \in H(\mathbb{B})$ and φ is a holomorphic self-map of $\mathbb{B}$ such that $uC_\varphi : H^p_\alpha(\mathbb{B}) \to H^q_\beta(\mathbb{B})$ is bounded. Then the qth power of the essential norm $\|uC_\varphi\|_{e,H^p_\alpha(\mathbb{B})\to H^q_\beta(\mathbb{B})}$ is comparable to*

$$\limsup_{|w|\to 1^-} \sup_{0<r<1} (1-r)^\beta \int_{\partial\mathbb{B}} |u(r\zeta)|^q \left\{ \frac{1-|w|^2}{\left|1-\langle \varphi(r\zeta), w\rangle\right|^2} \right\}^{q(\alpha+n)/p} d\sigma(\zeta). \tag{2.25}$$

Hence $uC_\varphi : H^p_\alpha(\mathbb{B}) \to H^q_\beta(\mathbb{B})$ is compact if and only if

$$\lim_{|w|\to 1^-} \sup_{0<r<1} (1-r)^\beta \int_{\partial\mathbb{B}} |u(r\zeta)|^q \left\{ \frac{1-|w|^2}{\left|1-\langle \varphi(r\zeta), w\rangle\right|^2} \right\}^{q(\alpha+n)/p} d\sigma(\zeta) = 0. \tag{2.26}$$

Proof. To prove a lower estimate

$$\|uC_\varphi\|^q_{e,H^p_\alpha(\mathbb{B})\to H^q_\beta(\mathbb{B})} \geq \limsup_{|w|\to 1^-}\sup_{0<r<1}(1-r)^\beta\int_{\partial\mathbb{B}}|u(r\zeta)|^q\left\{\frac{1-|w|^2}{|1-\langle\varphi(r\zeta),w\rangle|^2}\right\}^{q(\alpha+n)/p}d\sigma(\zeta), \tag{2.27}$$

we consider the test functions f_w defined in (2.11). The family $\{f_w\}_{w\in\mathbb{B}}$ is bounded in $H^p_\alpha(\mathbb{B})$, say by L, and $f_w \to 0$ uniformly on compact subsets of $\mathbb{B}$ as $|w| \to 1^-$. Thus by Lemma 2.7 we have that $f_w \to 0$ weakly in $H^p_\alpha(\mathbb{B})$ as $|w| \to 1^-$, so that $\|\mathcal{K}f_w\|_{H^q_\beta} \to 0$ as $|w| \to 1^-$ for every compact operator $\mathcal{K} : H^p_\alpha(\mathbb{B}) \to H^q_\beta(\mathbb{B})$. Hence

$$\begin{aligned} L\|uC_\varphi-\mathcal{K}\|_{H^p_\alpha(\mathbb{B})\to H^q_\beta(\mathbb{B})} &\geq \limsup_{|w|\to 1^-}\|(uC_\varphi-\mathcal{K})f_w\|_{H^q_\beta} \\ &\geq \limsup_{|w|\to 1^-}\|uC_\varphi f_w\|_{H^q_\beta}. \end{aligned} \tag{2.28}$$

This inequality and (2.12) give the lower estimate for $\|uC_\varphi\|^q_{e,H^p_\alpha(\mathbb{B})\to H^q_\beta(\mathbb{B})}$.

Next we prove an upper estimate. Take $f \in H^p_\alpha(\mathbb{B})$ with $\|f\|_{H^p_\alpha} \leq 1$. Fix $\varepsilon > 0$ and put

$$M_1 := \limsup_{|w|\to 1^-}\sup_{0<r<1}(1-r)^\beta\int_{\partial\mathbb{B}}|u(r\zeta)|^q\left\{\frac{1-|w|^2}{|1-\langle\varphi(r\zeta),w\rangle|^2}\right\}^{q(\alpha+n)/p}d\sigma(\zeta). \tag{2.29}$$

Then we can choose $R_0 \in (0,1)$ such that

$$\sup_{0<r<1}(1-r)^\beta\int_{\partial\mathbb{B}}|u(r\zeta)|^q\left\{\frac{1-|w|^2}{|1-\langle\varphi(r\zeta),w\rangle|^2}\right\}^{q(\alpha+n)/p}d\sigma(\zeta) < M_1+\varepsilon, \tag{2.30}$$

for $w \in \mathbb{B}$ with $|w| > R_0$. Fix $r \in (0,1)$ and $R \in (R_0,1)$. By the same argument as in the proof of inequality (2.20) in Theorem 2.5, we obtain that

$$\int_{\mathbb{B}}\left|(R_jf)_R(z)\right|^q d\mu^r_{u,\varphi}(z) \leq C\frac{(1-R)^{q\alpha/p}}{(1-r)^\beta}(M_1+\varepsilon)\|(R_jf)_R\|^q_{H^p}, \tag{2.31}$$

where the positive constant C is independent of r, R and a positive integer j. Since f_R is in the ball algebra, Lemma 2.2 gives

$$\|(R_jf)_R\|^q_{H^p} = \|R_j(f_R)\|^q_{H^p} \leq \sup_{j\geq 1}\|R_j\|^q_{H^p(\mathbb{B})\to H^p(\mathbb{B})}\|f_R\|^q_{H^p}. \tag{2.32}$$

Combining this with (2.31), we have

$$\begin{aligned}(1-r)^{\beta}\int_{\partial\mathbb{B}}|uC_{\varphi}(R_jf_R)(r\zeta)|^q d\sigma(\zeta) &\le C'(M_1+\varepsilon)\left[(1-R)^{\alpha}\int_{\partial\mathbb{B}}\left|f_R(\zeta)\right|^p d\sigma(\zeta)\right]^{q/p}\\ &\le C'(M_1+\varepsilon)\|f\|^q_{H^p_\alpha},\end{aligned} \tag{2.33}$$

and so we have

$$\|uC_{\varphi}(R_jf_R)\|^q_{H^q_\beta}\le C'(M_1+\varepsilon)\|f\|^q_{H^p_\alpha}. \tag{2.34}$$

Letting $R\to 1^-$, by Lemma 2.1, we obtain

$$\|uC_{\varphi}R_jf\|^q_{H^q_\beta}\le C'(M_1+\varepsilon)\|f\|^q_{H^p_\alpha}. \tag{2.35}$$

Since $\varepsilon>0$ is arbitrary, this estimate and Lemma 2.3 imply

$$\begin{aligned}\|uC_{\varphi}\|^q_{e,H^p_\alpha(\mathbb{B})\to H^q_\beta(\mathbb{B})} &\le \liminf_{j\to\infty}\|uC_{\varphi}R_j\|^q_{H^p_\alpha(\mathbb{B})\to H^q_\beta(\mathbb{B})}\\ &\le C'\limsup_{|w|\to 1^-}\sup_{0<r<1}(1-r)^{\beta}\int_{\partial\mathbb{B}}|u(r\zeta)|^q\left\{\frac{1-|w|^2}{\left|1-\langle\varphi(r\zeta),w\rangle\right|^2}\right\}^{q(\alpha+n)/p}d\sigma(\zeta),\end{aligned} \tag{2.36}$$

which completes the proof. □

Remark 2.9. In the above proof, we used Lemma 2.2. This lemma required the assumption $1<p<\infty$. Hence we cannot have an upper estimate for $\|uC_{\varphi}\|_{e,H^p_\alpha(\mathbb{B})\to H^q_\beta(\mathbb{B})}$ in the case $0<p\le 1$. However, Proposition 2.6 shows that the compactness of $uC_{\varphi}:H^p_\alpha(\mathbb{B})\to H^q_\beta(\mathbb{B})$ $(0<p\le q<\infty)$ is equivalent to

$$\lim_{|w|\to 1^-}\sup_{0<r<1}(1-r)^{\beta}\int_{\partial\mathbb{B}}|u(r\zeta)|^q\left\{\frac{1-|w|^2}{\left|1-\langle\varphi(r\zeta),w\rangle\right|^2}\right\}^{q(\alpha+n)/p}d\sigma(\zeta)=0. \tag{2.37}$$

3. Integral-Type Operators

Here we study the boundedness and compactness of the integral-type operators P^g_φ between weighted Hardy spaces on the unit ball.

For $f\in H(\mathbb{B})$ with the Taylor expansion $f(z)=\sum_{|\gamma|\ge 0}a_\gamma z^\gamma$, let $\mathfrak{R}f(z)=\sum_{|\gamma|\ge 0}|\gamma|a_\gamma z^\gamma$ be the radial derivative of f.

The following lemma was proved in [24] (see also [25]).

Lemma 3.1. *Assume that φ is a holomorphic self-map of $\mathbb{B}$, $g \in H(\mathbb{B})$ and $g(0) = 0$. Then for every $f \in H(\mathbb{B})$ one holds*

$$\mathfrak{R}\left[P_{\varphi}^{g}(f)\right](z) = f(\varphi(z))g(z). \tag{3.1}$$

A positive continuous function ω on the interval $[0,1)$ is called normal [40] if there is a $\delta \in [0,1)$ and a and b, $0 < a < b$ such that

$$\begin{aligned} &\frac{\omega(r)}{(1-r)^{a}} \text{ is decreasing on } [\delta,1) \text{ and } \lim_{r\to 1}\frac{\omega(r)}{(1-r)^{a}} = 0, \\ &\frac{\omega(r)}{(1-r)^{b}} \text{ is increasing on } [\delta,1) \text{ and } \lim_{r\to 1}\frac{\omega(r)}{(1-r)^{b}} = \infty. \end{aligned} \tag{3.2}$$

If it is said that a function $\omega : \mathbb{B} \to [0,\infty)$ is normal, it is also assume that it is radial.

Lemma 3.2. *Assume that $0 < q \leq \infty$, m is a positive integer and ω is normal. Then for every $f \in H(\mathbb{B})$*

$$\sup_{0<r<1}\omega(r)M_q(f,r) \asymp |f(0)| + \sup_{0<r<1}(1-r)^{m}\omega(r)M_q(\mathfrak{R}^{m}f,r), \tag{3.3}$$

where

$$M_q(f,r) = \left(\int_{\partial\mathbb{B}}|f(r\zeta)|^{q}d\sigma(\zeta)\right)^{1/q}, \qquad M_{\infty}(f,r) = \sup_{\zeta\in\partial\mathbb{B}}|f(r\zeta)|. \tag{3.4}$$

Proof. The proof of the lemma in the case $1 \leq q \leq \infty$ can be found in [27, Theorem 2]. However, due to an overlook, the proof for the case $q \in (0,1)$ has a gap. Hence we will give a correct proof here in the case.

We may assume that $f(0) = 0$, otherwise we can consider the functions $h(z) = f(z) - f(0)$. Also we may assume that $\delta = 0$, to avoid some minor technical difficulties.

By [27, Lemma 1], for each fixed $q \in (0,1]$, there is a positive constant C depending only on q and the dimension n such that

$$M_q(f,r) \leq \frac{C}{r}\left(\int_{0}^{r}(r-t)^{q-1}M_q^{q}(\mathfrak{R}f,t)dt\right)^{1/q}, \tag{3.5}$$

for every $r \in (0,1)$ and $f \in H(\mathbb{B})$ such that $f(0) = 0$.

From (3.5) and the fact that ω is normal, we have

$$\begin{aligned}
\sup_{0\le r<1}\omega(r)M_q(f,r) &\le C\sup_{0\le r<1}\omega(r)\frac{1}{r}\left(\int_0^r (r-t)^{q-1}M_q^q(\mathfrak{R}f,t)\,dt\right)^{1/q}\\
&\le C\sup_{0\le r<1}(1-r)^a\frac{1}{r}\left(\int_0^r (r-t)^{q-1}\frac{\omega^q(t)}{(1-t)^{aq}}M_q^q(\mathfrak{R}f,t)\,dt\right)^{1/q}\\
&\le C\sup_{0\le r<1}(1-r)^a\frac{1}{r}\left(\int_0^r \frac{(r-t)^{q-1}}{(1-t)^{aq+q}}\,dt\right)^{1/q}\sup_{0\le t<1}(1-t)\omega(t)M_q(\mathfrak{R}f,t)\\
&= C\sup_{0\le r<1}(1-r)^a\left(\int_0^1 \frac{(1-u)^{q-1}}{(1-ur)^{aq+q}}\,du\right)^{1/q}\sup_{0\le t<1}(1-t)\omega(t)M_q(\mathfrak{R}f,t).
\end{aligned}\tag{3.6}$$

By [40, page 291, Lemma 6] there exists a positive constant C such that

$$\int_0^1 \frac{(1-u)^{q-1}}{(1-ur)^{aq+q}}\,du \le \frac{C}{(1-r)^{aq}},\tag{3.7}$$

for every $r\in(0,1)$. Combining this with (3.6), we have

$$\begin{aligned}
\sup_{0\le r<1}\omega(r)M_q(f,r) &\le C\sup_{0\le r<1}(1-r)^a\left(\frac{1}{(1-r)^{aq}}\right)^{1/q}\sup_{0\le t<1}(1-t)\omega(t)M_q(\mathfrak{R}f,t)\\
&= C\sup_{0\le t<1}(1-t)\omega(t)M_q(\mathfrak{R}f,t).
\end{aligned}\tag{3.8}$$

The reverse inequality is proved by the following inequality:

$$(1-r)M_q(\mathfrak{R}f,r)\le CM_q\left(f,\frac{1+r}{2}\right)\tag{3.9}$$

and the fact that $\omega(r)\asymp\omega(1+r/2)$ for ω normal (see [27]). Hence, we obtain the result for the case $m=1$.

For $m\ge 2$ it should be only noticed that $(1-r)^m\omega(r)$ is still normal, that $\mathfrak{R}^m f(0)=0$ for every integer $m\ge 1$, and use the method of induction. □

Theorem 3.3. *Let $0<p\le q<\infty$ and $\alpha,\beta>0$. Suppose that $g\in H(\mathbb{B})$ with $g(0)=0$ and φ is a holomorphic self-map of $\mathbb{B}$. Then $P_\varphi^g: H_\alpha^p(\mathbb{B})\to H_\beta^q(\mathbb{B})$ is bounded if and only if*

$$\sup_{w\in\mathbb{B}}\sup_{0<r<1}(1-r)^{\beta+q}\int_{\partial\mathbb{B}}\left|g(r\zeta)\right|^q\left\{\frac{1-|w|^2}{\left|1-\langle\varphi(r\zeta),w\rangle\right|^2}\right\}^{q(\alpha+n)/p}d\sigma(\zeta)<\infty.\tag{3.10}$$

Proof. Take $f \in H_\alpha^p(\mathbb{B})$ with $\|f\|_{H_\alpha^p} \le 1$. Since the function $(1-r)^{\beta/q}$ for $\beta > 0$ and $0 < q < \infty$ is normal, Lemma 3.2 gives

$$\sup_{0<r<1}(1-r)^{\beta/q}M_q\left(P_\varphi^g f, r\right) \asymp \left|P_\varphi^g f(0)\right| + \sup_{0<r<1}(1-r)^{(\beta/q)+1}M_q\left(\Re\left[P_\varphi^g f\right], r\right). \tag{3.11}$$

The assumption $g(0) = 0$ implies $P_\varphi^g f(0) = 0$, and Lemma 3.1 shows $\Re[P_\varphi^g f] = gC_\varphi f$. Hence we obtain

$$\sup_{0<r<1}(1-r)^{\beta}M_q^q\left(P_\varphi^g f, r\right) \asymp \sup_{0<r<1}(1-r)^{\beta+q}M_q^q(gC_\varphi f, r), \tag{3.12}$$

and so we obtain $\|P_\varphi^g f\|_{H_\beta^q}^q \asymp \|gC_\varphi f\|_{H_{\beta+q}^q}^q$. This implies that the boundedness of $P_\varphi^g : H_\alpha^p(\mathbb{B}) \to H_\beta^q(\mathbb{B})$ is equivalent to the boundedness of $gC_\varphi : H_\alpha^p(\mathbb{B}) \to H_{\beta+q}^q(\mathbb{B})$. So Theorem 2.5 shows that the condition

$$\sup_{w\in\mathbb{B}}\sup_{0<r<1}(1-r)^{\beta+q}\int_{\partial\mathbb{B}}\left|g(r\zeta)\right|^q\left\{\frac{1-|w|^2}{\left|1-\langle\varphi(r\zeta),w\rangle\right|^2}\right\}^{q(\alpha+n)/p}d\sigma(\zeta) < \infty \tag{3.13}$$

is a necessary and sufficient condition for the boundedness of $P_\varphi^g : H_\alpha^p(\mathbb{B}) \to H_\beta^q(\mathbb{B})$. This completes the proof. □

The next proposition is proved similar to Proposition 2.6.

Proposition 3.4. *Let $0 < p, q < \infty$, and $\alpha, \beta > 0$. Suppose that $g \in H(\mathbb{B})$ with $g(0) = 0$ and φ is a holomorphic self-map of $\mathbb{B}$ which induce the bounded operator $P_\varphi^g : H_\alpha^p(\mathbb{B}) \to H_\beta^q(\mathbb{B})$. Then $P_\varphi^g : H_\alpha^p(\mathbb{B}) \to H_\beta^q(\mathbb{B})$ is compact if and only if for every bounded sequence $\{f_j\}_{j\in\mathbb{N}}$ in $H_\alpha^p(\mathbb{B})$ which converges to 0 uniformly on compact subsets of $\mathbb{B}$, $\{P_\varphi^g f_j\}_{j\in\mathbb{N}}$ converges to 0 in $H_\beta^q(\mathbb{B})$.*

Theorem 3.5. *Let $0 < p \le q < \infty$ and $\alpha, \beta > 0$. Suppose that $g \in H(\mathbb{B})$ with $g(0) = 0$ and φ is a holomorphic self-map of $\mathbb{B}$ which induce the bounded operator $P_\varphi^g : H_\alpha^p(\mathbb{B}) \to H_\beta^q(\mathbb{B})$. Then $P_\varphi^g : H_\alpha^p(\mathbb{B}) \to H_\beta^q(\mathbb{B})$ is compact if and only if*

$$\lim_{|w|\to 1^-}\sup_{0<r<1}(1-r)^{\beta+q}\int_{\partial\mathbb{B}}\left|g(r\zeta)\right|^q\left\{\frac{1-|w|^2}{\left|1-\langle\varphi(r\zeta),w\rangle\right|^2}\right\}^{q(\alpha+n)/p}d\sigma(\zeta) = 0. \tag{3.14}$$

Proof. First we assume that condition (3.14) holds. Take a bounded sequence $\{f_j\}_{j\in\mathbb{N}} \subset H_\alpha^p(\mathbb{B})$ which converges to 0 uniformly on compact subsets of $\mathbb{B}$. Theorem 2.8 and the remark in Section 2 show that $gC_\varphi : H_\alpha^p(\mathbb{B}) \to H_{\beta+q}^q(\mathbb{B})$ is compact. Thus Proposition 2.6 implies that

$$\lim_{j\to\infty}\|gC_\varphi f_j\|_{H_{\beta+q}^q} = 0. \tag{3.15}$$

From (3.15) and since $\|P_\varphi^g f_j\|_{H_\beta^q}^q \asymp \|gC_\varphi f_j\|_{H_{\beta+q}^q}^q$, we have that $\|P_\varphi^g f_j\|_{H_\beta^q}^q \to 0$ as $j \to \infty$. By Proposition 3.4, we see that $P_\varphi^g : H_\alpha^p(\mathbb{B}) \to H_\beta^q(\mathbb{B})$ is compact.

To prove the necessity of the condition in (3.14), we consider the family of test functions f_w which is defined by (2.11). Hence we have

$$\begin{aligned}\|P_\varphi^g f_w\|_{H_\beta^q}^q &\asymp \|gC_\varphi f_w\|_{H_{\beta+q}^q}^q \\ &= \sup_{0<r<1}(1-r)^{\beta+q}\int_{\partial\mathbb{B}}\left|g(r\zeta)\right|^q\left\{\frac{1-|w|^2}{\left|1-\langle\varphi(r\zeta),w\rangle\right|^2}\right\}^{q(\alpha+n)/p}d\sigma(\zeta),\end{aligned} \tag{3.16}$$

for all $w \in \mathbb{B}$. Since $\{f_w\}_{w\in\mathbb{B}}$ is a bounded sequence in $H_\alpha^p(\mathbb{B})$ and $f_w \to 0$ uniformly on compact subsets of $\mathbb{B}$ as $|w| \to 1^-$, the compactness of P_φ^g and Proposition 3.4 show that $\|P_\varphi^g f_w\|_{H_\beta^q}^q \to 0$ as $|w| \to 1^-$. This fact along with (3.16) implies the condition in (3.14), finishing the proof of the theorem. □

Theorem 3.6. *Let $1 < p \le q < \infty$ and $\alpha, \beta > 0$. Suppose that $g \in H(\mathbb{B})$ with $g(0) = 0$ and φ is a holomorphic self-map of $\mathbb{B}$ which induce the bounded operator $P_\varphi^g : H_\alpha^p(\mathbb{B}) \to H_\beta^q(\mathbb{B})$. Then the qth power of the essential norm of P_φ^g is comparable to*

$$\limsup_{|w|\to 1^-}\sup_{0<r<1}(1-r)^{\beta+q}\int_{\partial\mathbb{B}}\left|g(r\zeta)\right|^q\left\{\frac{1-|w|^2}{\left|1-\langle\varphi(r\zeta),w\rangle\right|^2}\right\}^{q(\alpha+n)/p}d\sigma(\zeta). \tag{3.17}$$

Proof. To prove a lower estimate, we take an arbitrary compact operator $\mathcal{K} : H_\alpha^p(\mathbb{B}) \to H_\beta^q(\mathbb{B})$. Since Lemma 2.7 implies that the family of functions f_w defined by (2.11) converges to 0 weakly in $H_\alpha^p(\mathbb{B})$ as $|w| \to 1^-$, we obtain

$$C\|P_\varphi^g - \mathcal{K}\|_{H_\alpha^p(\mathbb{B})\to H_\beta^q(\mathbb{B})} \ge \limsup_{|w|\to 1^-}\left(\|P_\varphi^g f_w\|_{H_\beta^q} - \|\mathcal{K} f_w\|_{H_\beta^q}\right) \ge \limsup_{|w|\to 1^-}\|P_\varphi^g f_w\|_{H_\beta^q}. \tag{3.18}$$

Combining this with (3.16), we have

$$C\|P_\varphi^g\|_{e,H_\alpha^p(\mathbb{B})\to H_\beta^q(\mathbb{B})}^q \ge \limsup_{|w|\to 1^-}\sup_{0<r<1}(1-r)^{\beta+q}\int_{\partial\mathbb{B}}\left|g(r\zeta)\right|^q\left\{\frac{1-|w|^2}{\left|1-\langle\varphi(r\zeta),w\rangle\right|^2}\right\}^{q(\alpha+n)/p}d\sigma(\zeta), \tag{3.19}$$

which is a lower estimate.

By some modification of Lemma 2.3 and the application of Lemmas 3.1 and 3.2, we get

$$\begin{aligned}\|P_\varphi^g\|_{e,H_\alpha^p(\mathbb{B})\to H_\beta^q(\mathbb{B})}^q &\le \liminf_{j\to\infty}\sup_{\|f\|_{H_\alpha^p}\le 1}\|P_\varphi^g R_j f\|_{H_\beta^q}^q \\ &\le C\liminf_{j\to\infty}\sup_{\|f\|_{H_\alpha^p}\le 1}\|gC_\varphi R_j f\|_{H_{\beta+q}^q}^q.\end{aligned} \tag{3.20}$$

As in the proof of Theorem 2.8, we obtain

$$\liminf_{j\to\infty} \sup_{\|f\|_{H_\alpha^p}\le 1} \|gC_\varphi R_j f\|_{H_{\beta+q}^q}^q \le C \limsup_{|w|\to 1^-} \sup_{0<r<1} (1-r)^{\beta+q} \times \int_{\partial\mathbb{B}} |g(r\zeta)|^q \left\{ \frac{1-|w|^2}{|1-\langle \varphi(r\zeta), w\rangle|^2} \right\}^{q(\alpha+n)/p} d\sigma(\zeta), \tag{3.21}$$

and so we have an upper estimate for $\|P_\varphi^g\|_{e,H_\alpha^p(\mathbb{B})\to H_\beta^q(\mathbb{B})}^q$. □

4. The Case $P_\varphi^g : H_\alpha^\infty(\mathbb{B}) \to H_\beta^\infty(\mathbb{B})$

When $p=\infty$ and $\alpha>0$, we define the weighted-type space $H_\alpha^\infty(\mathbb{B})$ as follows:

$$H_\alpha^\infty(\mathbb{B}) = \left\{ f \in H(\mathbb{B}) : \sup_{0<r<1} (1-r)^\alpha M_\infty(f,r) < \infty \right\}. \tag{4.1}$$

It is easy to see that $f \in H_\alpha^\infty(\mathbb{B})$ if and only if $\sup_{z\in\mathbb{B}}(1-|z|)^\alpha |f(z)| < \infty$, so we define the norm $\|f\|_{H_\alpha^\infty}$ on $H_\alpha^\infty(\mathbb{B})$ by this supremum.

Furthermore we consider the subspace $H_{\alpha,0}^\infty(\mathbb{B})$ defined by

$$H_{\alpha,0}^\infty(\mathbb{B}) = \left\{ f \in H(\mathbb{B}) : \lim_{r\to 1^-} (1-r)^\alpha M_\infty(f,r) = 0 \right\}. \tag{4.2}$$

Theorem 4.1. *Let $\alpha,\beta>0$. Suppose that $g\in H(\mathbb{B})$ with $g(0)=0$ and φ is a holomorphic self-map of $\mathbb{B}$. Then $P_\varphi^g : H_\alpha^\infty(\mathbb{B})$ (or $H_{\alpha,0}^\infty(\mathbb{B})$) $\to H_\beta^\infty(\mathbb{B})$ is bounded if and only if*

$$\sup_{z\in\mathbb{B}} \frac{(1-|z|)^{\beta+1}|g(z)|}{(1-|\varphi(z)|)^\alpha} < \infty. \tag{4.3}$$

In this case, the operator norm $\|P_\varphi^g\|_{H_\alpha^\infty(\mathbb{B})\ (\text{or}\ H_{\alpha,0}^\infty(\mathbb{B}))\to H_\beta^\infty(\mathbb{B})}$ is comparable to the above supremum.

Proof. By the definition of the space $H_\alpha^\infty(\mathbb{B})$, $f\in H_\alpha^\infty(\mathbb{B})$ satisfies the growth condition

$$|f(w)| \le (1-|w|)^{-\alpha}\|f\|_{H_\alpha^\infty} \quad (w\in\mathbb{B}), \tag{4.4}$$

so it follows from Lemma 3.1 and Lemma 3.2 that

$$\|P_\varphi^g f\|_{H_\beta^\infty} \asymp \sup_{z\in\mathbb{B}} (1-|z|)^{\beta+1}|gC_\varphi f(z)| \le \|f\|_{H_\alpha^\infty} \sup_{z\in\mathbb{B}} \frac{(1-|z|)^{\beta+1}|g(z)|}{(1-|\varphi(z)|)^\alpha}, \tag{4.5}$$

for every $f\in H_\alpha^\infty(\mathbb{B})$.

Hence we obtain

$$\|P_\varphi^g\|_{H_\alpha^\infty(\mathbb{B})\ (\text{or}\ \ H_{\alpha,0}^\infty(\mathbb{B}))\to H_\beta^\infty(\mathbb{B})} \le C\sup_{z\in\mathbb{B}}\frac{(1-|z|)^{\beta+1}|g(z)|}{(1-|\varphi(z)|)^\alpha}. \tag{4.6}$$

Now we prove the reverse inequality. For $w\in\mathbb{B}$, we put

$$f_w(z)=\frac{1}{(1-\langle z,w\rangle)^\alpha}. \tag{4.7}$$

Note that $f_w\in H_{\alpha,0}^\infty(\mathbb{B})$ for each $w\in\mathbb{B}$ and moreover $\sup_{w\in\mathbb{B}}\|f_w\|_{H_\alpha^\infty}\le 1$.

When $\varphi(z)\neq 0$, we have

$$\begin{aligned}\|P_\varphi^g\|_{H_{\alpha,0}^\infty(\mathbb{B})\to H_\beta^\infty(\mathbb{B})} &\ge \|P_\varphi^g f_{t(\varphi(z)/|\varphi(z)|)}\|_{H_\beta^\infty} \asymp \sup_{w\in\mathbb{B}}(1-|w|)^{\beta+1}\left|gC_\varphi f_{t(\varphi(z)/|\varphi(z)|)}(w)\right| \\ &\ge (1-|z|)^{\beta+1}|g(z)|\left|f_{t(\varphi(z)/|\varphi(z)|)}(\varphi(z))\right| \\ &= \frac{(1-|z|)^{\beta+1}|g(z)|}{(1-t|\varphi(z)|)^\alpha},\end{aligned} \tag{4.8}$$

for all $t\in(0,1)$. Letting $t\to 1^-$ in (4.8), we have

$$\|P_\varphi^g\|_{H_{\alpha,0}^\infty(\mathbb{B})\to H_\beta^\infty(\mathbb{B})} \ge C\frac{(1-|z|)^{\beta+1}|g(z)|}{(1-|\varphi(z)|)^\alpha}. \tag{4.9}$$

For the constant function $1\in H_{\alpha,0}^\infty(\mathbb{B})$ we obtain

$$\|P_\varphi^g 1\|_{H_\beta^q} \asymp \sup_{w\in\mathbb{B}}(1-|w|)^{\beta+1}|gC_\varphi 1(w)| \ge (1-|z|)^{\beta+1}|g(z)|. \tag{4.10}$$

Inequality (4.10) shows that the estimate in (4.9) also holds when $\varphi(z)=0$.

Hence, from (4.9) we obtain

$$\|P_\varphi^g\|_{H_{\alpha,0}^\infty(\mathbb{B})\to H_\beta^\infty(\mathbb{B})} \ge C\sup_{z\in\mathbb{B}}\frac{(1-|z|)^{\beta+1}|g(z)|}{(1-|\varphi(z)|)^\alpha}, \tag{4.11}$$

which along with the obvious inequality

$$\|P_\varphi^g\|_{H_\alpha^\infty(\mathbb{B})\to H_\beta^\infty(\mathbb{B})} \ge \|P_\varphi^g\|_{H_{\alpha,0}^\infty(\mathbb{B})\to H_\beta^\infty(\mathbb{B})} \tag{4.12}$$

completes the proof of the theorem. □

For the compactness of $P_\varphi^g : H_\alpha^\infty(\mathbb{B})\ \ (or\ H_{\alpha,0}^\infty(\mathbb{B}))\ \to\ H_\beta^\infty(\mathbb{B})$, we can also prove the following proposition which is similar to Proposition 2.6.

Proposition 4.2. *Let $\alpha, \beta > 0$. Suppose that $g \in H(\mathbb{B})$ with $g(0) = 0$ and φ is a holomorphic self-map of $\mathbb{B}$ which induce the bounded operator $P_\varphi^g : H_\alpha^\infty(\mathbb{B})$ (or $H_{\alpha,0}^\infty(\mathbb{B})$) $\to H_\beta^\infty(\mathbb{B})$. Then $P_\varphi^g : H_\alpha^\infty(\mathbb{B})$ (or $H_{\alpha,0}^\infty(\mathbb{B})$) $\to H_\beta^\infty(\mathbb{B})$ is compact if and only if for every bounded sequence $\{f_j\}_{j\in\mathbb{N}}$ in $H_\alpha^\infty(\mathbb{B})$ (or $H_{\alpha,0}^\infty(\mathbb{B})$) which converges to 0 uniformly on compact subsets of $\mathbb{B}$, $\{P_\varphi^g f_j\}_{j\in\mathbb{N}}$ converges to 0 in $H_\beta^\infty(\mathbb{B})$.*

Theorem 4.3. *Let $\alpha, \beta > 0$. Suppose that $g \in H(\mathbb{B})$ with $g(0) = 0$ and φ is a holomorphic self-map of $\mathbb{B}$ such that $P_\varphi^g : H_\alpha^\infty(\mathbb{B})$ (or $H_{\alpha,0}^\infty(\mathbb{B})$) $\to H_\beta^\infty(\mathbb{B})$ is bounded. Then the essential norm $\|P_\varphi^g\|_{e,H_\alpha^\infty(\mathbb{B})\ (or\ H_{\alpha,0}^\infty(\mathbb{B}))\to H_\beta^\infty(\mathbb{B})}$ is comparable to*

$$\limsup_{|\varphi(z)|\to 1^-} \frac{(1-|z|)^{\beta+1}|g(z)|}{(1-|\varphi(z)|)^\alpha}. \tag{4.13}$$

In particular, $P_\varphi^g : H_\alpha^\infty(\mathbb{B})$ (or $H_{\alpha,0}^\infty(\mathbb{B})$) $\to H_\beta^\infty(\mathbb{B})$ is compact if and only if

$$\lim_{|\varphi(z)|\to 1^-} \frac{(1-|z|)^{\beta+1}|g(z)|}{(1-|\varphi(z)|)^\alpha} = 0. \tag{4.14}$$

Proof. First we consider the family $\{f_w\}_{w\in\mathbb{B}}$ where

$$f_w(z) = \frac{1-|w|}{(1-\langle z, w\rangle)^{\alpha+1}}. \tag{4.15}$$

We can easily check that $f_w \in H_{\alpha,0}^\infty(\mathbb{B})$, $\|f_w\|_{H_\alpha^\infty} \le 1$ for all $w \in \mathbb{B}$ and $f_w \to 0$ uniformly on compact subsets of $\mathbb{B}$ as $|w| \to 1^-$. Hence [40, page 296, Theorem 2] implies that $f_w \to 0$ weakly in $H_{\alpha,0}^\infty(\mathbb{B})$ as $|w| \to 1^-$.

If $\|\varphi\|_\infty < 1$, then as in the proof of [26, Theorem 3] it can be seen that the operator $P_\varphi^g : H_\alpha^\infty(\mathbb{B})$ (or $H_{\alpha,0}^\infty(\mathbb{B})$) $\to H_\beta^\infty(\mathbb{B})$ is compact, so that

$$\|P_\varphi^g\|_{e,H_\alpha^\infty(\mathbb{B})\ (or\ H_{\alpha,0}^\infty(\mathbb{B}))\to H_\beta^\infty(\mathbb{B})} = 0. \tag{4.16}$$

On the other hand, the limit in (4.13) is vacuously equal to zero, from which the result follows in this case. If $\|\varphi\|_\infty = 1$, then take a sequence $\{\varphi(z_j)\}_{j\in\mathbb{N}}$ in $\mathbb{B}$ with $|\varphi(z_j)| \to 1$ as $j \to \infty$ and put $F_j(z) = f_{\varphi(z_j)}(z)$ for each $j \in \mathbb{N}$. Then $\{F_j\}_{j\in\mathbb{N}}$ is a bounded sequence in $H_{\alpha,0}^\infty(\mathbb{B})$

and $\{F_j\}_{j\in\mathbb{N}}$ converges to 0 weakly in $H^\infty_{\alpha,0}(\mathbb{B})$, as $j \to \infty$. Hence for every compact operator $\mathcal{K} : H^\infty_{\alpha,0}(\mathbb{B}) \to H^\infty_\beta(\mathbb{B})$ we have $\|\mathcal{K}F_j\|_{H^\infty_\beta} \to 0$ as $j \to \infty$. So we have

$$\begin{aligned}
\|P^g_\varphi - \mathcal{K}\|_{H^\infty_{\alpha,0}(\mathbb{B})\to H^\infty_\beta(\mathbb{B})} &\geq \limsup_{j\to\infty} \|P^g_\varphi F_j - \mathcal{K}F_j\|_{H^\infty_\beta} \\
&\geq \limsup_{j\to\infty} \|P^g_\varphi F_j\|_{H^\infty_\beta} \\
&\asymp \limsup_{j\to\infty} \sup_{w\in\mathbb{B}} (1-|w|)^{\beta+1}|g(w)|\,|F_j(\varphi(w))| \\
&\geq \limsup_{j\to\infty} (1-|z_j|)^{\beta+1}|g(z_j)||F_j(\varphi(z_j))| \\
&\geq \frac{1}{2^{\alpha+1}} \limsup_{j\to\infty} \frac{(1-|z_j|)^{\beta+1}|g(z_j)|}{(1-|\varphi(z_j)|)^\alpha},
\end{aligned} \tag{4.17}$$

for all compact operators $\mathcal{K} : H^\infty_{\alpha,0}(\mathbb{B}) \to H^\infty_\beta(\mathbb{B})$. Taking the infimum over the set of all compact operators $\mathcal{K} : H^\infty_{\alpha,0}(\mathbb{B}) \to H^\infty_\beta(\mathbb{B})$, we obtain

$$\|P^g_\varphi\|_{e,H^\infty_{\alpha,0}(\mathbb{B})\to H^\infty_\beta(\mathbb{B})} \geq C \limsup_{j\to\infty} \frac{(1-|z_j|)^{\beta+1}|g(z_j)|}{(1-|\varphi(z_j)|)^\alpha}. \tag{4.18}$$

Combining this with the estimate $\|P^g_\varphi\|_{e,H^\infty_\alpha(\mathbb{B})\to H^\infty_\beta(\mathbb{B})} \geq \|P^g_\varphi\|_{e,H^\infty_{\alpha,0}(\mathbb{B})\to H^\infty_\beta(\mathbb{B})}$, we have

$$\|P^g_\varphi\|_{e,H^\infty_\alpha(\mathbb{B})\to H^\infty_\beta(\mathbb{B})} \geq C \limsup_{|\varphi(z)|\to 1^-} \frac{(1-|z|)^{\beta+1}|g(z)|}{(1-|\varphi(z)|)^\alpha}. \tag{4.19}$$

Next we prove an upper estimate. Assume that $\{r_l\}_{l\in\mathbb{N}} \subset (0,1)$ is a sequence which increasingly converges to 1. For this $\{r_l\}_{l\in\mathbb{N}}$, we define the operators defined by

$$P^g_{r_l\varphi} f(z) = \int_0^1 g(tz) f(r_l\varphi(tz)) \frac{dt}{t}. \tag{4.20}$$

As in the proof of [26, Theorem 3], Proposition 4.2 shows that $P^g_{r_l\varphi} : H^\infty_\alpha(\mathbb{B}) \to H^\infty_\beta(\mathbb{B})$ is compact for each $l \in \mathbb{N}$.

Put

$$M_2 := \limsup_{|\varphi(z)|\to 1^-} \frac{(1-|z|)^{\beta+1}|g(z)|}{(1-|\varphi(z)|)^\alpha}, \tag{4.21}$$

and fix $\varepsilon > 0$. Then we can choose $R \in (0,1)$ such that

$$\frac{(1-|z|)^{\beta+1}|g(z)|}{(1-|\varphi(z)|)^{\alpha}} < M_2 + \varepsilon, \tag{4.22}$$

if $R < |\varphi(z)| < 1$. Take $f \in H_\alpha^\infty(\mathbb{B})$ with $\|f\|_{H_\alpha^\infty} \le 1$ and an integer $l \in \mathbb{N}$. By Lemma 3.1 and Lemma 3.2, we have

$$\begin{aligned}
\|P_\varphi^g f - P_{r_l\varphi}^g f\|_{H_\beta^\infty} &\asymp \sup_{z\in\mathbb{B}}(1-|z|)^{\beta+1}\left|\Re\left[P_\varphi^g f\right](z) - \Re\left[P_{r_l\varphi}^g f\right](z)\right| \\
&= \sup_{z\in\mathbb{B}}(1-|z|)^{\beta+1}|g(z)||f(\varphi(z)) - f(r_l\varphi(z))| \\
&= \sup_{|\varphi(z)|\le R}(1-|z|)^{\beta+1}|g(z)||f(\varphi(z)) - f(r_l\varphi(z))| \\
&\quad + \sup_{R<|\varphi(z)|<1}(1-|z|)^{\beta+1}|g(z)||f(\varphi(z)) - f(r_l\varphi(z))|.
\end{aligned} \tag{4.23}$$

By using the mean value theorem and the asymptotic relation

$$\sup_{z\in\mathbb{B}}(1-|z|)^{\alpha+1}|\nabla f(z)| \asymp \sup_{z\in\mathbb{B}}(1-|z|)^{\alpha+1}|\Re f(z)|, \tag{4.24}$$

we obtain

$$\begin{aligned}
\sup_{|\varphi(z)|\le R}|f(\varphi(z)) - f(r_l\varphi(z))| &\le \sup_{|\varphi(z)|\le R}(1-r_l)|\varphi(z)|\sup_{|w|\le R}|\nabla f(w)| \\
&\le \frac{(1-r_l)R}{(1-R)^{\alpha+1}}\sup_{w\in\mathbb{B}}(1-|w|)^{\alpha+1}|\nabla f(w)| \\
&\asymp \frac{(1-r_l)R}{(1-R)^{\alpha+1}}\sup_{w\in\mathbb{B}}(1-|w|)^{\alpha+1}|\Re f(w)| \\
&\asymp \frac{(1-r_l)R}{(1-R)^{\alpha+1}}\|f\|_{H_\alpha^\infty}.
\end{aligned} \tag{4.25}$$

Since the boundedness of $P_\varphi^g : H_\alpha^\infty(\mathbb{B}) \to H_\beta^\infty(\mathbb{B})$ implies that $P_\varphi^g 1 \in H_\beta^\infty(\mathbb{B})$, we see

$$\sup_{z\in\mathbb{B}}(1-|z|)^{\beta+1}|g(z)| < \infty, \tag{4.26}$$

and so we have

$$\sup_{\|f\|_{H_\alpha^\infty}\le 1}\sup_{|\varphi(z)|\le R}(1-|z|)^{\beta+1}|g(z)||f(\varphi(z)) - f(r_l\varphi(z))| \le C\frac{(1-r_l)R}{(1-R)^{\alpha+1}}\sup_{z\in\mathbb{B}}(1-|z|)^{\beta+1}|g(z)|. \tag{4.27}$$

On the other hand, the monotonicity of $M_\infty(f,r)$ shows

$$\left|f\left(r_l\varphi(z)\right)\right| \le \left(1-\left|\varphi(z)\right|\right)^{-\alpha}\|f_{r_l}\|_{H_\alpha^\infty} \le \left(1-\left|\varphi(z)\right|\right)^{-\alpha}\|f\|_{H_\alpha^\infty}. \tag{4.28}$$

Thus we have

$$\begin{aligned}\sup_{\|f\|_{H_\alpha^\infty}\le 1}\sup_{R<\left|\varphi(z)\right|<1}(1-|z|)^{\beta+1}\left|g(z)\right|\left|f\left(\varphi(z)\right)-f\left(r_l\varphi(z)\right)\right| &\le 2\sup_{R<\left|\varphi(z)\right|<1}\frac{(1-|z|)^{\beta+1}\left|g(z)\right|}{\left(1-\left|\varphi(z)\right|\right)^{\alpha}}\\ &\le 2(M_2+\varepsilon).\end{aligned} \tag{4.29}$$

From (4.23), (4.27), (4.29), and the compactness of $P_{r_l\varphi}^g$, we obtain

$$\begin{aligned}\|P_\varphi^g\|_{e,H_\alpha^\infty(\mathbb{B})\to H_\beta^\infty(\mathbb{B})} &\le \|P_\varphi^g-P_{r_l\varphi}^g\|_{H_\alpha^\infty(\mathbb{B})\to H_\beta^\infty(\mathbb{B})}\\ &\le C\frac{(1-r_l)R}{(1-R)^{\alpha+1}}\sup_{z\in\mathbb{B}}(1-|z|)^{\beta+1}\left|g(z)\right|+2(M_2+\varepsilon).\end{aligned} \tag{4.30}$$

Letting $l\to\infty$ and $\varepsilon\to 0$, we have

$$\|P_\varphi^g\|_{e,H_\alpha^\infty(\mathbb{B})\to H_\beta^\infty(\mathbb{B})} \le 2\limsup_{\left|\varphi(z)\right|\to 1^-}\frac{(1-|z|)^{\beta+1}\left|g(z)\right|}{\left(1-\left|\varphi(z)\right|\right)^{\alpha}}. \tag{4.31}$$

This completes the proof. □

When $P_\varphi^g : H_\alpha^\infty(\mathbb{B})\ \ (or\ H_{\alpha,0}^\infty(\mathbb{B}))\to H_{\beta,0}^\infty(\mathbb{B})$ is bounded, we see that $g\in H_{\beta+1,0}^\infty(\mathbb{B})$. By a standard argument as in the proof of [26, Corollary 3], we have

$$\limsup_{|z|\to 1^-}\frac{(1-|z|)^{\beta+1}\left|g(z)\right|}{(1-|\varphi(z)|)^{\alpha}} = \limsup_{\left|\varphi(z)\right|\to 1^-}\frac{(1-|z|)^{\beta+1}\left|g(z)\right|}{\left(1-\left|\varphi(z)\right|\right)^{\alpha}}, \tag{4.32}$$

and so

$$\|P_\varphi^g\|_{e,H_\alpha^\infty(\mathbb{B})\left(\text{or } H_{\alpha,0}^\infty(\mathbb{B})\right)\to H_{\beta,0}^\infty(\mathbb{B})} \asymp \limsup_{|z|\to 1^-}\frac{(1-|z|)^{\beta+1}\left|g(z)\right|}{\left(1-\left|\varphi(z)\right|\right)^{\alpha}}. \tag{4.33}$$

Hence we obtain the following characterization for the compactness of the operator $P_\varphi^g : H_\alpha^\infty(\mathbb{B})\ \ (or\ H_{\alpha,0}^\infty(\mathbb{B}))\to H_{\beta,0}^\infty(\mathbb{B})$.

Corollary 4.4. *Let $\alpha, \beta > 0$. Suppose that $g \in H(\mathbb{B})$ with $g(0) = 0$ and φ is a holomorphic self-map of $\mathbb{B}$ such that $P_\varphi^g : H_\alpha^\infty(\mathbb{B})$ (or $H_{\alpha,0}^\infty(\mathbb{B})$) $\to H_{\beta,0}^\infty(\mathbb{B})$ is bounded. Then $P_\varphi^g : H_\alpha^\infty(\mathbb{B})$ (or $H_{\alpha,0}^\infty(\mathbb{B})$) $\to H_{\beta,0}^\infty(\mathbb{B})$ is compact if and only if*

$$\lim_{|z|\to 1^-} \frac{(1-|z|)^{\beta+1}|g(z)|}{(1-|\varphi(z)|)^\alpha} = 0. \tag{4.34}$$

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