Research Article

On p-Adic Analogue of q-Bernstein Polynomials and Related Integrals

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1. Introduction

Let \( C[0,1] \) denote the set of continuous functions on \([0,1]\). For \( 0 < q < 1 \) and \( f \in C[0,1] \), Kim introduced the \( q \)-extension of Bernstein linear operator of order \( n \) for \( f \) as follows:

\[
\mathbb{B}_{n,q}(f \mid x) = \sum_{k=0}^{n} f\left( \frac{k}{n} \right) \binom{n}{k} [x]_q^k [1-x]_q^{n-k} = \sum_{k=0}^{n} f\left( \frac{k}{n} \right) B_k^n(x,q),
\]

where \([x]_q = (1 - q^x)/(1 - q)\) (see [1]). Here \( \mathbb{B}_{n,q}(f \mid x) \) is called Kim’s \( q \)-Bernstein operator of order \( n \) for \( f \). For \( k, n \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\} \), \( B_k^n(x,q) = \left( \frac{x}{q} \right) \binom{n}{k} [x]_q^k [1-x]_q^{n-k} \) are called the Kim’s \( q \)-Bernstein polynomials of degree \( n \) (see [2–6]).
In [7], Carlitz defined a set of numbers \( \xi_k = \xi_k(q) \) inductively by

\[
\xi_0 = 1, \quad (q \xi + 1)^k - \xi_k = \begin{cases} 
1 & \text{if } k = 1, \\
0 & \text{if } k > 1,
\end{cases}
\] (1.2)

with the usual convention of replacing \( \xi^k \) by \( \xi_k \). These numbers are \( q \)-analogues of ordinary Bernoulli numbers \( B_k \), but they do not remain finite for \( q = 1 \). So he modified the definition as follows:

\[
\beta_{0,q} = 1, \quad q(q \beta + 1)^k - \beta_{k,q} = \begin{cases} 
1 & \text{if } k = 1, \\
0 & \text{if } k > 1,
\end{cases}
\] (1.3)

with the usual convention of replacing \( \beta^k \) by \( \beta_{k,q} \) (see [7]). These numbers \( \beta_{n,q} \) are called the \( n \)th Carlitz \( q \)-Bernoulli numbers. And Carlitz’ \( q \)-Bernoulli polynomials are defined by

\[
\beta_{k,q}(x) = (q^x \beta + [x]_q)^k = \sum_{i=0}^{k} \binom{k}{i} \beta_i q^i [x]_q^{k-i}.
\] (1.4)

As \( q \to 1 \), we have \( \beta_{k,q} \to B_k \) and \( \beta_{k,q}(x) \to B_k(x) \), where \( B_k \) and \( B_k(x) \) are the ordinary Bernoulli numbers and polynomials, respectively.

Let \( p \) be a fixed prime number. Throughout this paper, \( \mathbb{Z}, \mathbb{Q}, \mathbb{Z}_p, \mathbb{Q}_p, \) and \( \mathbb{C}_p \) will denote the ring of rational integers, the field of rational numbers, the ring of \( p \)-adic integers, the field of \( p \)-adic rational numbers and the completion of algebraic closure of \( \mathbb{Q}_p \), respectively. Let \( \nu_p \) be the normalized exponential valuation of \( \mathbb{C}_p \) such that \( |p|_p = p^{-\nu_p(p)} = 1/p \).

Let \( q \) be regarded as either a complex number \( q \in \mathbb{C} \) or a \( p \)-adic number \( q \in \mathbb{C}_p \). If \( q \in \mathbb{C} \), we assume \( |q| < 1 \), and if \( q \in \mathbb{C}_p \), we normally assume \( |1 - q|_p < 1 \).

We say that \( f \) is a uniformly differentiable function at a point \( a \in \mathbb{Z}_p \) and denote this property by \( f \in \text{UD}(\mathbb{Z}_p) \) if the difference quotient \( F_f(x, y) = (f(x) - f(y))/(x - y) \) has a limit \( f'(a) \) as \((x, y) \to (a, a)\) (see [1, 3, 8–13]).

For \( f \in \text{UD}(\mathbb{Z}_p) \), let us begin with the expression

\[
\frac{1}{[p^N]_q} \sum_{0 \leq x < p^N} q^x f(x) = \sum_{0 \leq x < p^N} f(x) \mu_q \left( x + p^N \mathbb{Z}_p \right),
\] (1.5)

representing a \( q \)-analogue of the Riemann sums for \( f \) (see [11]). The integral of \( f \) on \( \mathbb{Z}_p \) is defined as the limit as \( N \to \infty \) of the sums (if exists). The \( p \)-adic \( q \)-integral on a function \( f \in \text{UD}(\mathbb{Z}_p) \) is defined by

\[
I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N \to \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x) q^x,
\] (1.6)

(see [11]).
As was shown in [3], Carlitz’s $q$-Bernoulli numbers can be represented by $p$-adic $q$-integral on $\mathbb{Z}_p$ as follows:

$$\int_{\mathbb{Z}_p} [x]^m_q d\mu_q(x) = \beta_{m,q}, \quad \text{for} \ m \in \mathbb{Z}_+. \quad (1.7)$$

Also, Carlitz’s $q$-Bernoulli polynomials $\beta_{m,q}(x)$ can be represented

$$\beta_{m,q}(x) = \int_{\mathbb{Z}_p} [x+y]^m_q d\mu_q(y), \quad \text{for} \ m \in \mathbb{Z}_+, \quad (1.8)$$

(see [3]).

In this paper, we consider the $p$-adic analogue of Kim’s $q$-Bernstein polynomials on $\mathbb{Z}_p$ and give some properties of the several type Kim’s $q$-Bernoulli polynomials to represent the $p$-adic $q$-integral on $\mathbb{Z}_p$ of these polynomials. Finally, we derive some relations on the $p$-adic $q$-integral of the products of several type Kim’s $q$-Bernoulli polynomials and the powers of them on $\mathbb{Z}_p$.

## 2. $q$-Bernstein Polynomials Associated with $p$-Adic $q$-Integral on $\mathbb{Z}_p$

In this section, we assume that $q \in \mathbb{C}_p$ with $|1-q|_p < 1$.

From (1.5), (1.7) and (1.8), we note that

$$\int_{\mathbb{Z}_p} [1-x+x_1]^n_1 d\mu_{1/q}(x_1) = \frac{q^n}{(q-1)^{n-1}} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^l \frac{l+1}{q^{l+1}-1}, \quad (2.1)$$

$$\int_{\mathbb{Z}_p} [x+x_1]^n_1 d\mu_q(x_1) = \frac{1}{(1-q)^{n-1}} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^l \frac{l+1}{1-q^{l+1}}. \quad$$

By (2.1), we get

$$(-1)^n q^n \int_{\mathbb{Z}_p} [x+x_1]^n_1 d\mu_q(x_1) = \int_{\mathbb{Z}_p} [1-x+x_1]^n_1 d\mu_{1/q}(x_1). \quad (2.2)$$

Therefore, we obtain the following theorem.

**Theorem 2.1.** For $n \in \mathbb{Z}_+$, one has

$$\int_{\mathbb{Z}_p} [1-x+x_1]^n_1 d\mu_{1/q}(x_1) = (-1)^n q^n \int_{\mathbb{Z}_p} [x+x_1]^n_1 d\mu_q(x_1). \quad (2.3)$$
By the definition of Carlitz's $q$-Bernoulli numbers and polynomials, we get
\[ q^2 \beta_{n,q}(2) - (n + 1)q^2 + q = q(q\beta + 1)^n = \beta_{n,q} \quad \text{if } n > 1. \] (2.4)

Thus, we have the following proposition.

**Proposition 2.2.** For $n \in \mathbb{N}$ with $n > 1$, one has
\[ \beta_{n,q}(2) = \frac{1}{q^2} \beta_{n,q} + n + 1 - \frac{1}{q}. \] (2.5)

It is easy to show that
\[ (1 - x)^n = \left(1 - [x]_q\right)^n = (-1)^n q^n [x - 1]_q^n. \] (2.6)

Hence, we have
\[ \int_{\mathbb{Z}_p} [1 - x]^{n/4} d\mu_q(x) = (-1)^n q^n \int_{\mathbb{Z}_p} [x - 1]^{n/4} d\mu_q(x). \] (2.7)

By (1.8), we get
\[ \int_{\mathbb{Z}_p} [1 - x]^{n/4} d\mu_q(x) = (-1)^n q^n \beta_{n,q}(-1). \] (2.8)

By Theorem 2.1 and (2.8), we see that
\[ \int_{\mathbb{Z}_p} [1 - x]^{n/4} d\mu_q(x) = (-1)^n q^n \beta_{n,q}(-1) = \beta_{n,1/4}(2). \] (2.9)

From (2.9) and Proposition 2.2, we have
\[ \int_{\mathbb{Z}_p} [1 - x]^{n/4} d\mu_q(x) = \beta_{n,1/4}(2) = q^2 \beta_{n,1/4} + n + 1 - q. \] (2.10)

By (1.7) and (2.10), we obtain the following theorem.

**Theorem 2.3.** For $n \in \mathbb{N}$ with $n > 1$, one has
\[ \int_{\mathbb{Z}_p} [1 - x]^{n/4} d\mu_q(x) + q^2 \int_{\mathbb{Z}_p} [x]^{n/4} d\mu_{1/4}(x) + n + 1 - q. \] (2.11)
Taking the $p$-adic $q$-integral on $\mathbb{Z}_p$ for one Kim’s $q$-Bernstein polynomials, we get

$$
\int_{\mathbb{Z}_p} B_{k,n}(x, q) d\mu_q(x) = \binom{n}{k} \int_{\mathbb{Z}_p} [x]_q^k [1 - x]_q^{n-k} d\mu_q(x) = \binom{n}{k} \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l \int_{\mathbb{Z}_p} [x]_q^{k+l} d\mu_q(x) (2.12)
$$

and, by the $q$-symmetric property of $B_{k,n}(x, q)$, we see that

$$
\int_{\mathbb{Z}_p} B_{k,n}(x, q) d\mu_q(x) = \int_{\mathbb{Z}_p} B_{n-k,n}(1 - x, q) d\mu_q(x) \quad (2.13)
$$

For $n > k + 1$, by Theorem 2.3 and (2.13), one has

$$
\int_{\mathbb{Z}_p} B_{k,n}(x, q) d\mu_q(x) = \binom{n}{k} \sum_{l=0}^{k} (-1)^{k+l} \binom{k}{l} \left[ n - l + 1 - q + q^2 \int_{\mathbb{Z}_p} [x]^{n-l}_q d\mu_q(x) \right] \quad (2.14)
$$

Let $m, n, k \in \mathbb{Z}_p$ with $m + n > 2k + 1$. Then the $p$-adic $q$-integral for the multiplication of two Kim’s $q$-Bernstein polynomials on $\mathbb{Z}_p$ can be given by the following relation:

$$
\int_{\mathbb{Z}_p} B_{k,n}(x, q) B_{k,m}(x, q) d\mu_q(x) = \binom{n}{k} \binom{m}{k} \int_{\mathbb{Z}_p} [x]_q^{2k} [1 - x]_q^{n+m-2k} d\mu_q(x) \quad (2.15)
$$
By Theorem 2.3 and (2.15), we get

\[
\int_{Z_p} B_{k,n}(x, q) B_{k,m}(x, q) d\mu_q(x) \\
= \binom{n}{k} \binom{m}{k} \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{l+2k} \left[ n + m - l + 1 - q + q^2 \int_{Z_p} [x]^{n+m-l}_{1/q} d\mu_1/q(x) \right] \\
= \binom{n}{k} \binom{m}{k} \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{l+2k} \left[ n + m - l + 1 - q + q^2 \beta_{n+m-l,1/q} \right].
\]  

(2.16)

By the simple calculation, we easily get

\[
\int_{Z_p} B_{k,n}(x, q) B_{k,m}(x, q) d\mu_q(x) = \binom{n}{k} \binom{m}{k} \int_{Z_p} [x]_q^{2k} [1 - x]^{n+m-2k}_{1/q} d\mu_q(x) \\
= \binom{n}{k} \binom{m}{k} \sum_{l=0}^{n+m-2k} \binom{n+m-2k}{l} (-1)^{l} \int_{Z_p} [x]_q^{l+2k} d\mu_q(x) \\
= \binom{n}{k} \binom{m}{k} \sum_{l=0}^{n+m-2k} \binom{n+m-2k}{l} (-1)^{l} \beta_{l+2k,q}.
\]  

(2.17)

Continuing this process, we obtain

\[
\int_{Z_p} \left( \prod_{i=1}^{s} B_{k,n_i}(x, q) \right) d\mu_q(x) = \left( \prod_{i=1}^{s} \binom{n_i}{k} \right) \int_{Z_p} [x]_q^{sk} [1 - x]^{n_1+\ldots+n_s-sk}_{1/q} d\mu_q(x) \\
= \left( \prod_{i=1}^{s} \binom{n_i}{k} \right) \sum_{l=0}^{sk} \binom{sk}{l} (-1)^{sk+l} \int_{Z_p} [x]_q^{l} [1 - x]^{n_1+\ldots+n_s-l}_{1/q} d\mu_q(x).
\]  

(2.18)

Let \( s \in \mathbb{N} \) and \( n_1, \ldots, n_s, k \in \mathbb{Z}_+ \) with \( n_1 + n_2 + \ldots + n_s > sk + 1 \). By Theorem 2.3 and (2.18), we get

\[
\int_{Z_p} \left( \prod_{i=1}^{s} B_{k,n_i}(x, q) \right) d\mu_q(x) \\
= \left( \prod_{i=1}^{s} \binom{n_i}{k} \right) \sum_{l=0}^{sk} \binom{sk}{l} (-1)^{sk+l} \left\{ \sum_{i=1}^{s} n_i - l + 1 - q + q^2 \int_{Z_p} [x]_q^{n_i+\ldots+n_s-l}_{1/q} d\mu_1/q(x) \right\} \\
= \left( \prod_{i=1}^{s} \binom{n_i}{k} \right) \sum_{l=0}^{sk} \binom{sk}{l} (-1)^{sk+l} \left\{ \sum_{i=1}^{s} n_i - l + 1 - q + q^2 \beta_{n_i+\ldots+n_s-l,1/q} \right\}.
\]  

(2.19)
From the definition of binomial coefficient, we note that

\[
\int_{\mathbb{Z}_p} \left( \prod_{i=1}^{s} B_{k,n_i}(x, q) \right) d\mu_q(x)
= \left( \prod_{i=1}^{s} \binom{n_i}{k} \right) \sum_{l=0}^{\infty} \binom{sk}{l} (-1)^{sk+l} \left\{ \sum_{i=1}^{s} n_i - l + 1 - q + q^2 \beta_{n_1+\cdots+n_s-l,1/q} \right\}
\]

(2.20)

where \( s \in \mathbb{N} \) and \( n_1, \ldots, n_s, k \in \mathbb{Z}_+ \).

By (2.19) and (2.20), we obtain the following theorem.

**Theorem 2.4.** (I) For \( s \in \mathbb{N} \) and \( n_1, \ldots, n_s, k \in \mathbb{N} \) with \( n_1 + n_2 + \cdots + n_s > sk + 1 \), one has

\[
\int_{\mathbb{Z}_p} \left( \prod_{i=1}^{s} B_{k,n_i}(x, q) \right) d\mu_q(x)
= \left( \prod_{i=1}^{s} \binom{n_i}{k} \right) \sum_{l=0}^{\infty} \binom{sk}{l} (-1)^{sk+l} \left\{ \sum_{i=1}^{s} n_i - l + 1 - q + q^2 \beta_{n_1+\cdots+n_s-l,1/q} \right\}
\]

(2.21)

(II) For \( s \in \mathbb{N} \) and \( n_1, \ldots, n_s, k \in \mathbb{Z}_+ \), one has

\[
\int_{\mathbb{Z}_p} \left( \prod_{i=1}^{s} B_{k,n_i}(x, q) \right) d\mu_q(x) = \left( \prod_{i=1}^{s} \binom{n_i}{k} \right) \sum_{l=0}^{\infty} \binom{sk}{l} (-1)^{sk+l} \left\{ \sum_{i=1}^{s} n_i - l + 1 - q + q^2 \beta_{n_1+\cdots+n_s-l,1/q} \right\}
\]

(2.22)

By Theorem 2.4, we obtain the following corollary.

**Corollary 2.5.** For \( s \in \mathbb{N} \) and \( n_1, \ldots, n_s, k \in \mathbb{N} \) with \( n_1 + n_2 + \cdots + n_s > sk + 1 \), one has

\[
\sum_{l=0}^{sk} \binom{sk}{l} (-1)^{sk+l} \left\{ \sum_{i=1}^{s} n_i - l + 1 - q + q^2 \beta_{n_1+\cdots+n_s-l,1/q} \right\}
= \sum_{l=0}^{n_1+\cdots+n_s-sk} \binom{n_1 + \cdots + n_s - sk}{l} (-1)^l \beta_{sk+l,q}.
\]

(2.23)
Let \( s \in \mathbb{N} \) and \( m_1, \ldots, m_s, n_1, \ldots, n_s, k \in \mathbb{Z}_+ \) with \( m_1 n_1 + \cdots + m_s n_s > (m_1 + \cdots + m_s) k + 1 \). Then one has
\[
\int_{\mathbb{Z}_p} \left( \prod_{i=1}^s \beta_{k,n_i}(x,q) \right) d\mu_q(x) = \left( \prod_{i=1}^s \binom{n_i}{k} \right) \sum_{l=0}^{k \sum_{i=1}^s m_i} \left( k \sum_{i=1}^s m_i \right) (-1)^{k \sum_{i=1}^s m_i - l} \times \int_{\mathbb{Z}_p} [1 - x]^{\sum_{i=1}^s n_i m_i - l} d\mu_q(x)
\]
\[
= \left( \prod_{i=1}^s \binom{n_i}{k} \right) \sum_{l=0}^{k \sum_{i=1}^s m_i} \left( k \sum_{i=1}^s m_i \right) (-1)^{k \sum_{i=1}^s m_i - l} \times \left\{ \left( \sum_{i=1}^s m_i n_i - l + 1 \right) - q + q^2 \int_{\mathbb{Z}_p} [x]^{\sum_{i=1}^s n_i m_i - l} d\mu_q(x) \right\}
\]
\[
= \left( \prod_{i=1}^s \binom{n_i}{k} \right) \sum_{l=0}^{k \sum_{i=1}^s m_i} \left( k \sum_{i=1}^s m_i \right) (-1)^{k \sum_{i=1}^s m_i - l} \times \left\{ \left( \sum_{i=1}^s m_i n_i - l + 1 \right) - q + q^2 \beta_{n_1,m_1,\ldots,n_s,m_s,1/q} \right\}.
\]
(2.24)

From the definition of binomial coefficient, one has
\[
\int_{\mathbb{Z}_p} \left( \prod_{i=1}^s \beta_{k,n_i}(x,q) \right) d\mu_q(x)
\]
\[
= \left( \prod_{i=1}^s \binom{n_i}{k} \right) \sum_{l=0}^{k \sum_{i=1}^s m_i - k \sum_{i=1}^s m_i} \left( \sum_{i=1}^s n_i m_i - k \sum_{i=1}^s m_i \right) (-1)^l \times \int_{\mathbb{Z}_p} [x]^{(m_1+n_1+\cdots+m_s)k+l} d\mu_q(x)
\]
\[
= \left( \prod_{i=1}^s \binom{n_i}{k} \right) \sum_{l=0}^{k \sum_{i=1}^s m_i - k \sum_{i=1}^s m_i} \left( \sum_{i=1}^s n_i m_i - k \sum_{i=1}^s m_i \right) \times (-1)^l \beta_{(m_1+n_1+\cdots+m_s)k+l,q}.
\]
(2.25)

By (2.24) and (2.25), we obtain the following theorem.
Theorem 2.6. For \( s \in \mathbb{N} \) and \( m_1, \ldots, m_s, n_1, \ldots, n_s, k \in \mathbb{Z}_s \) with \( m_1 n_1 + \cdots + m_s n_s > (m_1 + \cdots + m_s)k + 1 \), one has

\[
\sum_{l=0}^{k} \left( \sum_{i=1}^{s} m_i \right) \left( \sum_{l=1}^{s} m_i - l \right)^{-\sum_{i=1}^{s} n_i m_i - k \sum_{i=1}^{s} m_i} \left( \frac{s}{l} \right) \beta_{n_1 m_1 + \cdots + n_s m_s, l, 1/q} \right) \xrightarrow{q \to 1} \sum_{i=1}^{s} \sum_{l=0}^{s-1} m_i \left( \sum_{i=1}^{s} n_i m_i - k \sum_{i=1}^{s} m_i \right) \left( -1 \right)^{l} \beta_{(m_1 + \cdots + m_s)k + l, 1/q}.
\]

(2.26)

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References
