Research Article

Convergence Theorems on a New Iteration Process for Two Asymptotically Nonexpansive Nonself-Mappings with Errors in Banach Spaces

Murat Ozdemir, Sezgin Akbulut, and Hukmi Kiziltunc

Department of Mathematics, Faculty of Science, Ataturk University, Erzurum 25240, Turkey

Correspondence should be addressed to Murat Ozdemir, mozdemir@atauni.edu.tr

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We introduce a new two-step iterative scheme for two asymptotically nonexpansive nonself-mappings in a uniformly convex Banach space. Weak and strong convergence theorems are established for this iterative scheme in a uniformly convex Banach space. The results presented extend and improve the corresponding results of Chidume et al. (2003), Wang (2006), Shahzad (2005), and Thianwan (2008).

1. Introduction

Let E be a real normed space and K be a nonempty subset of E . A mapping $T : K \rightarrow K$ is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in K$. A mapping $T : K \rightarrow K$ is called asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \rightarrow 1$ such that $\|T^n x - T^n y\| \leq k_n \|x - y\|$ for all $x, y \in K$ and $n \geq 1$. T is called uniformly L -Lipschitzian if there exists a real number $L > 0$ such that $\|T^n x - T^n y\| \leq L \|x - y\|$ for all $x, y \in K$ and $n \geq 1$. It is easy to see that if T is an asymptotically nonexpansive, then it is uniformly L -Lipschitzian with the uniform Lipschitz constant $L = \sup\{k_n : n \geq 1\}$.

Iterative techniques for nonexpansive and asymptotically nonexpansive mappings in Banach spaces including Mann type and Ishikawa type iteration processes have been studied extensively by various authors; see [1–8]. However, if the domain of T , $D(T)$, is a proper subset of E (and this is the case in several applications), and T maps $D(T)$ into E , then the iteration processes of Mann type and Ishikawa type studied by the authors mentioned above, and their modifications introduced may fail to be well defined.

A subset K of E is said to be a retract of E if there exists a continuous map $P : E \rightarrow K$ such that $Px = x$, for all $x \in K$. Every closed convex subset of a uniformly convex Banach

space is a retract. A map $P : E \rightarrow K$ is said to be a retraction if $P^2 = P$. It follows that if a map P is a retraction, then $Py = y$ for all $y \in R(P)$, the range of P .

The concept of asymptotically nonexpansive nonself-mappings was firstly introduced by Chidume et al. [4] as the generalization of asymptotically nonexpansive self-mappings. The asymptotically nonexpansive nonself-mapping is defined as follows.

Definition 1.1 (see [4]). Let K be a nonempty subset of real normed linear space E . Let $P : E \rightarrow K$ be the nonexpansive retraction of E onto K . A nonself mapping $T : K \rightarrow E$ is called asymptotically nonexpansive if there exists sequence $\{k_n\} \subset [1, \infty)$, $k_n \rightarrow 1$ ($n \rightarrow \infty$) such that

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq k_n\|x - y\| \quad (1.1)$$

for all $x, y \in K$ and $n \geq 1$. T is said to be uniformly L -Lipschitzian if there exists a constant $L > 0$ such that

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq L\|x - y\| \quad (1.2)$$

for all $x, y \in K$ and $n \geq 1$.

In [4], they study the following iterative sequence:

$$x_{n+1} = P\left((1 - \alpha_n)x_n + \alpha_n T(PT)^{n-1}x_n\right), \quad x_1 \in K, \quad n \geq 1 \quad (1.3)$$

to approximate some fixed point of T under suitable conditions. In [9], Wang generalized the iteration process (1.3) as follows:

$$\begin{aligned} x_{n+1} &= P\left((1 - \alpha_n)x_n + \alpha_n T_1(PT_1)^{n-1}y_n\right), \\ y_n &= P\left((1 - \alpha'_n)x_n + \alpha'_n T_2(PT_2)^{n-1}x_n\right), \quad x_1 \in K, \quad n \geq 1, \end{aligned} \quad (1.4)$$

where $T_1, T_2 : K \rightarrow E$ are asymptotically nonexpansive nonself-mappings and $\{\alpha_n\}, \{\alpha'_n\}$ are sequences in $[0, 1]$. He studied the strong and weak convergence of the iterative scheme (1.4) under proper conditions. Meanwhile, the results of [9] generalized the results of [4].

In [10], Shahzad studied the following iterative sequence:

$$x_{n+1} = P\left((1 - \alpha_n)x_n + \alpha_n TP\left[(1 - \beta_n)x_n + \beta_n Tx_n\right]\right), \quad x_1 \in K, \quad n \geq 1, \quad (1.5)$$

where $T : K \rightarrow E$ is a nonexpansive nonself-mapping and K is a nonempty closed convex nonexpansive retract of a real uniformly convex Banach space E with P , nonexpansive retraction.

Recently, Thianwan [11] generalized the iteration process (1.5) as follows:

$$\begin{aligned} x_{n+1} &= P((1 - \alpha_n - \gamma_n)x_n + \alpha_n TP((1 - \beta_n)y_n + \beta_n Ty_n) + \gamma_n u_n), \\ y_n &= P((1 - \alpha'_n - \gamma'_n)x_n + \alpha'_n TP((1 - \beta'_n)x_n + \beta'_n Tx_n) + \gamma'_n v_n), \quad x_1 \in K, \quad n \geq 1, \end{aligned} \quad (1.6)$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha'_n\}$, $\{\beta'_n\}$, $\{\gamma'_n\}$ are appropriate sequences in $[0, 1]$ and $\{u_n\}$, $\{v_n\}$ are bounded sequences in K . He proved weak and strong convergence theorems for nonexpansive nonself-mappings in uniformly convex Banach spaces.

The purpose of this paper, motivated by the Wang [9], Thianwan [11] and some others, is to construct an iterative scheme for approximating a fixed point of asymptotically nonexpansive nonself-mappings (provided that such a fixed point exists) and to prove some strong and weak convergence theorems for such maps.

Let E be a normed space, K a nonempty convex subset of E , $P : E \rightarrow K$ the nonexpansive retraction of E onto K , and $T_1, T_2 : K \rightarrow E$ be two asymptotically nonexpansive nonself-mappings. Then, for given $x_1 \in K$ and $n \geq 1$, we define the sequence $\{x_n\}$ by the iterative scheme:

$$\begin{aligned} x_{n+1} &= P((1 - \alpha_n - \gamma_n)x_n + \alpha_n T_1(PT_1)^{n-1}P((1 - \beta_n)y_n + \beta_n T_1(PT_1)^{n-1}y_n) + \gamma_n u_n), \\ y_n &= P((1 - \alpha'_n - \gamma'_n)x_n + \alpha'_n T_2(PT_2)^{n-1}P((1 - \beta'_n)x_n + \beta'_n T_2(PT_2)^{n-1}x_n) + \gamma'_n v_n), \end{aligned} \quad (1.7)$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha'_n\}$, $\{\beta'_n\}$, $\{\gamma'_n\}$ are appropriate sequences in $[0, 1]$ satisfying $\alpha_n + \beta_n + \gamma_n = 1 = \alpha'_n + \beta'_n + \gamma'_n$ and $\{u_n\}$, $\{v_n\}$ are bounded sequences in K . Clearly, the iterative scheme (1.7) is generalized by the iterative schemes (1.4) and (1.6).

Now, we recall the well-known concepts and results.

Let E be a Banach space with dimension $E \geq 2$. The modulus of E is the function $\delta_E : (0, 2] \rightarrow [0, 1]$ defined by

$$\delta_E(\varepsilon) = \inf \left\{ 1 - \left\| \frac{1}{2}(x + y) \right\| : \|x\| = \|y\| = 1, \quad \varepsilon = \|x - y\| \right\}. \quad (1.8)$$

A Banach space E is uniformly convex if and only if $\delta_E(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$.

A Banach space E is said to satisfy Opial's condition [12] if for any sequence $\{x_n\}$ in E , $x_n \rightharpoonup x$ implies that

$$\limsup_{n \rightarrow \infty} \|x_n - x\| < \limsup_{n \rightarrow \infty} \|x_n - y\| \quad (1.9)$$

for all $y \in E$ with $y \neq x$, where $x_n \rightharpoonup x$ denotes that $\{x_n\}$ converges weakly to x .

The mapping $T : K \rightarrow E$ with $F(T) \neq \emptyset$ is said to satisfy condition (A) [13] if there is a nondecreasing function $f : [0, \infty) \rightarrow [0, \infty)$ with $f(0) = 0$, $f(t) > 0$ for all $t \in (0, \infty)$ such that

$$\|x - Tx\| \geq f(d(x, F(T))) \quad (1.10)$$

for all $x \in K$, where $d(x, F(T)) = \inf\{\|x - p\| : p \in F(T)\}$; (see [13, page 337]) for an example of nonexpansive mappings satisfying condition (A).

Two mappings $T_1, T_2 : K \rightarrow E$ are said to satisfy condition (A') [14] if there is a nondecreasing function $f : [0, \infty) \rightarrow [0, \infty)$ with $f(0) = 0$, $f(t) > 0$ for all $t \in (0, \infty)$ such that

$$\frac{1}{2}(\|x - T_1x\| + \|x - T_2x\|) \geq f(d(x, F(T))) \quad (1.11)$$

for all $x \in K$, where $d(x, F(T)) = \inf\{\|x - p\| : p \in F(T) = F(T_1) \cap F(T_2)\}$.

Note that condition (A') reduces to condition (A) when $T_1 = T_2$ and hence is more general than the demicompactness of T_1 and T_2 [13]. A mapping $T : K \rightarrow K$ is called: (1) demicompact if any bounded sequence $\{x_n\}$ in K such that $\{x_n - Tx_n\}$ converges has a convergent subsequence, (2) semicompact (or hemicompact) if any bounded sequence $\{x_n\}$ in K such that $\{x_n - Tx_n\} \rightarrow 0$ as $n \rightarrow \infty$ has a convergent subsequence. Every demicompact mapping is semicompact but the converse is not true in general.

Senter and Dotson [13] have approximated fixed points of a nonexpansive mapping T by Mann iterates, whereas Maiti and Ghosh [14] and Tan and Xu [5] have approximated the fixed points using Ishikawa iterates under the condition (A) of Senter and Dotson [13]. Tan and Xu [5] pointed out that condition (A) is weaker than the compactness of K . Khan and Takahashi [6] have studied the two mappings case for asymptotically nonexpansive mappings under the assumption that the domain of the mappings is compact. We shall use condition (A') instead of compactness of K to study the strong convergence of $\{x_n\}$ defined in (1.7).

In the sequel, we need the following usefull known lemmas to prove our main results.

Lemma 1.2 (see [5]). *Let $\{a_n\}$, $\{b_n\}$, and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1 + \delta_n)a_n + b_n, \quad n \geq 1. \quad (1.12)$$

If $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} \delta_n < \infty$, then

(i) $\lim_{n \rightarrow \infty} a_n$ exists;

(ii) *In particular, if $\{a_n\}$ has a subsequence which converges strongly to zero, then $\lim_{n \rightarrow \infty} a_n = 0$.*

Lemma 1.3 (see [2]). *Suppose that E is a uniformly convex Banach space and $0 < p \leq t_n \leq q < 1$ for all $n \geq 1$. Suppose further that $\{x_n\}$ and $\{y_n\}$ are sequences of E such that*

$$\limsup_{n \rightarrow \infty} \|x_n\| \leq r, \quad \limsup_{n \rightarrow \infty} \|y_n\| \leq r, \quad \lim_{n \rightarrow \infty} \|t_n x_n + (1 - t_n) y_n\| = r \quad (1.13)$$

hold for some $r \geq 0$. Then $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$.

Lemma 1.4 (see [4]). *Let E be a uniformly convex Banach space, K a nonempty closed convex subset of E , and $T : K \rightarrow E$ be a nonexpansive mapping. Then, $(I - T)$ is demiclosed at zero, that is, if $x_n \rightharpoonup x$ weakly and $x_n - Tx_n \rightarrow 0$ strongly, then $x \in F(T)$, where $F(T)$ is the set fixed point of T .*

2. Main Results

We shall make use of the following lemmas.

Lemma 2.1. *Let E be a normed space and let K be a nonempty closed convex subset of E which is also a nonexpansive retract of E . Let $T_1, T_2 : K \rightarrow E$ be two asymptotically nonexpansive nonself-mappings of E with sequences $\{k_n\}, \{l_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty} (k_n - 1) < \infty$, $\sum_{n=1}^{\infty} (l_n - 1) < \infty$, respectively and $F(T_1) \cap F(T_2) := \{x \in K : T_1 x = T_2 x = x\} \neq \emptyset$. Suppose that $\{u_n\}, \{v_n\}$ are bounded sequences in K such that $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \gamma'_n < \infty$. Starting from an arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ by the recursion (1.7). Then, $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists for all $p \in F(T_1) \cap F(T_2)$.*

Proof. Let $p \in F(T_1) \cap F(T_2)$. Since $\{u_n\}$ and $\{v_n\}$ are bounded sequences in K , we have

$$r = \max \left\{ \sup_{n \geq 1} \|u_n - p\|, \sup_{n \geq 1} \|v_n - p\| \right\}. \quad (2.1)$$

Set $\sigma_n = (1 - \beta_n)y_n + \beta_n T_1 (PT_1)^{n-1} y_n$ and $\delta_n = (1 - \beta'_n)x_n + \beta'_n T_2 (PT_2)^{n-1} x_n$. Firstly, we note that

$$\begin{aligned} \|\sigma_n - p\| &= \|(1 - \beta_n)y_n + \beta_n T_1 (PT_1)^{n-1} y_n - p\| \\ &\leq \beta_n \|T_1 (PT_1)^{n-1} y_n - p\| + (1 - \beta_n) \|y_n - p\| \\ &\leq \beta_n k_n \|y_n - p\| + (1 - \beta_n) \|y_n - p\| \\ &\leq k_n \|y_n - p\|, \end{aligned} \quad (2.2)$$

$$\begin{aligned} \|\delta_n - p\| &= \|(1 - \beta'_n)x_n + \beta'_n T_2 (PT_2)^{n-1} x_n - p\| \\ &\leq \beta'_n \|T_2 (PT_2)^{n-1} x_n - p\| + (1 - \beta'_n) \|x_n - p\| \\ &\leq \beta'_n l_n \|x_n - p\| + (1 - \beta'_n) \|x_n - p\| \\ &\leq l_n \|x_n - p\|. \end{aligned} \quad (2.3)$$

From (1.7) and (2.3), we have

$$\begin{aligned} \|y_n - p\| &= \|P((1 - \alpha'_n - \gamma'_n)x_n + \alpha'_n T_2 (PT_2)^{n-1} P\delta_n + \gamma'_n v_n) - p\| \\ &\leq \|(1 - \alpha'_n - \gamma'_n)x_n + \alpha'_n T_2 (PT_2)^{n-1} P\delta_n + \gamma'_n v_n - p\| \\ &\leq \alpha'_n \|T_2 (PT_2)^{n-1} P\delta_n - p\| + (1 - \alpha'_n - \gamma'_n) \|x_n - p\| + \gamma'_n \|v_n - p\| \\ &\leq \alpha'_n l_n \|\delta_n - p\| + (1 - \alpha'_n - \gamma'_n) \|x_n - p\| + \gamma'_n \|v_n - p\| \\ &\leq \alpha'_n l_n^2 \|x_n - p\| + (1 - \alpha'_n - \gamma'_n) \|x_n - p\| + \gamma'_n r \\ &\leq l_n^2 \|x_n - p\| + \gamma'_n r. \end{aligned} \quad (2.4)$$

Substituting (2.4) into (2.2), we obtain

$$\|\sigma_n - p\| \leq k_n \|y_n - p\| \leq k_n l_n^2 \|x_n - p\| + k_n \gamma'_n r. \quad (2.5)$$

It follows from (1.7) and (2.5) that

$$\begin{aligned} \|x_{n+1} - p\| &= \left\| P \left((1 - \alpha_n - \gamma_n)x_n + \alpha_n T_1 (PT_1)^{n-1} P\sigma_n + \gamma_n u_n \right) - p \right\| \\ &\leq \left\| (1 - \alpha_n - \gamma_n)x_n + \alpha_n T_1 (PT_1)^{n-1} P\sigma_n + \gamma_n u_n - p \right\| \\ &\leq \alpha_n \left\| T_1 (PT_1)^{n-1} P\sigma_n - p \right\| + (1 - \alpha_n - \gamma_n) \|x_n - p\| + \gamma_n \|u_n - p\| \\ &\leq \alpha_n k_n \|\sigma_n - p\| + (1 - \alpha_n - \gamma_n) \|x_n - p\| + \gamma_n \|u_n - p\| \\ &\leq \alpha_n (k_n^2 l_n^2 \|x_n - p\| + k_n^2 \gamma'_n r) + (1 - \alpha_n - \gamma_n) \|x_n - p\| + \gamma_n r \\ &\leq k_n^2 l_n^2 \|x_n - p\| + k_n^2 \gamma'_n r + \gamma_n r \\ &= \left(1 + (l_n^2 - 1)(k_n^2 - 1) + (l_n^2 - 1) + (k_n^2 - 1) \right) \|x_n - p\| + (k_n^2 \gamma'_n + \gamma_n) r. \end{aligned} \quad (2.6)$$

Note that $\sum_{n=1}^{\infty} k_n - 1 < \infty$ and $\sum_{n=1}^{\infty} l_n - 1 < \infty$ are equivalent to $\sum_{n=1}^{\infty} k_n^2 - 1 < \infty$ and $\sum_{n=1}^{\infty} l_n^2 - 1 < \infty$, respectively. Since $\sum_{n=1}^{\infty} \gamma_n < \infty$ and $\sum_{n=1}^{\infty} \gamma'_n < \infty$, we have $\sum_{n=1}^{\infty} (k_n^2 \gamma'_n + \gamma_n) r < \infty$. We obtained from (2.6) and Lemma 1.2 that $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists for all $p \in F(T)$. This completes the proof. $\square$

Lemma 2.2. *Let E be a normed space and let K be a nonempty closed convex subset of E which is also a nonexpansive retract of E . Let $T_1, T_2 : K \rightarrow E$ be nonself uniformly L_1 -Lipschitzian, L_2 -Lipschitzian, respectively. Suppose that $\{u_n\}, \{v_n\}$ are bounded sequences in K such that $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \gamma'_n < \infty$. Starting from an arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ by the recursion (1.7) and set $C_n = \|x_n - T_1(PT_1)^{n-1}x_n\|$, $C'_n = \|x_n - T_2(PT_2)^{n-1}x_n\|$ for all $n \geq 1$. If $\lim_{n \rightarrow \infty} C_n = \lim_{n \rightarrow \infty} C'_n = 0$, then*

$$\lim_{n \rightarrow \infty} \|x_n - T_1 x_n\| = \lim_{n \rightarrow \infty} \|x_n - T_2 x_n\| = 0. \quad (2.7)$$

Proof. Since $\{u_n\}, \{v_n\}$ are bounded, it follows from Lemma 2.1 that $\{u_n - x_n\}$ and $\{v_n - x_n\}$ are all bounded. We set

$$\begin{aligned} r_1 &= \sup\{\|u_n - x_n\| : n \geq 1\}, & r_2 &= \sup\{\|v_n - x_n\| : n \geq 1\}, \\ r_3 &= \sup\{\|u_{n-1} - x_{n-1}\| : n \geq 1\}, & r &= \max\{r_i : i = 1, 2, 3\}. \end{aligned} \quad (2.8)$$

Let $\sigma_n = (1 - \beta_n)y_n + \beta_n T_1(PT_1)^{n-1}y_n$ and $\delta_n = (1 - \beta'_n)x_n + \beta'_n T_2(PT_2)^{n-1}x_n$. Then, we have

$$\begin{aligned}
 \|\sigma_n - x_n\| &= \left\| (1 - \beta_n)y_n + \beta_n T_1(PT_1)^{n-1}y_n - x_n \right\| \\
 &\leq \beta_n \left\| T_1(PT_1)^{n-1}y_n - T_1(PT_1)^{n-1}x_n \right\| \\
 &\quad + \beta_n \left\| T_1(PT_1)^{n-1}x_n - x_n \right\| + (1 - \beta_n) \|y_n - x_n\| \\
 &\leq (L_1 + 1) \|y_n - x_n\| + C_n,
 \end{aligned} \tag{2.9}$$

$$\begin{aligned}
 \|\delta_n - x_n\| &= \left\| (1 - \beta'_n)x_n + \beta'_n T_2(PT_2)^{n-1}x_n - x_n \right\| \\
 &\leq \beta'_n \left\| T_2(PT_2)^{n-1}x_n - x_n \right\| \\
 &\leq C'_n.
 \end{aligned} \tag{2.10}$$

We find the following from (1.7) and (2.10):

$$\begin{aligned}
 \|y_n - x_n\| &= \left\| P \left((1 - \alpha'_n - \gamma'_n)x_n + \alpha'_n T_2(PT_2)^{n-1}P\delta_n + \gamma'_n v_n \right) - x_n \right\| \\
 &\leq \left\| (1 - \alpha'_n - \gamma'_n)x_n + \alpha'_n T_2(PT_2)^{n-1}P\delta_n + \gamma'_n v_n - x_n \right\| \\
 &\leq \alpha'_n \left\| T_2(PT_2)^{n-1}P\delta_n - T_2(PT_2)^{n-1}x_n \right\| \\
 &\quad + \alpha'_n \left\| T_2(PT_2)^{n-1}x_n - x_n \right\| + \gamma'_n \|v_n - x_n\| \\
 &\leq L_2 \|\delta_n - x_n\| + C'_n + \gamma'_n r \\
 &\leq L_2 C'_n + C'_n + \gamma'_n r \\
 &= (L_2 + 1)C'_n + \gamma'_n r.
 \end{aligned} \tag{2.11}$$

Substituting (2.11) into (2.9), we get

$$\|\sigma_n - x_n\| \leq (L_1 + 1)(L_2 + 1)C'_n + (L_1 + 1)\gamma'_n r + C_n. \tag{2.12}$$

It follows from (1.7) and (2.12) that

$$\begin{aligned}
\|x_{n+1} - x_n\| &\leq \left\| P\left((1 - \alpha_n - \gamma_n)x_n + \alpha_n T_1(PT_1)^{n-1}P\sigma_n + \gamma_n u_n\right) - x_n \right\| \\
&\leq \left\| T_1(PT_1)^{n-1}P\sigma_n - x_n \right\| + \gamma_n \|u_n - x_n\| \\
&\leq \left\| T_1(PT_1)^{n-1}P\sigma_n - T_1(PT_1)^{n-1}x_n \right\| + \left\| T_1(PT_1)^{n-1}x_n - x_n \right\| + \gamma_n r \\
&\leq L_1 \|\sigma_n - x_n\| + C_n + \gamma_n r \\
&\leq L_1((L_1 + 1)(L_2 + 1)C'_n + (L_1 + 1)\gamma'_n r + C_n) + C_n + \gamma_n r \\
&= (L_1 + 1)C_n + L_1(L_1 + 1)(L_2 + 1)C'_n + L_1(L_1 + 1)\gamma'_n r + \gamma_n r.
\end{aligned} \tag{2.13}$$

Using (2.11) and (2.13), we obtain

$$\begin{aligned}
\|\sigma_{n-1} - x_n\| &= \left\| (1 - \beta_{n-1})y_{n-1} + \beta_{n-1}T_1(PT_1)^{n-2}y_{n-1} - x_n \right\| \\
&\leq \beta_{n-1} \left\| T_1(PT_1)^{n-2}y_{n-1} - T_1(PT_1)^{n-2}x_{n-1} \right\| + \beta_{n-1} \left\| T_1(PT_1)^{n-2}x_{n-1} - x_{n-1} \right\| \\
&\quad + \beta_{n-1} \|x_n - x_{n-1}\| + (1 - \beta_{n-1}) \|y_{n-1} - x_n\| \\
&\leq L_1 \|y_{n-1} - x_{n-1}\| + C_{n-1} + \|x_n - x_{n-1}\| \\
&\quad + \|y_{n-1} - x_{n-1}\| + \|x_n - x_{n-1}\| \\
&\leq (L_1 + 1)[(L_2 + 1)C'_{n-1} + \gamma'_{n-1}r] \\
&\quad + 2 \left[\begin{aligned} &(L_1 + 1)C_{n-1} + L_1(L_1 + 1)(L_2 + 1)C'_{n-1} \\ &+ L_1(L_1 + 1)\gamma'_{n-1}r + \gamma_{n-1}r \end{aligned} \right] + C_{n-1} \\
&= (2L_1 + 3)C_{n-1} + (2L_1 + 1)(L_1 + 1)(L_2 + 1)C'_{n-1} \\
&\quad + (2L_1 + 1)(L_1 + 1)\gamma'_{n-1}r + 2\gamma_{n-1}r.
\end{aligned} \tag{2.14}$$

Combine (2.13) with (2.14) yields that

$$\begin{aligned}
\|x_n - (PT_1)^{n-1}x_n\| &= \|x_n - T_1(PT_1)^{n-2}x_n\| \\
&\leq \|(1 - \alpha_{n-1} - \gamma_{n-1})x_{n-1} + \alpha_{n-1}T_1(PT_1)^{n-2}P\sigma_{n-1} + \gamma_{n-1}u_{n-1} - T_1(PT_1)^{n-2}x_n\| \\
&\leq \alpha_{n-1}\|T_1(PT_1)^{n-2}P\sigma_{n-1} - T_1(PT_1)^{n-2}x_n\| \\
&\quad + (1 - \alpha_{n-1})\|x_{n-1} - T_1(PT_1)^{n-2}x_n\| + \gamma_{n-1}\|u_{n-1} - x_{n-1}\| \\
&\leq \|T_1(PT_1)^{n-2}P\sigma_{n-1} - T_1(PT_1)^{n-2}x_n\| \\
&\quad + \|x_{n-1} - T_1(PT_1)^{n-2}x_n\| + \gamma_{n-1}r \\
&\leq L_1\|\sigma_{n-1} - x_n\| + \|x_{n-1} - T_1(PT_1)^{n-2}x_{n-1}\| \\
&\quad + \|T_1(PT_1)^{n-2}x_n - T_1(PT_1)^{n-2}x_{n-1}\| + \gamma_{n-1}r \\
&\leq L_1 \left[(2L_1 + 3)C_{n-1} + (2L_1 + 1)(L_1 + 1)(L_2 + 1)C'_{n-1} \right. \\
&\quad \left. + (2L_1 + 1)(L_1 + 1)\gamma'_{n-1}r + 2\gamma_{n-1}r \right] \\
&\quad + C_{n-1} + (L_1 + 1)C_{n-1} + L_1(L_1 + 1)(L_2 + 1)C'_{n-1} \\
&\quad + L_1(L_1 + 1)\gamma'_{n-1}r + 2\gamma_{n-1}r \\
&= 2(L_1 + 1)^2C_{n-1} + 2L_1(L_1 + 1)^2(L_2 + 1)C'_{n-1} \\
&\quad + 2L_1(L_1 + 1)^2\gamma'_{n-1}r + 2(L_1 + 1)\gamma_{n-1}r,
\end{aligned} \tag{2.15}$$

from which it follows that

$$\begin{aligned}
\|x_n - T_1x_n\| &= \|x_n - T_1(PT_1)^{n-1}x_n + T_1(PT_1)^{n-1}x_n - T_1x_n\| \\
&\leq \|x_n - T_1(PT_1)^{n-1}x_n\| + \|T_1(PT_1)^{n-1}x_n - T_1x_n\| \\
&\leq C_n + L_1\|(PT_1)^{n-1}x_n - x_n\| \\
&\leq C_n + 2L_1(L_1 + 1)^2C_{n-1} + 2L_1^2(L_1 + 1)^2(L_2 + 1)C'_{n-1} \\
&\quad + 2L_1^2(L_1 + 1)^2\gamma'_{n-1}r + 2L_1(L_1 + 1)\gamma_{n-1}r.
\end{aligned} \tag{2.16}$$

It follows from $\lim_{n \rightarrow \infty} C_n = \lim_{n \rightarrow \infty} C'_n = 0$ that $\lim_{n \rightarrow \infty} \|x_n - T_1 x_n\| = 0$. Similarly, we can show that $\lim_{n \rightarrow \infty} \|x_n - T_2 x_n\| = 0$. This completes the proof. $\square$

Lemma 2.3. *Let E be a real uniformly convex Banach space and let K be a nonempty closed convex subset of E which is also a nonexpansive retract of E . Let $T_1, T_2 : K \rightarrow E$ be two asymptotically nonexpansive nonself-mappings of E with sequences $\{k_n\}, \{l_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty} (k_n - 1) < \infty$, $\sum_{n=1}^{\infty} (l_n - 1) < \infty$, respectively, and $F(T_1) \cap F(T_2) \neq \emptyset$. Suppose that $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\alpha'_n\}, \{\beta'_n\}, \{\gamma'_n\}$ are appropriate sequences in $[0, 1]$ satisfying $\alpha_n + \beta_n + \gamma_n = 1 = \alpha'_n + \beta'_n + \gamma'_n$, and $\{u_n\}, \{v_n\}$ are bounded sequences in K such that $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \gamma'_n < \infty$. Moreover, $0 < a \leq \alpha_n, \alpha'_n, \beta_n, \beta'_n \leq b < 1$ for all $n \geq 1$ and some $a, b \in (0, 1)$. Starting from an arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ by the recursion (1.7). Then,*

$$\lim_{n \rightarrow \infty} \|x_n - T_1 x_n\| = \lim_{n \rightarrow \infty} \|x_n - T_2 x_n\| = 0. \quad (2.17)$$

Proof. Let $\sigma_n = (1 - \beta_n)y_n + \beta_n T_1 (PT_1)^{n-1} y_n$ and $\delta_n = (1 - \beta'_n)x_n + \beta'_n T_2 (PT_2)^{n-1} x_n$. By Lemma 2.1, we see that $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists. Assume that $\lim_{n \rightarrow \infty} \|x_n - p\| = c$. If $c = 0$, then by the continuity of T_1 and T_2 the conclusion follows. Now, suppose $c > 0$. Taking lim sup on both sides in the inequalities (2.2), (2.3), and (2.4), we have

$$\limsup_{n \rightarrow \infty} \|\sigma_n - p\| \leq c, \quad \limsup_{n \rightarrow \infty} \|\delta_n - p\| \leq c, \quad \limsup_{n \rightarrow \infty} \|y_n - p\| \leq c, \quad (2.18)$$

respectively. Next, we consider

$$\begin{aligned} \|T_1 (PT_1)^{n-1} P\sigma_n - p + \gamma_n(u_n - x_n)\| &\leq \|T_1 (PT_1)^{n-1} P\sigma_n - p\| + \gamma_n \|u_n - x_n\| \\ &\leq k_n \|\sigma_n - p\| + \gamma_n r. \end{aligned} \quad (2.19)$$

Taking lim sup on both sides in the above inequality and using (2.18), we get

$$\limsup_{n \rightarrow \infty} \|T_1 (PT_1)^{n-1} P\sigma_n - p + \gamma_n(u_n - x_n)\| \leq c. \quad (2.20)$$

Observe that

$$\|x_n - p + \gamma_n(u_n - x_n)\| \leq \|x_n - p\| + \gamma_n \|u_n - x_n\| \leq \|x_n - p\| + \gamma_n r, \quad (2.21)$$

which implies that

$$\limsup_{n \rightarrow \infty} \|x_n - p + \gamma_n(u_n - x_n)\| \leq c. \quad (2.22)$$

$\limsup_{n \rightarrow \infty} \|x_{n+1} - p\| = c$ means that

$$\liminf_{n \rightarrow \infty} \left\| \alpha_n \left(T_1 (PT_1)^{n-1} P\sigma_n - p + \gamma_n(u_n - x_n) \right) + (1 - \alpha_n)(x_n - p + \gamma_n(u_n - x_n)) \right\| \geq c. \quad (2.23)$$

On the other hand, by using (2.23) and (2.5), we have

$$\begin{aligned}
& \left\| \alpha_n \left(T_1 (PT_1)^{n-1} P\sigma_n - p + \gamma_n (u_n - x_n) \right) + (1 - \alpha_n) (x_n - p + \gamma_n (u_n - x_n)) \right\| \\
& \leq \alpha_n \left\| T_1 (PT_1)^{n-1} P\sigma_n - p \right\| + (1 - \alpha_n) \|x_n - p\| + \gamma_n \|u_n - x_n\| \\
& \leq \alpha_n k_n \|\sigma_n - p\| + (1 - \alpha_n) \|x_n - p\| + \gamma_n \|u_n - x_n\| \\
& \leq \alpha_n k_n \left(k_n l_n^2 \|x_n - p\| + k_n \gamma'_n r \right) + (1 - \alpha_n) \|x_n - p\| + \gamma_n r \\
& \leq k_n^2 l_n^2 \|x_n - p\| + k_n^2 \gamma'_n r + \gamma_n r.
\end{aligned} \tag{2.24}$$

Therefore, we have

$$\limsup_{n \rightarrow \infty} \left\| \alpha_n \left(T_1 (PT_1)^{n-1} P\sigma_n - p + \gamma_n (u_n - x_n) \right) + (1 - \alpha_n) (x_n - p + \gamma_n (u_n - x_n)) \right\| \leq c. \tag{2.25}$$

Combining (2.23) with (2.25), we obtain

$$\lim_{n \rightarrow \infty} \left\| \alpha_n \left(T_1 (PT_1)^{n-1} P\sigma_n - p + \gamma_n (u_n - x_n) \right) + (1 - \alpha_n) (x_n - p + \gamma_n (u_n - x_n)) \right\| = c. \tag{2.26}$$

Hence, applying Lemma 1.3, we find

$$\lim_{n \rightarrow \infty} \left\| T_1 (PT_1)^{n-1} P\sigma_n - x_n \right\| = 0. \tag{2.27}$$

Note that

$$\|x_n - p\| \leq \left\| T_1 (PT_1)^{n-1} P\sigma_n - p \right\| + \left\| T_1 (PT_1)^{n-1} P\sigma_n - x_n \right\| \leq k_n \|\sigma_n - p\| \tag{2.28}$$

which yields that

$$c \leq \liminf_{n \rightarrow \infty} \|\sigma_n - p\| \leq \limsup_{n \rightarrow \infty} \|\sigma_n - p\| \leq c. \tag{2.29}$$

That is, $\lim_{n \rightarrow \infty} \|\sigma_n - p\| = c$. This implies that

$$\liminf_{n \rightarrow \infty} \left\| \beta_n \left(T_1 (PT_1)^{n-1} y_n - p \right) + (1 - \beta_n) (y_n - p) \right\| \geq c. \tag{2.30}$$

Similarly, we have

$$\begin{aligned} & \left\| \beta_n \left(T_1 (PT_1)^{n-1} y_n - p \right) + (1 - \beta_n) (y_n - p) \right\| \\ & \leq \beta_n \left\| T_1 (PT_1)^{n-1} y_n - p \right\| + (1 - \beta_n) \|y_n - p\| \leq k_n \|y_n - p\|, \end{aligned} \quad (2.31)$$

$$\limsup_{n \rightarrow \infty} \left\| \beta_n \left(T_1 (PT_1)^{n-1} y_n - p \right) + (1 - \beta_n) (y_n - p) \right\| \leq c. \quad (2.32)$$

Combining (2.30) with (2.32), we obtain

$$\lim_{n \rightarrow \infty} \left\| \beta_n \left(T_1 (PT_1)^{n-1} y_n - p \right) + (1 - \beta_n) (y_n - p) \right\| = c. \quad (2.33)$$

On the other hand, we have

$$\left\| T_1 (PT_1)^{n-1} y_n - p \right\| \leq k_n \|y_n - p\|, \quad (2.34)$$

$$\limsup_{n \rightarrow \infty} \left\| T_1 (PT_1)^{n-1} y_n - p \right\| \leq c. \quad (2.35)$$

Hence, using (2.32), (2.33), (2.35), and Lemma 1.3, we find

$$\lim_{n \rightarrow \infty} \left\| T_1 (PT_1)^{n-1} y_n - y_n \right\| = 0. \quad (2.36)$$

Note that from (2.36), we have

$$\begin{aligned} \|\sigma_n - p\| &= \left\| (1 - \beta_n) y_n + \beta_n T_1 (PT_1)^{n-1} y_n - p \right\| \\ &\leq (1 - \beta_n) \|y_n - p\| + \beta_n \left\| T_1 (PT_1)^{n-1} y_n - p \right\| \\ &\leq (1 - \beta_n) \|y_n - p\| + \beta_n \left\| T_1 (PT_1)^{n-1} y_n - y_n \right\| + \beta_n \|y_n - p\| \\ &= \|y_n - p\| \end{aligned} \quad (2.37)$$

which yields that

$$c \leq \liminf_{n \rightarrow \infty} \|y_n - p\| \leq \limsup_{n \rightarrow \infty} \|y_n - p\| \leq c. \quad (2.38)$$

That is, $\lim_{n \rightarrow \infty} \|y_n - p\| = c$.

Again, $\lim_{n \rightarrow \infty} \|y_n - p\| = c$ means that

$$\liminf_{n \rightarrow \infty} \left\| \alpha'_n \left(T_2 (PT_2)^{n-1} P\delta_n - p + \gamma'_n (v_n - x_n) \right) + (1 - \alpha'_n) (x_n - p + \gamma'_n (v_n - x_n)) \right\| \geq c. \quad (2.39)$$

By using (2.39) and (2.3), we obtain

$$\begin{aligned}
& \left\| \alpha'_n \left(T_2 (PT_2)^{n-1} P\delta_n - p + \gamma'_n (v_n - x_n) \right) + (1 - \alpha'_n) (x_n - p + \gamma'_n (v_n - x_n)) \right\| \\
& \leq \alpha'_n \left\| T_2 (PT_2)^{n-1} P\delta_n - p \right\| + (1 - \alpha'_n) \|x_n - p\| + \gamma'_n \|v_n - x_n\| \\
& \leq \alpha'_n l_n \|\delta_n - p\| + (1 - \alpha'_n) \|x_n - p\| + \gamma'_n \|v_n - x_n\| \\
& \leq \alpha'_n l_n^2 \|x_n - p\| + (1 - \alpha'_n) \|x_n - p\| + \gamma'_n r \\
& \leq l_n^2 \|x_n - p\| + \gamma'_n r.
\end{aligned} \tag{2.40}$$

Therefore, we have

$$\limsup_{n \rightarrow \infty} \left\| \alpha'_n \left(T_2 (PT_2)^{n-1} P\delta_n - p + \gamma'_n (v_n - x_n) \right) + (1 - \alpha'_n) (x_n - p + \gamma'_n (v_n - x_n)) \right\| \leq c. \tag{2.41}$$

Combining (2.39) with (2.41), we obtain

$$\lim_{n \rightarrow \infty} \left\| \alpha'_n \left(T_2 (PT_2)^{n-1} P\delta_n - p + \gamma'_n (v_n - x_n) \right) + (1 - \alpha'_n) (x_n - p + \gamma'_n (v_n - x_n)) \right\| = c. \tag{2.42}$$

On the other hand, we have

$$\begin{aligned}
\left\| T_2 (PT_2)^{n-1} P\delta_n - p + \gamma'_n (v_n - x_n) \right\| & \leq \left\| T_2 (PT_2)^{n-1} P\delta_n - p \right\| + \gamma'_n \|v_n - x_n\| \\
& \leq l_n \|\delta_n - p\| + \gamma'_n r
\end{aligned} \tag{2.43}$$

which implies that

$$\limsup_{n \rightarrow \infty} \left\| T_2 (PT_2)^{n-1} P\delta_n - p + \gamma'_n (v_n - x_n) \right\| \leq c. \tag{2.44}$$

Notice that

$$\|x_n - p + \gamma'_n (v_n - x_n)\| \leq \|x_n - p\| + \gamma'_n \|v_n - x_n\| \leq \|x_n - p\| + \gamma'_n r, \tag{2.45}$$

which implies that

$$\limsup_{n \rightarrow \infty} \|x_n - p + \gamma'_n (v_n - x_n)\| \leq c. \tag{2.46}$$

Using (2.42), (2.44), (2.46), and Lemma 1.3, we find

$$\lim_{n \rightarrow \infty} \left\| T_2 (PT_2)^{n-1} P\delta_n - x_n \right\| = 0. \tag{2.47}$$

Observe that

$$\|x_n - p\| \leq \|T_2(PT_2)^{n-1}P\delta_n - x_n\| + \|T_2(PT_2)^{n-1}P\delta_n - p\| \leq l_n\|\delta_n - p\| \quad (2.48)$$

which yields that

$$c \leq \liminf_{n \rightarrow \infty} \|\delta_n - p\| \leq \limsup_{n \rightarrow \infty} \|\delta_n - p\| \leq c. \quad (2.49)$$

That is, $\lim_{n \rightarrow \infty} \|\delta_n - p\| = c$. This implies that

$$\liminf_{n \rightarrow \infty} \left\| \beta'_n \left(T_2(PT_2)^{n-1}x_n - p \right) + (1 - \beta'_n)(x_n - p) \right\| \geq c. \quad (2.50)$$

Similarly, we have

$$\begin{aligned} & \left\| \beta'_n \left(T_2(PT_2)^{n-1}x_n - p \right) + (1 - \beta'_n)(x_n - p) \right\| \\ & \leq \beta'_n \|T_2(PT_2)^{n-1}x_n - p\| + (1 - \beta'_n) \|x_n - p\| \leq l_n \|x_n - p\|, \end{aligned} \quad (2.51)$$

$$\limsup_{n \rightarrow \infty} \left\| \beta'_n \left(T_2(PT_2)^{n-1}x_n - p \right) + (1 - \beta'_n)(x_n - p) \right\| \leq c. \quad (2.52)$$

Combining (2.50) with (2.52), we obtain

$$\lim_{n \rightarrow \infty} \left\| \beta'_n \left(T_2(PT_2)^{n-1}x_n - p \right) + (1 - \beta'_n)(x_n - p) \right\| = c. \quad (2.53)$$

On the other hand, we have

$$\|T_2(PT_2)^{n-1}x_n - p\| \leq l_n \|x_n - p\|, \quad (2.54)$$

$$\limsup_{n \rightarrow \infty} \|T_2(PT_2)^{n-1}x_n - p\| \leq c,$$

$$\limsup_{n \rightarrow \infty} \|x_n - p\| \leq c. \quad (2.55)$$

Hence, using (2.53), (2.54), (2.55), and Lemma 1.3, we find

$$\lim_{n \rightarrow \infty} \|T_2(PT_2)^{n-1}x_n - x_n\| = 0. \quad (2.56)$$

In addition, from $y_n = P((1 - \alpha'_n - \gamma'_n)x_n + \alpha'_n T_2(P T_2)^{n-1} P \delta_n + \gamma'_n v_n)$ and (2.47), we have

$$\begin{aligned}
 \|y_n - x_n\| &= \left\| P\left((1 - \alpha'_n - \gamma'_n)x_n + \alpha'_n T_2(P T_2)^{n-1} P \delta_n + \gamma'_n v_n\right) - x_n \right\| \\
 &\leq \alpha'_n \|T_2(P T_2)^{n-1} P \delta_n - x_n\| + \gamma'_n \|v_n - x_n\| \\
 &\leq \|T_2(P T_2)^{n-1} P \delta_n - x_n\| + \gamma'_n r. \\
 &\longrightarrow 0, \quad (\text{as } n \longrightarrow \infty).
 \end{aligned} \tag{2.57}$$

Hence, from (2.36) and (2.57), we find

$$\begin{aligned}
 \|T_1(P T_1)^{n-1} x_n - x_n\| &\leq \|T_1(P T_1)^{n-1} x_n - T_1(P T_1)^{n-1} y_n\| \\
 &\quad + \|T_1(P T_1)^{n-1} y_n - y_n\| + \|y_n - x_n\| \\
 &\leq k_n \|y_n - x_n\| + \|T_1(P T_1)^{n-1} y_n - y_n\| + \|y_n - x_n\| \\
 &\longrightarrow 0, \quad (\text{as } n \longrightarrow \infty).
 \end{aligned} \tag{2.58}$$

That is,

$$\lim_{n \rightarrow \infty} \|T_1(P T_1)^{n-1} x_n - x_n\| = 0. \tag{2.59}$$

Since T_1 and T_2 are uniformly L_1 -Lipschitzian and uniformly L_2 -Lipschitzian, respectively, for some $L_1, L_2 \geq 0$, it follows from (2.56), (2.59), and Lemma 2.2 that

$$\lim_{n \rightarrow \infty} \|x_n - T_1 x_n\| = \lim_{n \rightarrow \infty} \|x_n - T_2 x_n\| = 0. \tag{2.60}$$

This completes the proof. $\square$

Theorem 2.4. *Let E be a real uniformly convex Banach space and let K be a nonempty closed convex subset of E which is also a nonexpansive retract of E . Let $T_1, T_2 : K \rightarrow E$ be two asymptotically nonexpansive nonself-mappings of E with sequences $\{k_n\}, \{l_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty} (k_n - 1) < \infty$, $\sum_{n=1}^{\infty} (l_n - 1) < \infty$, respectively, and $F(T_1) \cap F(T_2) \neq \emptyset$. Suppose that $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\alpha'_n\}, \{\beta'_n\}, \{\gamma'_n\}$ are appropriate sequences in $[0, 1]$ satisfying $\alpha_n + \beta_n + \gamma_n = 1 = \alpha'_n + \beta'_n + \gamma'_n$, and $\{u_n\}, \{v_n\}$ are bounded sequences in K such that $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \gamma'_n < \infty$. Moreover, $0 < a \leq \alpha_n, \alpha'_n, \beta_n, \beta'_n \leq b < 1$ for all $n \geq 1$ and some $a, b \in (0, 1)$. If one of T_1 and T_2 is completely continuous, then the sequence $\{x_n\}$ defined by the recursion (1.7) converges strongly to some common fixed point of T_1 and T_2 .*

Proof. By Lemma 2.1, $\{x_n\}$ is bounded. In addition, by Lemma 2.3; $\lim_{n \rightarrow \infty} \|x_n - T_1 x_n\| = \lim_{n \rightarrow \infty} \|x_n - T_2 x_n\| = 0$; then $\{T_1 x_n\}$ and $\{T_2 x_n\}$ are also bounded. If T_1 is completely continuous, there exists subsequence $\{T_1 x_{n_j}\}$ of $\{T_1 x_n\}$ such that $T_1 x_{n_j} \rightarrow p$ as $j \rightarrow \infty$. It follows from Lemma 2.3 that $\lim_{j \rightarrow \infty} \|x_{n_j} - T_1 x_{n_j}\| = \lim_{j \rightarrow \infty} \|x_{n_j} - T_2 x_{n_j}\| = 0$. So by the continuity of T_1 and Lemma 1.4, we have $\lim_{j \rightarrow \infty} \|x_{n_j} - p\| = 0$ and $p \in F(T_1) \cap F(T_2)$.

Furthermore, by Lemma 2.1, we get that $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists. Thus $\lim_{n \rightarrow \infty} \|x_n - p\| = 0$. The proof is completed. $\square$

The following result gives a strong convergence theorem for two asymptotically nonexpansive nonself-mappings in a uniformly convex Banach space satisfying condition (A').

Theorem 2.5. *Let E be a real uniformly convex Banach space and let K be a nonempty closed convex subset of E which is also a nonexpansive retract of E . Let $T_1, T_2 : K \rightarrow E$ be two asymptotically nonexpansive nonself-mappings of E with sequences $\{k_n\}, \{l_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty} (k_n - 1) < \infty$, $\sum_{n=1}^{\infty} (l_n - 1) < \infty$, respectively, and $F(T_1) \cap F(T_2) \neq \emptyset$. Suppose that $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\alpha'_n\}, \{\beta'_n\}, \{\gamma'_n\}$ are appropriate sequences in $[0, 1]$ satisfying $\alpha_n + \beta_n + \gamma_n = 1 = \alpha'_n + \beta'_n + \gamma'_n$, and $\{u_n\}, \{v_n\}$ are bounded sequences in K such that $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \gamma'_n < \infty$. Moreover, $0 < a \leq \alpha_n, \alpha'_n, \beta_n, \beta'_n \leq b < 1$ for all $n \geq 1$ and some $a, b \in (0, 1)$. Suppose that T_1 and T_2 satisfy condition (A'). Then, the sequence $\{x_n\}$ defined by the recursion (1.7) converges strongly to some common fixed point of T_1 and T_2 .*

Proof. By Lemma 2.1, we readily see that $\lim_{n \rightarrow \infty} \|x_n - p\|$ and so, $\lim_{n \rightarrow \infty} d(x_n, F(T_1) \cap F(T_2))$ exists for all $p \in F(T_1) \cap F(T_2)$. Also, by Lemma 2.3, $\lim_{n \rightarrow \infty} \|T_1 x_n - x_n\| = \lim_{n \rightarrow \infty} \|T_2 x_n - x_n\| = 0$. It follows from condition (A') that

$$\lim_{n \rightarrow \infty} f(d(x_n, F(T_1) \cap F(T_2))) \leq \lim_{n \rightarrow \infty} \left(\frac{1}{2} (\|x_n - T_1 x_n\| + \|x_n - T_2 x_n\|) \right) = 0. \quad (2.61)$$

That is,

$$\lim_{n \rightarrow \infty} f(d(x_n, F(T_1) \cap F(T_2))) = 0. \quad (2.62)$$

Since $f : [0, \infty) \rightarrow [0, \infty)$ is a nondecreasing function satisfying $f(0) = 0$, $f(t) > 0$ for all $t \in (0, \infty)$, therefore, we have

$$\lim_{n \rightarrow \infty} d(x_n, F(T_1) \cap F(T_2)) = 0. \quad (2.63)$$

Now we can take a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and sequence $\{y_j\} \subset F$ such that $\|x_{n_j} - y_j\| < 2^{-j}$ for all integers $j \geq 1$. Using the proof method of Tan and Xu [5], we have

$$\|x_{n_{j+1}} - y_j\| \leq \|x_{n_j} - y_j\| < 2^{-j}, \quad (2.64)$$

and hence

$$\|y_{j+1} - y_j\| \leq \|y_{j+1} - x_{n_{j+1}}\| + \|x_{n_{j+1}} - y_j\| \leq 2^{-(j+1)} + 2^{-j} < 2^{-j+1}. \quad (2.65)$$

We get that $\{y_j\}$ is a Cauchy sequence in F and so it converges. Let $y_j \rightarrow y$. Since F is closed, therefore, $y \in F$ and then $x_{n_j} \rightarrow y$. As $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists, $x_n \rightarrow y \in F(T_1) \cap F(T_2)$. Thereby completing the proof. $\square$

Remark 2.6. If $\gamma_n = \gamma'_n = \beta_n = \beta'_n = 0$, then the iterative scheme (1.7) reduces to the iterative scheme (1.4) of [9]. Moreover, the condition (A') is weaker than both the compactness of K and the semicompactness of the asymptotically nonexpansive nonself-mappings $T_1, T_2 : K \rightarrow E$. Also, the condition $0 < a \leq \alpha_n, \alpha'_n \leq b < 1$ for all $n \geq 1$ is weaker than the condition $0 < \varepsilon \leq \alpha_n, \alpha'_n \leq 1 - \varepsilon$, for all $n \geq 1$ and some $\varepsilon \in [0, 1)$. Hence, Theorems 2.4 and 2.5 generalize Theorems 3.3 and 3.4 in [9], respectively.

In the next result, we prove the weak convergence of the iterative scheme (1.7) for two asymptotically nonexpansive nonself-mappings in a uniformly convex Banach space satisfying Opial's condition.

Theorem 2.7. *Let E be a real uniformly convex Banach space and let K be a nonempty closed convex subset of E which is also a nonexpansive retract of E . Let $T_1, T_2 : K \rightarrow E$ be two asymptotically nonexpansive nonself-mappings of E with sequences $\{k_n\}, \{l_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty} (k_n - 1) < \infty$, $\sum_{n=1}^{\infty} (l_n - 1) < \infty$, respectively, and $F(T_1) \cap F(T_2) \neq \emptyset$. Suppose that $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\alpha'_n\}, \{\beta'_n\}, \{\gamma'_n\}$ are appropriate sequences in $[0, 1]$ satisfying $\alpha_n + \beta_n + \gamma_n = 1 = \alpha'_n + \beta'_n + \gamma'_n$, and $\{u_n\}, \{v_n\}$ are bounded sequences in K such that $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \gamma'_n < \infty$. Moreover, $0 < a \leq \alpha_n, \alpha'_n, \beta_n, \beta'_n \leq b < 1$ for all $n \geq 1$ and some $a, b \in (0, 1)$. Suppose that T_1 and T_2 satisfy Opial's condition. Then, the sequence $\{x_n\}$ defined by the recursion (1.7) converges weakly to some common fixed point of T and T_2 .*

Proof. Let $p \in F(T_1) \cap F(T_2)$. By Lemma 2.1, we see that $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists and $\{x_n\}$ bounded. Now we prove that $\{x_n\}$ has a unique weak subsequential limit in $F(T_1) \cap F(T_2)$. Firstly, suppose that subsequences $\{x_{n_k}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$ converge weakly to p_1 and p_2 , respectively. By Lemma 2.3, we have $\lim_{n \rightarrow \infty} \|x_{n_k} - T_1 x_{n_k}\| = 0$. And Lemma 1.4 guarantees that $(I - T_1)p_1 = 0$, that is, $T_1 p_1 = p_1$. Similarly, $T_2 p_1 = p_1$. Again in the same way, we can prove that $p_2 \in F(T_1) \cap F(T_2)$.

Secondly, assume $p_1 \neq p_2$, then by Opial's condition, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \|x_n - p_1\| &= \lim_{k \rightarrow \infty} \|x_{n_k} - p_1\| < \lim_{k \rightarrow \infty} \|x_{n_k} - p_2\| \\ &= \lim_{j \rightarrow \infty} \|x_{n_j} - p_2\| < \lim_{k \rightarrow \infty} \|x_{n_k} - p_1\| \\ &= \lim_{n \rightarrow \infty} \|x_n - p_1\|, \end{aligned} \quad (2.66)$$

which is a contradiction, hence, $p_1 = p_2$. Then, $\{x_n\}$ converges weakly to a common fixed point of T_1 and T_2 . This completes the proof. $\square$

Remark 2.8. The above Theorem generalizes Theorem 3.5 of Wang [9].

3. Case of Two Nonself-Nonexpansive Mappings

Let $T_1, T_2 : K \rightarrow E$ be two nonexpansive nonself-mappings. Then, the iterative scheme (1.7) is written as follows:

$$\begin{aligned} x_{n+1} &= P((1 - \alpha_n - \gamma_n)x_n + \alpha_n T_1 P(1 - \beta_n)y_n + \beta_n T_1 y_n + \gamma_n u_n), \\ y_n &= P((1 - \alpha'_n - \gamma'_n)x_n + \alpha'_n T_2 P(1 - \beta'_n)x_n + \beta'_n T_2 x_n + \gamma'_n v_n), \quad x_1 \in K, n \geq 1. \end{aligned} \quad (3.1)$$

Nothing prevents one from proving the results of the previous section for nonexpansive nonself-mappings case. Thus, one can easily prove the following.

Theorem 3.1. *Let E be a real uniformly convex Banach space and let K be a nonempty closed convex subset of E which is also a nonexpansive retract of E . Let $T_1, T_2 : K \rightarrow E$ be two nonexpansive nonself-mappings of E with sequences $\{k_n\}, \{l_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty} (k_n - 1) < \infty$, $\sum_{n=1}^{\infty} (l_n - 1) < \infty$, respectively, and $F(T_1) \cap F(T_2) \neq \emptyset$. Suppose that $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\alpha'_n\}, \{\beta'_n\}, \{\gamma'_n\}$ are appropriate sequences in $[0, 1]$ satisfying $\alpha_n + \beta_n + \gamma_n = 1 = \alpha'_n + \beta'_n + \gamma'_n$, and $\{u_n\}, \{v_n\}$ are bounded sequences in K such that $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \gamma'_n < \infty$. Moreover, $0 < a \leq \alpha_n, \alpha'_n, \beta_n, \beta'_n \leq b < 1$ for all $n \geq 1$ and some $a, b \in (0, 1)$. Suppose that T_1 and T_2 satisfy condition (A'). Then, the sequence $\{x_n\}$ defined by the recursion (3.1) converges strongly to some common fixed point of T_1 and T_2 .*

Theorem 3.2. *Let E be a real uniformly convex Banach space and let K be a nonempty closed convex subset of E which is also a nonexpansive retract of E . Let $T_1, T_2 : K \rightarrow E$ be two nonexpansive nonself-mappings of E with sequences $\{k_n\}, \{l_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty} (k_n - 1) < \infty$, $\sum_{n=1}^{\infty} (l_n - 1) < \infty$, respectively, and $F(T_1) \cap F(T_2) \neq \emptyset$. Suppose that $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\alpha'_n\}, \{\beta'_n\}, \{\gamma'_n\}$ are appropriate sequences in $[0, 1]$ satisfying $\alpha_n + \beta_n + \gamma_n = 1 = \alpha'_n + \beta'_n + \gamma'_n$, and $\{u_n\}, \{v_n\}$ are bounded sequences in K such that $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \gamma'_n < \infty$. Moreover, $0 < a \leq \alpha_n, \alpha'_n, \beta_n, \beta'_n \leq b < 1$ for all $n \geq 1$ and some $a, b \in (0, 1)$. Suppose that T_1 and T_2 satisfy Opial's condition. Then, the sequence $\{x_n\}$ defined by the recursion (3.1) converges weakly to some common fixed point of T and T_2 .*

Remark 3.3. If $T_1 = T_2 = T$ and T is a nonexpansive nonself-mapping, then the iterative scheme (3.1) reduces to the iterative scheme (1.6) of Thianwan [11]. Then, Theorems 3.1-3.2 generalize Theorems 2.4 and 2.6 in [11], respectively.

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