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Research Article

Solutions for m-Point BVP with Sign Changing Nonlinearity

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We study the existence of positive solutions for the following nonlinear m-point boundary value problem for an increasing homeomorphism and homomorphism with sign changing nonlinearity: $\{(\phi(u'(t)))' + a(t)f(t,u(t)) = 0, 0 < t < 1, u'(0) = \sum_{i=1}^{m-2} a_i u'(\xi_i), u(1) = \sum_{i=1}^{k} b_i u(\xi_i) - \sum_{i=k+1}^{s} b_i u(\xi_i) - \sum_{i=s+1}^{m-2} b_i u'(\xi_i)$, where $\phi : R \to R$ is an increasing homeomorphism and homomorphism and $\phi(0) = 0$. The nonlinear term f may change sign. As an application, an example to demonstrate our results is given. The conclusions in this paper essentially extend and improve the known results.

1. Introduction

In this paper, we study the existence of positive solutions of the following nonlinear m-point boundary value problem with sign changing nonlinearity:

$$\begin{gathered}(\phi(u'(t)))' + a(t)f(t,u(t)) = 0, \quad 0 < t < 1, \\ u'(0) = \sum_{i=1}^{m-2} a_i u'(\xi_i), \qquad u(1) = \sum_{i=1}^{k} b_i u(\xi_i) - \sum_{i=k+1}^{s} b_i u(\xi_i) - \sum_{i=s+1}^{m-2} b_i u'(\xi_i),\end{gathered} \tag{1.1}$$

where $\phi : R \to R$ is an increasing homeomorphism and homomorphism and $\phi(0) = 0$; $\xi_i \in (0,1)$ with $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$ and a_i, b_i, a, f satisfy

(H_1) $a_i, b_i \in [0,+\infty)$, $0 < \sum_{i=1}^{k} b_i - \sum_{i=k+1}^{s} b_i < 1, 0 < \sum_{i=1}^{m-2} a_i < 1$;

(H_2) $a(t) : (0,1) \to [0,+\infty)$ does not vanish identically on any subinterval of $[0,1]$ and satisfies

$$0 < \int_0^1 a(t)\mathrm{d}t < \infty; \tag{1.2}$$

(H_3) $f \in C([0,1] \times [0,+\infty), (-\infty,+\infty))$, $f(t,0) \geq 0$ and $f(t,0) \neq 0$.

Definition 1.1. A projection $\phi : R \to R$ is called an increasing homeomorphism and homomorphism, if the following conditions are satisfied:

(i) if $x \leq y$, then $\phi(x) \leq \phi(y)$, for all $x, y \in R$;

(ii) ϕ is a continuous bijection and its inverse mapping is also continuous;

(iii) $\phi(xy) = \phi(x)\phi(y)$, for all $x, y \in R$.

The study of multipoint boundary value problems for linear second-order ordinary differential equations was initiated by Il'in and Moiseev [1, 2]. Motivated by the study of [1, 2], Gupta [3] studied certain three-point boundary value problems for nonlinear ordinary differential equations. Since then, more general nonlinear multipoint boundary value problems have been studied by several authors. We refer the reader to [4–12] for some references along this line. Multipoint boundary value problems describe many phenomena in the applied mathematical sciences. For example, the vibrations of a guy wire of a uniform cross-section and composed of N parts of different densities can be set up as a multipoint boundary value problems (see Moshinsky [13]); many problems in the theory of elastic stability can be handled by the method of multipoint boundary value problems (see Timoshenko [14]).

In 2001, Ma [6] studied m-point boundary value problem (BVP):

$$\begin{gathered} u''(t) + h(t)f(u) = 0, \quad 0 \leq t \leq 1, \\ u(0) = 0, \qquad u(1) = \sum_{i=1}^{m-2} \alpha_i u'(\xi_i), \end{gathered} \tag{1.3}$$

where $\alpha_i > 0$ $(i = 1,2,\ldots,m-2)$, $\sum_{i=1}^{m-2}\alpha_i < 1$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, and $f \in C([0,+\infty),[0,+\infty))$, $h \in C([0,1],[0,+\infty))$. Author established the existence of positive solutions under the condition that f is either superlinear or sublinear.

In [11], we considered the existence of positive solutions for the following nonlinear four-point singular boundary value problem with p-Laplacian:

$$\begin{gathered} \left(\phi_p\left(u'(t)\right)\right)' + a(t)f(u(t)) = 0, \quad 0 < t < 1, \\ \alpha\phi_p(u(0)) - \beta\phi_p\left(u'(\xi)\right) = 0, \qquad \gamma\phi_p(u(1)) + \delta\phi_p\left(u'(\eta)\right) = 0, \end{gathered} \tag{1.4}$$

where $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $\phi_q = (\phi_p)^{-1}$, $1/p + 1/q = 1$, $\alpha > 0$, $\beta \geq 0$, $\gamma > 0$, $\delta \geq 0$, $\xi, \eta \in (0,1)$, $\xi < \eta$, $a : (0,1) \to [0,\infty)$. By using the fixed-point theorem of cone, the existence of positive solution and many positive solutions for nonlinear singular boundary value problem p-Laplacian is obtained.

Recently, Ma et al. [5] used the monotone iterative technique in cones to prove the existence of at least one positive solution for m-point boundary value problem (BVP):

$$\begin{gathered}(\phi_p(u'(t)))' + a(t)f(t,u(t)) = 0, \quad 0 < t < 1, \\ u'(0) = \sum_{i=1}^{m-2} a_i u'(\xi_i), \qquad u(1) = \sum_{i=1}^{m-2} b_i u(\xi_i),\end{gathered} \tag{1.5}$$

where $0 < \sum_{i=1}^{m-2} b_i < 1$, $0 < \sum_{i=1}^{m-2} a_i < 1$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, $a(t) \in L^1[0,1]$, $f \in C([0,1] \times [0,+\infty), [0,+\infty))$.

In [9], Wang and Hou investigated the following m-point BVP:

$$\begin{gathered}(\phi_p(u'(t)))' + f(t,u(t)) = 0, \quad t \in (0,1), \\ \phi_p(u'(0)) = \sum_{i=1}^{n-2} a_i \phi_p(u'(\xi_i)), \qquad u(1) = \sum_{i=1}^{n-2} b_i u(\xi_i),\end{gathered} \tag{1.6}$$

where $\phi_p(u) = |u|^{p-2}u$, $p > 1$, $\xi_i \in (0,1)$ with $0 < \xi_1 < \xi_2 < \cdots < \xi_{n-2} < 1$ and a_i, b_i satisfy $a_i, b_i \in [0,+\infty)$, $0 < \sum_{i=1}^{n-2} a_i < 1$, $0 < \sum_{i=1}^{n-2} b_i < 1$.

However, in all the above-mentioned paper, the authors discuss the boundary value problem (BVP) under the key conditions that the nonlinear term is positive continuous function. Motivated by the results mentioned above, in this paper we study the existence of positive solutions of m-point boundary value problem (1.1) for an increasing homeomorphism and homomorphism with sign changing nonlinearity. We generalize the results in [4–12].

By a positive solution of BVP (1.1), we understand a function u which is positive on $(0,1)$ and satisfies the differential equation as well as the boundary conditions in BVP (1.1).

2. The Preliminary Lemmas

In this section, we present some lemmas which are important to our main results.

Lemma 2.1. *Let (H_1) and (H_2) hold. Then for $u \geq 0 \in C[0,1]$, the problem*

$$\begin{gathered}(\phi(u'(t)))' + a(t)f(t,u(t)) = 0, \quad 0 < t < 1, \\ u'(0) = \sum_{i=1}^{m-2} a_i u'(\xi_i), \qquad u(1) = \sum_{i=1}^{k} b_i u(\xi_i) - \sum_{i=k+1}^{s} b_i u(\xi_i) - \sum_{i=s+1}^{m-2} b_i u'(\xi_i)\end{gathered} \tag{2.1}$$

has a unique solution $u(t)$ if and only if $u(t)$ can be express as the following equation:

$$u(t) = -\int_t^1 \omega_f(s)\mathrm{d}s + B, \tag{2.2}$$

where A, B satisfy

$$\phi^{-1}(A) = \sum_{i=1}^{m-2} a_i \phi^{-1}\left(A - \int_0^{\xi_i} a(s) f(s, u(s)) \mathrm{d}s\right), \tag{2.3}$$

$$\begin{aligned} B = -\frac{1}{1 - \sum_{i=1}^{k} b_i + \sum_{i=k+1}^{s} b_i} \Bigg[& \sum_{i=1}^{k} b_i \int_{\xi_i}^{1} \omega_f(s) \mathrm{d}s - \sum_{i=k+1}^{s} b_i \int_{\xi_i}^{1} \omega_f(s) \mathrm{d}s \\ & + \sum_{i=s+1}^{m-2} b_i \phi^{-1}\left(A - \int_0^{\xi_i} a(s) f(s, u(s)) \mathrm{d}s\right)\Bigg], \end{aligned} \tag{2.4}$$

where

$$\omega_f(s) = \phi^{-1}\left(-\int_0^{s} a(r) f(r, u(r)) \mathrm{d}r + A\right). \tag{2.5}$$

Define $l = \phi(\sum_{i=1}^{m-2} a_i)/(1 - \phi(\sum_{i=1}^{m-2} a_i)) \in (0,1)$, *then there exists a unique* $A \in [-l\int_0^1 a(s) f(s, u(s)) \mathrm{d}s, 0]$ *satisfying* (2.3).

Proof. The method of the proof is similar to [5, Lemma 2.1], we omit the details. □

Lemma 2.2. *Let* (H_1) *and* (H_2) *hold. If* $u \in C^+[0,1]$, *the unique solution of the problem* (2.1) *satisfies*

$$u(t) \geq 0, \quad t \in [0,1]. \tag{2.6}$$

Proof. According to Lemma 2.1, we first have

$$-A + \int_0^{s} a(r) f(t, u(r)) \mathrm{d}r \geq 0. \tag{2.7}$$

So

$$\begin{aligned} u(1) &= B \\ &= -\frac{1}{1 - \sum_{i=1}^{k} b_i + \sum_{i=k+1}^{s} b_i} \Bigg[\sum_{i=1}^{k} b_i \int_{\xi_i}^{1} \omega_f(s) \mathrm{d}s - \sum_{i=k+1}^{s} b_i \int_{\xi_i}^{1} \omega_f(s) \mathrm{d}s \\ &\qquad + \sum_{i=s+1}^{m-2} b_i \phi^{-1}\left(A - \int_0^{\xi_i} a(s) f(t, u(s)) \mathrm{d}s\right)\Bigg] \\ &= \frac{1}{1 - \sum_{i=1}^{k} b_i + \sum_{i=k+1}^{s} b_i} \Bigg[\sum_{i=1}^{k} b_i \int_{\xi_i}^{1} -\omega_f(s) \mathrm{d}s - \sum_{i=k+1}^{s} b_i \int_{\xi_i}^{1} -\omega_f(s) \mathrm{d}s \\ &\qquad + \sum_{i=s+1}^{m-2} b_i \phi^{-1}\left(-A + \int_0^{\xi_i} a(s) f(t, u(s)) \mathrm{d}s\right)\Bigg] \end{aligned}$$

$$
\begin{aligned}
&\geq \frac{1}{1-\sum_{i=1}^{k} b_i + \sum_{i=k+1}^{s} b_i}\left[\sum_{i=1}^{k} b_i \int_{\xi_k}^{1} -\omega_f(s)\mathrm{d}s - \sum_{i=k+1}^{s} b_i \int_{\xi_k}^{1} -\omega_f(s)\mathrm{d}s\right] \\
&= \frac{\left(\sum_{i=1}^{k} b_i - \sum_{i=k+1}^{s} b_i\right)\int_{\xi_k}^{1} -\omega_f(s)\mathrm{d}s}{1-\sum_{i=1}^{k} b_i + \sum_{i=k+1}^{s} b_i} \\
&\geq 0.
\end{aligned}
\tag{2.8}
$$

If $t \in [0,1)$, we have

$$
\begin{aligned}
u(t) &= B - \int_t^1 \phi^{-1}\left(A - \int_0^s a(r) f(r,u(r))\mathrm{d}r\right)\mathrm{d}s \\
&= u(1) + \int_t^1 \phi^{-1}\left(-A + \int_0^s a(r) f(r,u(r))\mathrm{d}r\right)\mathrm{d}s \\
&\geq u(1) \\
&\geq 0.
\end{aligned}
\tag{2.9}
$$

So $u(t) \geq 0,\ t \in [0,1]$. The proof of Lemma 2.2 is completed. □

Lemma 2.3. *Let* (H_1) *and* (H_2) *hold. If* $u \in C^+[0,1]$, *the unique solution of the problem* (2.1) *satisfies*

$$
\inf_{t\in[0,1]} u(t) \geq \gamma \|u\|, \tag{2.10}
$$

where $\gamma = (\sum_{i=1}^{k} b_i - \sum_{i=k+1}^{s} b_i)(1-\xi_k)/(1-\sum_{i=1}^{k} b_i\xi_k + \sum_{i=k+1}^{s} b_i\xi_k) \in (0,1)$, $\|u\| = \max_{t\in[0,1]}|u(t)|$.

Proof. Clearly

$$
\begin{aligned}
u'(t) &= \phi^{-1}\left(A - \int_0^t a(s) f(s,u(s))\mathrm{d}s\right) \\
&= -\phi^{-1}\left(-A + \int_0^t a(s) f(s,u(s))\mathrm{d}s\right) \\
&\leq 0.
\end{aligned}
\tag{2.11}
$$

This implies that

$$
\|u\| = u(0), \qquad \min_{t\in[0,1]} u(t) = u(1). \tag{2.12}
$$

It is easy to see that $u'(t_2) \le u'(t_1)$, for any $t_1, t_2 \in [0,1]$ with $t_1 \le t_2$. Hence $u'(t)$ is a decreasing function on $[0,1]$. This means that the graph of $u(t)$ is concave down on $(0,1)$. So we have

$$u(\xi_k) - u(1)\xi_k \ge (1-\xi_k)u(0). \tag{2.13}$$

Together with $u(1) = \sum_{i=1}^{k} b_i u(\xi_i) - \sum_{i=k+1}^{s} b_i u(\xi_i) - \sum_{i=s+1}^{m-2} b_i u'(\xi_i)$ and $u'(t) \le 0$ on $[0,1]$, we get

$$\begin{aligned}
u(0) &\le \frac{\sum_{i=1}^{k} b_i u(\xi_k) - u(1)\sum_{i=1}^{k} b_i\xi_k - \sum_{i=k+1}^{s} b_i u(\xi_k) + u(1)\sum_{i=k+1}^{s} b_i\xi_k}{\left(\sum_{i=1}^{k} b_i - \sum_{i=k+1}^{s} b_i\right)(1-\xi_k)} \\
&\le \frac{\sum_{i=1}^{k} b_i u(\xi_i) - u(1)\sum_{i=1}^{k} b_i\xi_k - \sum_{i=k+1}^{s} b_i u(\xi_i) + u(1)\sum_{i=k+1}^{s} b_i\xi_k}{\left(\sum_{i=1}^{k} b_i - \sum_{i=k+1}^{s} b_i\right)(1-\xi_k)} \\
&\le \frac{u(1)\left(1 - \sum_{i=1}^{k} b_i\xi_k + \sum_{i=k+1}^{s} b_i\xi_k\right)}{\left(\sum_{i=1}^{k} b_i - \sum_{i=k+1}^{s} b_i\right)(1-\xi_k)} \\
&= \frac{u(1)}{\gamma}.
\end{aligned} \tag{2.14}$$

The proof of Lemma 2.3 is completed. □

Lemma 2.4 (see [8]). *Let K be a cone in a Banach space X. Let D be an open bounded subset of X with $D_K = D \cap K \ne \phi$ and $\overline{D_K} \ne K$. Assume that $A : \overline{D_K} \to K$ is a compact map such that $x \ne AK$ for $x \in \partial D_K$. Then the following results hold.*

(1) *If $\|Ax\| \le \|x\|$, $x \in \partial D_K$, then $i(A, D_K, K) = 1$.*

(2) *If there exists $x_0 \in K \setminus \{\theta\}$ such that $x \ne Ax + \lambda x_0$, for all $x \in \partial D_K$ and all $x > 0$, then $i(A, D_K, K) = 0$.*

(3) *Let U be open in X such that $\overline{U} \subset D_K$. If $i(A, D_K, K) = 1$ and $i(A, D_K, K) = 0$, then A has a fixed point in $D_K \setminus \overline{U}_K$. The same results hold, if $i(A, D_K, K) = 0$ and $i(A, D_K, K) = 1$.*

Let $E = C[0,1]$, then E is Banach space, with respect to the norm $\|u\| = \sup_{t\in[0,1]}|u(t)|$. Denote

$$K = \left\{u \mid u \in C[0,1],\ u(t) \ge 0,\ \inf_{t\in[0,1]} u(t) \ge \gamma\|u\|\right\}, \tag{2.15}$$

where γ is the same as in Lemma 2.3. It is obvious that K is a cone in $C[0,1]$.

We define $\varphi(t) = \min\{t, 1-t\},\ t \in (0,1)$ and

$$\begin{aligned} K_\rho &= \{u(t) \in K : \|u\| < \rho\}, \\ K_\rho^* &= \{u(t) \in K : \rho\varphi(t) < u(t) < \rho\}, \\ \Omega_\rho &= \left\{u(t) \in K : \min_{\xi_{m-2} \le t \le 1} u(t) < \gamma\rho\right\} \\ &= \left\{u(t) \in E : u \ge 0,\ \gamma\|u\| \le \min_{\xi_{m-2} \le t \le 1} u(t) < \gamma\rho\right\}. \end{aligned} \tag{2.16}$$

Lemma 2.5 (see [13]). *Ω_ρ defined above has the following properties:*

(a) $K_{\gamma\rho} \subset \Omega_\rho \subset K_\rho$;

(b) *Ω_ρ is open relative to K;*

(c) *$X \in \partial\Omega_\rho$ if and only if $\min_{\xi_{m-2} \le t \le 1c} x(t) = \gamma\rho$;*

(d) *If $x \in \partial\Omega_\rho$, then $\gamma\rho \le x(t) \le \rho$ for $t \in [\xi_{m-2}, 1]$.*

Now, for the convenience, one introduces the following notations:

$$\begin{aligned} f_{\gamma\rho}^\rho &= \min\left\{\min_{\xi_{m-2} \le t \le 1} \frac{f(t,u)}{\phi(\rho)} : u \in [\gamma\rho, \rho]\right\}, \\ f_0^\rho &= \max\left\{\max_{0 \le t \le 1} \frac{f(t,u)}{\phi(\rho)} : u \in [0, \rho]\right\}, \\ f_{\varphi(t)\rho}^\rho &= \max\left\{\max_{0 \le t \le 1} \frac{f(t,u)}{\phi(\rho)} : u \in [\varphi(t)\rho, \rho]\right\}, \\ f^\alpha &= \lim_{u \to \alpha} \sup \max_{0 \le t \le 1} \frac{f(t,u)}{\phi(u)}, \\ f_\alpha &= \lim_{u \to \alpha} \inf \min_{\xi_{m-2} \le t \le 1} \frac{f(t,u)}{\phi(u)}, \quad (\alpha := \infty \text{ or } 0^+), \\ m &= \left\{\frac{\left(1 + \sum_{i=k+1}^{s} b_i + \sum_{i=s+1}^{m-2} b_i\right)\phi^{-1}\left((l+1)\int_0^1 a(s)\mathrm{d}s\right)}{1 - \sum_{i=1}^{k} b_i + \sum_{i=k+1}^{s} b_i}\right\}^{-1}, \\ M &= \left\{\frac{\sum_{i=1}^{k} b_i - \sum_{i=k+1}^{s} b_i}{1 - \sum_{i=1}^{k} b_i + \sum_{i=k+1}^{s} b_i} \int_{\xi_k}^{1} \phi^{-1}\left(\int_0^s a(r)\mathrm{d}r\right)\mathrm{d}s\right\}^{-1}. \end{aligned} \tag{2.17}$$

3. The Main Result

In the rest of the section, we also assume the following conditions.

(A_1) There exist $\rho_1, \rho_2 \in (0, +\infty)$ with $\rho_1 < \gamma\rho_2$ such that

$$\begin{aligned} &(1) \quad f(t,u) > 0, \quad t \in [0,1],\ u \in [\rho_1\varphi(t), +\infty), \\ &(2) \quad f^{\rho_1}_{\varphi(t)\rho_1} \le \phi(m), \qquad f^{\rho_2}_{\gamma\rho_2} \ge \phi(M\gamma). \end{aligned} \tag{3.1}$$

(A_2) There exist $\rho_1, \rho_2 \in (0, +\infty)$ with $\rho_1 < \rho_2$ such that

$$\begin{aligned} &(3) \quad f(t,u) > 0, \quad t \in [0,1],\ u \in [\min\{\gamma\rho_1, \rho_2\varphi(t)\}, +\infty), \\ &(4) \quad f^{\rho_1}_{\gamma\rho_1} \ge \phi(M\gamma), \qquad f^{\rho_2}_{\varphi(t)\rho_2} \le \phi(m). \end{aligned} \tag{3.2}$$

(A_3) There exist $\rho_1, \rho_2, \rho_3 \in (0, +\infty)$ with $\rho_1 < \gamma\rho_2$ and $\rho_2 < \rho_3$ such that

$$\begin{aligned} &(1) \quad f(t,u) > 0, \quad t \in [0,1],\ u \in [\rho_1\varphi(t), +\infty), \\ &(2) \quad f^{\rho_1}_{\varphi(t)\rho_1} \le \phi(m), \qquad f^{\rho_2}_{\gamma\rho_2} \ge \phi(M\gamma), \qquad f^{\rho_3}_{\varphi(t)\rho_3} \le \phi(m). \end{aligned} \tag{3.3}$$

(A_4) There exist $\rho_1, \rho_2, \rho_3 \in (0, +\infty)$ with $\rho_1 < \rho_2 < \gamma\rho_3$ such that

$$\begin{aligned} &(3) \quad f(t,u) > 0, \quad t \in [0,1],\ u \in [\min\{\gamma\rho_1, \rho_2\varphi(t)\}, +\infty), \\ &(4) \quad f^{\rho_1}_{\gamma\rho_1} \ge \phi(M\gamma), \qquad f^{\rho_2}_{\varphi(t)\rho_2} \le \phi(m), \qquad f^{\rho_3}_{\gamma\rho_3} \ge \phi(M\gamma). \end{aligned} \tag{3.4}$$

(A_5) There exist $\rho', \rho \in (0, +\infty)$ with $\rho' < \gamma\rho$ such that

$$\begin{aligned} &(1) \quad f(t,u) > 0, \quad t \in [0,1],\ u \in [\rho'\varphi(t), +\infty), \\ &(2) \quad f^{\rho'}_{\varphi(t)\rho'} \le \phi(m), \quad f^{\rho}_{\gamma\rho} \ge \phi(M\gamma), \quad 0 \le f^{\infty} < \phi(m). \end{aligned} \tag{3.5}$$

(A_6) There exist $\rho', \rho \in (0, +\infty)$ with $\rho' < \rho$ such that

$$\begin{aligned} &(3) \quad f(t,u) > 0, \quad t \in [0,1],\ u \in [\min\{\gamma\rho', \rho\varphi(t)\}, +\infty), \\ &(4) \quad f^{\rho'}_{\gamma\rho'} \ge \phi(M\gamma), \quad f^{\rho}_{\varphi(t)\rho} \le \phi(m), \quad \phi(M) < f_{\infty} \le \infty. \end{aligned} \tag{3.6}$$

Our main results are the following theorems.

Theorem 3.1. *Assume that* (H_1), (H_2), (H_3), (A_3) *hold. Then* BVP (1.1) *has at least three positive solutions.*

Theorem 3.2. *Assume that* (H_1), (H_2), (H_3), (A_4) *hold. Then* BVP (1.1) *has at least two positive solutions.*

Theorem 3.3. *Assume that* (H_1), (H_2), (H_3) *hold and also assume that* (A_1) *or* (A_2) *hold. Then* BVP (1.1) *has at least a positive solution.*

Theorem 3.4. *Assume that* (H_1), (H_2), (H_3) *hold and also assume that* (A_5) *or* (A_6) *hold. Then* BVP (1.1) *has at least two positive solutions.*

Proof of Theorem 3.1. Without loss of generality, we suppose that (A_3) hold. Denote

$$f^*(t,u) = \begin{cases} f(t,u), & u \geq \rho_1\varphi(t), \\ f(t,\rho_1\varphi(t)), & 0 \leq u < \rho_1\varphi(t), \end{cases} \tag{3.7}$$

it is easy to check that $f^*(t,u) \in C([0,1] \times [0,+\infty),(0,+\infty))$.

Now define an operator $T : K \to C[0,1]$ by setting

$$\begin{aligned}(Tu)(t) = -\frac{1}{1-\sum_{i=1}^{k} b_i + \sum_{i=k+1}^{s} b_i}\Bigg[&\sum_{i=1}^{k} b_i \int_{\xi_i}^{1} \omega(s)\mathrm{d}s - \sum_{i=k+1}^{s} b_i \int_{\xi_i}^{1} \omega(s)\mathrm{d}s \\ &+ \sum_{i=s+1}^{m-2} b_i \phi^{-1}\left(A - \int_0^{\xi_i} a(s) f^*(s,u(s))\mathrm{d}s\right)\Bigg] - \int_t^1 \omega(s)\mathrm{d}s,\end{aligned} \tag{3.8}$$

where

$$\omega(s) = \phi^{-1}\left(-\int_0^s a(r) f^*(\tau,u(r))\mathrm{d}r + A\right). \tag{3.9}$$

By Lemma 2.3, we have $T(K) \subset K$. So by applying Arzela-Ascoli's theorem, we can obtain that $T(K)$ is relatively compact. In view of Lebesgue's dominated convergence theorem, it is easy to prove that T is continuous. Hence, $T : K \to K$ is completely continuous.

Now, we consider the following modified BVP (1.1):

$$\begin{gathered}(\phi(u'))' + a(t) f^*(t,u(t)) = 0, \quad 0 < t < 1, \\ u'(0) = \sum_{i=1}^{m-2} a_i u'(\xi_i), \qquad u(1) = \sum_{i=1}^{k} b_i u(\xi_i) - \sum_{i=k+1}^{s} b_i u(\xi_i) - \sum_{i=s+1}^{m-2} b_i u'(\xi_i).\end{gathered} \tag{3.10}$$

Obviously, BVP (3.10) has a solution $u(t)$ if and only if u is a fixed point of the operator T. From the condition $(A_3)(2)$, we have

$$f^{*\rho_1}_{\varphi(t)\rho_1} \le \phi(m), \qquad f^{*\rho_2}_{\gamma\rho_2} \ge \phi(M\gamma), \qquad f^{*\rho_3}_{\varphi(t)\rho_3} \le \phi(m). \tag{3.11}$$

Next, we will show that $i(T, K^*_{\rho_1}, K) = 1$.

In fact, by $f^{*\rho_1}_{\varphi(t)\rho_1} \le \phi(m)$, for $\forall u \in \partial K^*_{\rho_1}$, we have

$$\begin{aligned}
(Tu)(t) &= -\frac{1}{1-\sum_{i=1}^{k} b_i + \sum_{i=k+1}^{s} b_i} \\
&\quad \times \left(\sum_{i=1}^{k} b_i \int_{\xi_i}^{1} \omega(s)\mathrm{d}s - \sum_{i=k+1}^{s} b_i \int_{\xi_i}^{1} \omega(s)\mathrm{d}s + \sum_{i=s+1}^{m-2} b_i \phi^{-1}\left(A - \int_0^{\xi_i} a(s) f^*(s,u(s))\mathrm{d}s \right) \right) \\
&\quad - \int_t^1 \omega(s)\mathrm{d}s \\
&\le \frac{1}{1-\sum_{i=1}^{k} b_i + \sum_{i=k+1}^{s} b_i} \left(\sum_{i=1}^{k} b_i \int_0^1 \phi^{-1}\left((l+1)\int_0^1 a(r) f^*(r,u(r)) \right)\mathrm{d}r \right)\mathrm{d}s \\
&\qquad\qquad + \sum_{i=s+1}^{m-2} b_i \phi^{-1}\left((l+1)\int_0^1 a(s) f^*(s,u(s))\mathrm{d}s \right) \Bigg) \\
&\quad + \int_0^1 \phi^{-1}\left((l+1)\int_0^1 a(r) f^*(r,u(r))\mathrm{d}r \right)\mathrm{d}s \\
&\le \frac{\left(1+\sum_{i=k+1}^{s} b_i + \sum_{i=s+1}^{m-2} b_i\right)\phi^{-1}\left((l+1)\int_0^1 a(s)\mathrm{d}s\right)}{1-\sum_{i=1}^{k} b_i + \sum_{i=k+1}^{s} b_i} \phi^{-1}\left(\phi(\rho_1)\phi(m)\right) \\
&= \rho_1 = \|u\|.
\end{aligned} \tag{3.12}$$

This implies that $\|Tu\| \le \|u\|$ for $u \in \partial K^*_\rho$. By Lemma 2.4(1), we have

$$i\left(T, K^*_{\rho_1}, K\right) = 1. \tag{3.13}$$

Furthermore, we will show that $i(T, K_{\rho_2}, K) = 1$.

Let $e(t) \equiv 1$, for $t \in [0,1]$, then $e \in \partial K_1$. We claim that

$$u \ne Tu + \lambda e, \quad u \in \partial\Omega_{\rho_2},\ \lambda > 0. \tag{3.14}$$

In fact, if not, there exist $u_0 \in \partial\Omega_2$ and $\lambda_0 > 0$ such that $u_0 = Tu_0 + \lambda_0 e$.

By (A_3) and Lemma 2.1, we have for $t \in [0,1]$,

$$-\int_0^s a(\tau)f^*(\tau,u(\tau))d\tau + A \le -\phi(\rho_2)\phi(M\gamma)\left(\int_0^s a(\tau)d\tau\right), \tag{3.15}$$

so that

$$\begin{aligned} -\omega(s) &= \phi^{-1}\left(-\int_0^s a(\tau)f^*(\tau,u(\tau))d\tau + A\right) \\ &\ge \rho_2 M\gamma\phi^{-1}\left[\int_0^s a(\tau)d\tau\right]. \end{aligned} \tag{3.16}$$

Then, we have that

$$\begin{aligned} u_0(t) &= Tu_0(t) + \lambda_0 e(t) \\ &\ge \frac{1}{1-\sum_{i=1}^k b_i + \sum_{i=k+1}^s b_i}\sum_{i=1}^k\left(b_i\int_{\xi_k}^1(-\omega(s))\mathrm{d}s - \sum_{i=k+1}^s b_i\int_{\xi_k}^1(-\omega(s))\mathrm{d}s\right) + \lambda_0 \\ &\ge \frac{\sum_{i=1}^k b_i - \sum_{i=k+1}^s b_i}{1-\sum_{i=1}^k b_i + \sum_{i=k+1}^s b_i}\rho_2 M\gamma\int_{\xi_k}^1\phi^{-1}\left(\int_0^s a(r)\mathrm{d}r\right)\mathrm{d}s + \lambda_0 \\ &= \gamma\rho_2 + \lambda_0. \end{aligned} \tag{3.17}$$

This implies that $\gamma\rho_2 \ge \gamma\rho_2 + \lambda_0$, this is a contradiction. Hence, by Lemma 2.4(2), it follows that

$$i(T,\Omega_{\rho_2},K) = 0. \tag{3.18}$$

Finally, similar to the proof of $i(T,K^*_{\rho_1},K) = 1$, we can show that $i(T,K^*_{\rho_3},K) = 1$.

By Lemma 2.5(a) and $\rho_1 < \gamma\rho_2$ and $\rho_2 < \rho_3$, we have $\overline{K}_{\rho_1} \subset K_{\gamma\rho_2} \subset \Omega_{\rho_2} \subset K_{\rho_2} \subset K_{\rho_3}$. It follows from Lemma 2.4(3) that T has three positive fixed points $u_1,\ u_2,\ u_3$ in $K^*_{\rho_1}, \Omega_{\rho_2}\setminus\overline{K^*}_{\rho_1},\ K^*_{\rho_3}$, respectively. Therefore, BVP (3.10) has three positive solutions $u_1,\ u_2,\ u_3$ in $K^*_{\rho_1}, \Omega_{\rho_2}\setminus\overline{K^*}_{\rho_1},\ K^*_{\rho_3}$, respectively.

Then, BVP (3.10) has three positive solutions $u_1, u_2, u_3 \in [\rho_1\varphi(t),\infty)$, which means that $u_1,\ u_2,\ u_3$ are also the positive solutions of BVP (1.1). □

Proof of Theorem 3.2. The proof of Theorem 3.2 is similar to that of Theorem 3.1, and so we omit it here. The proof of Theorem 3.2 is completed. □

Proof of Theorem 3.3. Theorem 3.3 is corollary of Theorem 3.1. The proof of Theorem 3.3 is completed. □

Proof of Theorem 3.4. We show that condition (A_5) implies condition (A_3). Let $k \in (f^{\infty}, \phi(m))$, then there exists $r > \rho$ such that $\max_{t\in[0,1]} f(t,u) \le k\phi(u)$, $u \in [r,\infty)$ since $0 \le f^{\infty} < \phi(m)$. Denote

$$\beta = \max\left\{\max_{t\in[0,1]} f(t,u) : \rho'\varphi(t) \le u \le r\right\}, \qquad \rho_3 > \max\left\{\phi^{-1}\left(\frac{\beta}{\phi(m)-k}\right), \rho\right\}. \tag{3.19}$$

Then we have

$$\max_{t\in[0,1]} f(t,u) \le k\phi(u) + \beta \le k\phi(\rho_3) + \beta \le \phi(m)\phi(\rho_3), \quad u \in [\rho'\varphi(t), \infty). \tag{3.20}$$

This implies that $f^{\rho_3}_{\varphi(t)\rho_3} \le \phi(m)$ and (A_3) holds. Similarly condition (A_6) implies condition (A_4).

By an argument similar to that Theorem 3.1, we can obtain the result of Theorem 3.4. The proof of Theorem 3.4 is completed. □

4. Examples

Example 4.1. Consider the following five-point boundary value problem with p-Laplacian:

$$\begin{gathered}
(\phi(u'))' + f(t,u) = 0, \quad 0 < t < 1, \\
u'(0) = \frac{1}{128}u'\left(\frac{1}{4}\right) + \frac{1}{256}u'\left(\frac{1}{2}\right) + \frac{1}{64}u'\left(\frac{3}{4}\right), \\
u(1) = \frac{1}{8}u\left(\frac{1}{4}\right) - \frac{1}{64}u\left(\frac{1}{2}\right),
\end{gathered} \tag{4.1}$$

where $a_1 = 1/128$, $a_2 = 1/256$, $a_3 = 1/64$, $b_1 = 1/8$, $b_2 = 1/64$, $b_3 = 0$, $\xi_1 = 1/4$, $\xi_2 = 1,/2$, $\xi_3 = 3/4$:

$$\begin{gathered}
\phi(u) = \begin{cases} -u^2, & u \le 0, \\ u^2, & u > 0, \end{cases} \\
f(t,u) = \begin{cases} \dfrac{1}{5}(1+t)\left(u(t) - \dfrac{\varphi(t)}{2}\right)^{30}, & (t,u) \in [0,1] \times (0,2], \\ \dfrac{1}{5}(1+t)\left(2 - \dfrac{\varphi(t)}{2}\right)^{30}, & (t,u) \in [0,1] \times (2,+\infty). \end{cases}
\end{gathered} \tag{4.2}$$

It is easy to check that $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous. It follows from a direct calculation that

$$m = \left\{ \frac{\left(1 + \sum_{i=k+1}^{s} b_i + \sum_{i=s+1}^{m-2} b_i\right)\phi^{-1}\left((l+1)\int_0^1 a(s)\mathrm{d}s\right)}{1 - \sum_{i=1}^{k} b_i + \sum_{i=k+1}^{s} b_i} \right\}^{-1} = 0.96,$$

$$\gamma = \frac{\left(\sum_{i=1}^{k} b_i - \sum_{i=k+1}^{s} b_i\right)(1-\xi_k)}{1 - \sum_{i=1}^{k} b_i\xi_k + \sum_{i=k+1}^{s} b_i\xi_k} = \frac{21}{250}, \tag{4.3}$$

$$M = \left\{ \frac{\sum_{i=1}^{k} b_i - \sum_{i=k+1}^{s} b_i}{1 - \sum_{i=1}^{k} b_i + \sum_{i=k+1}^{s} b_i} \int_{\xi_k}^{1} \phi^{-1}\left(\int_0^s a(r)\mathrm{d}r\right)\mathrm{d}s \right\}^{-1} = 0.76.$$

Choose $\rho_1 = 1,\ \rho_2 = 250$, it is easy to check that $\rho_1 < \gamma\rho_2$ and

$$f(t,u) > 0, \quad t \in [0,1],\ u \in [\varphi(t),+\infty),$$

$$f_{\rho_1\varphi(t)}^{\rho_1} = \max\left\{ \max_{0\le t\le 1} \frac{(1/5)(1+t)(u(t)-\varphi(t)/2)^{30}}{1^2} \right\} = \frac{(1/5)(1+1)1^{30}}{1^2} = \frac{2}{5}$$

$$< \phi(m) = m^2 = 0.92, \quad t \in [0,1],\ u \in [\varphi(t)\rho_1, \rho_1],$$

$$f_{\gamma\rho_2}^{\rho_2} = \min\left\{ \min_{3/4\le t\le 1} \frac{(1/5)(1+t)(2-\varphi(t)/2)^{30}}{250^2} \right\} = \frac{(1/5)(1+3/4)(2-1/2)^{30}}{250^2} = 1.0742$$

$$> \phi(M\gamma) = (M\gamma)^2 = 0.004, \quad t \in \left[\frac{3}{4},1\right],\ u \in [\gamma\rho_2, \rho_2]. \tag{4.4}$$

It follows that f satisfies the condition (A_1) of Theorem 3.3, then problems (1.1) have at least two positive solutions.

Remark 4.2. Let $\varphi(u) = u$, the problem is second-order m-point boundary value problem.

Remark 4.3. Let $\phi_p(s) = |s|^{p-2}s$, $p > 1$, the problem is boundary value problem with p-Laplacian operators.

Hence our results generalize boundary value problem with p-Laplacian operators.

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References

[1] V. A. Il'in and E. I. Moiseev, "Nonlocal boundary value problem of the second kind for a Sturm-Liouville operator," *Differential Equations*, vol. 23, no. 8, pp. 979–987, 1987.

[2] V. A. Il'in and E. I. Moiseev, "Nonlocal boundary value problem of the first kind for a Sturm-Liouville operator in its differential and finite difference aspects," *Differential Equations*, vol. 23, no. 7, pp. 803–810, 1987.

[3] C. P. Gupta, "Solvability of a three-point nonlinear boundary value problem for a second order ordinary differential equation," *Journal of Mathematical Analysis and Applications*, vol. 168, no. 2, pp. 540–551, 1992.

[4] H. Su, B. Wang, Z. Wei, and X. Zhang, "Positive solutions of four-point boundary value problems for higher-order p-Laplacian operator," *Journal of Mathematical Analysis and Applications*, vol. 330, no. 2, pp. 836–851, 2007.

[5] D.-X. Ma, Z.-J. Du, and W.-G. Ge, "Existence and iteration of monotone positive solutions for multipoint boundary value problem with p-Laplacian operator," *Computers & Mathematics with Applications*, vol. 50, no. 5-6, pp. 729–739, 2005.

[6] R. Ma, "Positive solutions for a nonlinear m-point boundary value problem," *Computers & Mathematics with Applications*, vol. 42, no. 6-7, pp. 755–765, 2001.

[7] H. Su, Z. Wei, and F. Xu, "The existence of countably many positive solutions for a system of nonlinear singular boundary value problems with the p-Laplacian operator," *Journal of Mathematical Analysis and Applications*, vol. 325, no. 1, pp. 319–332, 2007.

[8] D. J. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, vol. 5 of *Notes and Reports in Mathematics in Science and Engineering*, Academic Press, Boston, Mass, USA, 1988.

[9] Y. Wang and C. Hou, "Existence of multiple positive solutions for one-dimensional p-Laplacian," *Journal of Mathematical Analysis and Applications*, vol. 315, no. 1, pp. 144–153, 2006.

[10] H. Su, Z. Wei, and F. Xu, "The existence of positive solutions for nonlinear singular boundary value system with p-Laplacian," *Applied Mathematics and Computation*, vol. 181, no. 2, pp. 826–836, 2006.

[11] H. Su, Z. Wei, and B. Wang, "The existence of positive solutions for a nonlinear four-point singular boundary value problem with a p-Laplacian operator," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 66, no. 10, pp. 2204–2217, 2007.

[12] H. Su, "Positive solutions for n-order m-point p-Laplacian operator singular boundary value problems," *Applied Mathematics and Computation*, vol. 199, no. 1, pp. 122–132, 2008.

[13] M. Moshinsky, "Sobre los problems de condiciones a la frontiera en una dimension de carac- teristicas discontinuas," *Boletín de la Sociedad Matemática Mexicana*, vol. 7, pp. 1–25, 1950.

[14] S. P. Timoshenko, *Theory of Elastic Stability*, McGraw-Hill, New York, NY, USA, 2nd edition, 1961.