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Research Article

Existence of Positive Solutions of a Discrete Elastic Beam Equation

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Let T be an integer with $T \geq 5$ and let $\mathbb{T}_2 = \{2, 3, \ldots, T\}$. We consider the existence of positive solutions of the nonlinear boundary value problems of fourth-order difference equations $\Delta^4 u(t-2) - ra(t)f(u(t)) = 0$, $t \in \mathbb{T}_2$, $u(1) = u(T+1) = \Delta^2 u(0) = \Delta^2 u(T) = 0$, where r is a constant, $a : \mathbb{T}_2 \to (0, \infty)$, and $f : [0, \infty) \to [0, \infty)$ is continuous. Our approaches are based on the Krein-Rutman theorem and the global bifurcation theorem.

1. Introduction

An elastic beam in an equilibrium state whose both ends are simply supported can be described by the fourth-order boundary value problem of the form

$$\begin{gathered} y'''' = f(x, y, y''), \quad x \in (0,1), \\ y(0) = y(1) = y''(0) = y''(1) = 0; \end{gathered} \tag{1.1}$$

see Gupta [1, 2]. The existence of solutions of (1.3) and (1.4) has been extensively studied; see Gupta [1, 2], Aftabizadeh [3], Yang [4], Del Pino and Manásevich [5], Galewski [6], Yao [7], and the references therein. The existence and multiplicity of positive solutions of the boundary value problem of ordinary differential equations

$$\begin{gathered} y''''(x) - \lambda f(x, y(x)) = 0, \quad x \in (0,1), \\ y(0) = y(1) = y''(0) = y''(1) = 0 \end{gathered} \tag{1.2}$$

have also been studied by many authors; see Ma and Wang [8], Ma [9], Bai and Wang [10], Chai [11], Yao and Bai [12] for some references along this line.

Recently, the existence of solutions of boundary value problems (BVPs) of difference equations has received much attention; see Agarwal and Wong [13], Henderson [14], He and Yu [15], Zhang et al. [16], and the references therein. However, relatively little is known about the existence of positive solutions of fourth-order discrete boundary value problems. To our best knowledge, only He and Yu [15] and Zhang et al. [16] dealt with that. In [15], He and Yu studied the existence of positive solutions of the nonlinear fourth-order discrete boundary value problem

$$\Delta^4 u(t-2) - ra(t)f(u(t)) = 0, \quad t \in \mathbb{T}_2, \tag{1.3}$$

$$u(0) = u(T+2) = \Delta^2 u(0) = \Delta^2 u(T) = 0 \tag{1.4}$$

(where $T \geq 3$ is an integer, $\mathbb{T}_2 := \{2, \ldots, T\}$, r is a parameter, $a : \mathbb{T}_2 \to [0, \infty)$, $f \in C([0, \infty), [0, \infty))$ satisfies some growth conditions which are not optimal!). The likely reason is that the spectrum structure of the linear eigenvalue problem

$$\Delta^4 u(t-2) = \lambda a(t)u(t), \quad t \in \mathbb{T}_2, \tag{1.5}$$

$$u(0) = u(T+2) = \Delta^2 u(0) = \Delta^2 u(T) = 0 \tag{1.6}$$

is not clear. It has been pointed out in [15, 16] that (1.3) and (1.4) are equivalent to the integral equation of the form

$$u(t) = r\sum_{s=1}^{T+1} G(t,s) \sum_{s=2}^{T} G_1(s,j)a(j)f(u(j)) =: A_0 u(t), \quad j \in \mathbb{T}_2, \tag{1.7}$$

where

$$\begin{aligned} G(t,s) &= \frac{1}{T+2}\begin{cases} s(T+2-t), & 1 \leq s \leq t \leq T+2, \\ t(T+2-s), & 0 \leq t \leq s \leq T+1, \end{cases} \\ G_1(t,i) &= \frac{1}{T}\begin{cases} (T+1-t)(i-1), & 2 \leq i \leq t \leq T+1, \\ (T+1-i)(t-1), & 1 \leq t \leq i \leq T, \end{cases} \end{aligned} \tag{1.8}$$

and other results on the existence of positive solutions of (1.3) and (1.4) can be found in the two papers. Notice that in the integral (1.7), two distinct Green's functions, G and G_1, are used. This makes the construction of cones and the verification of strong positivity of A_0 more complex and difficult. Therefore, we think that the boundary condition (1.4) is not very suitable for the study of the positive solutions of fourth order difference equations.

It is the purpose of this paper to assume the fourth-order difference equation (1.3) subject to a new boundary condition of the form

$$u(1) = u(T+1) = \Delta^2 u(0) = \Delta^2 u(T) = 0. \tag{1.9}$$

This will make our approaches much more simple and natural, and only one Green's function is needed. However, the classical definitions of positive solutions are useless for (1.3) and (1.9) any more. We have to adopt the following new definition of *positive solutions*.

Definition 1.1. Denote

$$\mathbb{T}_1 := \{1, 2, \ldots, T+1\}, \qquad \mathbb{T}_0 := \{0, 1, \ldots, T+1, T+2\}. \tag{1.10}$$

A function $y : \mathbb{T}_0 \to \mathbb{R}^+$ is called a *positive solution* of (1.3) and (1.9) if y satisfies (1.3), (1.9), and $y(t) \geq 0$ on $\mathbb{T}_2$ and $y(t) \not\equiv 0$ on $\mathbb{T}_2$.

Remark 1.2. Notice that the fact $y : \mathbb{T}_0 \to \mathbb{R}^+$ is a positive solution of (1.3) and (1.9) does not mean that $y(t) \geq 0$ on $\mathbb{T}_0$. In fact, y satisfies

(1) $y(t) \geq 0$ for $t \in \mathbb{T}_2$,

(2) $y(1) = y(T+1) = 0$,

(3) $y(0) = -y(2)$, $y(T+2) = -y(T)$.

Remark 1.3. In [17], Eleo and Henderson defined a kind of *positive solutions* which actually are sign-change solutions. In Definition 1.1, a positive solution may allow to take nonpositive value at $t = 0$ and $t = T+2$. We think it is useful in this case that T is large enough.

In the rest of the paper, we will use global bifurcation technique; see Dancer [18, Theorem 2] or Ma and Xu [19, Lemma 2.1], to deal with (1.3) and (1.9). To do this, we have to study the spectrum properties of (1.5) and (1.9). This will be done in Section 2. Finally, in Section 3, we will state and prove our main result.

2. Eigenvalues

Recall

$$\mathbb{T}_2 := \{2, 3, \ldots, T\}, \qquad \mathbb{T}_1 := \{1, 2, \ldots, T+1\}, \qquad \mathbb{T}_0 := \{0, 1, \ldots, T+1, T+2\}. \tag{2.1}$$

Let

$$E := \{u \mid u : \mathbb{T}_0 \longrightarrow \mathbb{R}\}. \tag{2.2}$$

Then, $\dim E = T+3$, and E is a Banach space with the norm

$$\|u\|_E = \max\{|u(j)| \mid j \in \mathbb{T}_0\}. \tag{2.3}$$

Let

$$Y := \{u \mid u : \mathbb{T}_2 \longrightarrow \mathbb{R}\}. \tag{2.4}$$

Then Y is a Banach space with the norm

$$\|u\|_Y = \max\{|u(j)| \mid j \in \mathbb{T}_2\}. \tag{2.5}$$

Let

$$E_0 := \left\{u \in E \mid u(1) = u(T+1) = \Delta^2 u(0) = \Delta^2 u(T) = 0\right\}. \tag{2.6}$$

Then the operator $\chi : E_0 \to Y$

$$\chi(-u(2), 0, u(2), u(3), \ldots, u(T), 0, -u(T)) := (u(2), u(3), \ldots, u(T)), \tag{2.7}$$

is a homomorphism.

In this paper, we assume that

$(H1)$ $a : \mathbb{T}_2 \to (0, \infty)$.

Definition 2.1. We say that λ is an eigenvalue of linear problem

$$\begin{gathered} \Delta^4 u(t-2) = \lambda a(t) u(t), \quad t \in \mathbb{T}_2, \\ u(1) = u(T+1) = \Delta^2 u(0) = \Delta^2 u(T) = 0 \end{gathered} \tag{2.8}$$

if (2.8) has a nontrivial solution.

In the rest of this section, we will prove the existence of the first eigenvalue of (2.8).

Theorem 2.2. *Equation* (2.8) *has an algebraically simple eigenvalue* λ_1, *with an eigenfunction* φ *satisfying*

(i) $\varphi(t) > 0$ *on* $\mathbb{T}_2$;

(ii) $\varphi(0) = -\varphi(2)$; $\varphi(T+2) = -\varphi(T)$.

Moreover, there is no other eigenvalue whose eigenfunction is nonnegative on $\mathbb{T}_2$.

To prove Theorem 2.2, we need several preliminary results.

Lemma 2.3. *For each* $h = (h(2), \ldots, h(T))$, *the linear problem*

$$\begin{gathered} \Delta^4 u(t-2) = h(t), \quad t \in \mathbb{T}_2, \\ u(1) = u(T+1) = \Delta^2 u(0) = \Delta^2 u(T) = 0 \end{gathered} \tag{2.9}$$

has a unique solution

$$u(t) = \sum_{s=2}^{T} H(t,s) \sum_{j=2}^{T} H(s,j) h(j), \quad t \in \mathbb{T}_1, \tag{2.10}$$

where

$$H(t,s) = \frac{1}{T}\begin{cases} (t-1)(T+1-s), & 1 \le t \le s \le T, \\ (s-1)(T+1-t), & 2 \le s \le t \le T+1. \end{cases} \tag{2.11}$$

Proof. Let $\Delta^2 u(t-2) = w(t-1)$ for $t \in \mathbb{T}_2$. Then (2.9) is equivalent to the system

$$\begin{aligned} \Delta^2 w(t-1) &= h(t), \quad t \in \mathbb{T}_2, \\ \Delta^2 u(t-1) &= w(t), \quad t \in \mathbb{T}_2, \\ w(1) &= w(T+1) = 0, \\ u(1) &= u(T+1) = 0. \end{aligned} \tag{2.12}$$

From Kelly and Peterson [20, Theorem 6.8 and Example 6.12], it follows that

$$\begin{aligned} w(t) &= -\sum_{s=2}^{T} H(t,s)h(s), \quad t \in \mathbb{T}_1 \\ u(t) &= -\sum_{s=2}^{T} H(t,s)w(s), \quad t \in \mathbb{T}_1. \end{aligned} \tag{2.13}$$

Therefore, (2.10) holds. □

Denote

$$\rho := 4\sin^2\frac{\pi}{2T}, \qquad e(t) := \sin\frac{\pi(t-1)}{T}, \quad t \in \mathbb{T}_1. \tag{2.14}$$

Then

$$\begin{aligned} \Delta^2 e(t-1) + \rho e(t) &= 0, \quad t \in \mathbb{T}_2, \\ e(1) &= e(T+1) = 0. \end{aligned} \tag{2.15}$$

Notice that

$$\left\{ \left(\frac{t-1}{T}, e(t) \right) \mid t \in \mathbb{T}_1 \right\} \subset \{(x, \sin \pi x) \mid x \in [0,1]\}. \tag{2.16}$$

From the assumption $T \ge 5$, we have

$$\rho \le 4\sin^2\frac{\pi}{2\cdot 5} < 4\sin^2\frac{\pi}{6} = 1. \tag{2.17}$$

Lemma 2.4. *Let $v \in E$ satisfy $v(1) = v(T+1) = \Delta^2 v(0) = \Delta^2 v(T) = 0$ and*

$$-\gamma e(t) \le -\Delta^2 v(t-1) \le \gamma e(t), \quad t \in \mathbb{T}_2, \tag{2.18}$$

where $\gamma \in (0, \infty)$. Then

$$-\frac{\gamma}{\rho} e(t) \le v(t) \le \frac{\gamma}{\rho} e(t), \quad t \in \mathbb{T}_1. \tag{2.19}$$

Proof. From (2.18), we get

$$-\gamma \sum_{s=2}^{T} H(t,s) e(s) \le -\sum_{s=2}^{T} H(t,s) \Delta^2 v(t-1) \le \gamma \sum_{s=2}^{T} H(t,s) e(s), \quad t \in \mathbb{T}_2. \tag{2.20}$$

This is

$$-\frac{\gamma}{\rho} e(t) \le v(t) \le \frac{\gamma}{\rho} e(t), \quad t \in \mathbb{T}_2. \tag{2.21}$$

Combining this with the boundary conditions $v(1) = v(T+1) = \Delta^2 v(0) = \Delta^2 v(T) = 0$, it concludes that

$$-\frac{\gamma}{\rho} e(t) \le v(t) \le \frac{\gamma}{\rho} e(t), \quad t \in \mathbb{T}_1. \tag{2.22}$$

□

Let

$$X := \left\{ u \in E_0 \mid -\gamma e(t) \le -\Delta^2 u(t-1) \le \gamma e(t),\ t \in \mathbb{T}_1 \right\} \tag{2.23}$$

for some $\gamma \in (0, \infty)$. Since $\gamma < \gamma/\rho$, from Lemma 2.4 and (2.17), we may define

$$\|u\|_X := \inf \left\{ \frac{\gamma}{\rho} \mid -\gamma e(t) \le -\Delta^2 u(t-1) \le \gamma e(t),\ t \in \mathbb{T}_1 \right\}. \tag{2.24}$$

For any $x, y \in X$, we have from the definition of $\|\cdot\|_X$ that

$$-\rho e \|x\|_X \le -\Delta^2 x(t-1) \le \rho e \|x\|_X, \quad t \in \mathbb{T}_1, \quad -\rho e \|y\|_X \le -\Delta^2 y(t-1) \le \rho e \|y\|_X, \quad t \in \mathbb{T}_1. \tag{2.25}$$

It follows that

$$-\rho e \left(\|x\|_X + \|y\|_X \right) \le -\Delta^2 \left(x(t-1) + y(t-1) \right) \le \rho e \left(\|x\|_X + \|y\|_X \right), \quad t \in \mathbb{T}_1. \tag{2.26}$$

Thus, $x + y \in X$, and moreover,

$$\|x + y\|_X \le \|x\|_X + \|y\|_X. \tag{2.27}$$

Therefore, $\|\cdot\|_X$ is a norm of X, and $(X, \|\cdot\|_X)$ is a normed linear space. Since $\dim X = T - 1$, $(X, \|\cdot\|_X)$ is actually a Banach space. Let

$$P := \left\{u \in X \mid \Delta^2 u(t-1) \le 0 \text{ for } t \in \mathbb{T}_2;\ u(t) \ge 0 \text{ for } t \in \mathbb{T}_1\right\}. \tag{2.28}$$

Then the cone P is normal and has nonempty interior $\operatorname{int} P$.

Lemma 2.5. *For $u \in X$,*

$$\|u\|_{\infty,1} \le (T+1)^2 \left\|\Delta^2 u\right\|_{\infty,2}, \qquad \left\|\Delta^2 u\right\|_{\infty,2} \le \rho \|u\|_X, \tag{2.29}$$

where

$$\|u\|_{\infty,1} := \max\{|u(j)| \mid u \in \mathbb{T}_1\}, \quad \left\|\Delta^2 u\right\|_{\infty,2} := \max\left\{\left|\Delta^2 u(j-1)\right| \mid j \in \mathbb{T}_2\right\}. \tag{2.30}$$

Proof. (i) From the relation

$$u(t) = \sum_{s=2}^{T} H(t,s)\left(-\Delta^2 u(s-1)\right), \quad t \in \mathbb{T}_1, \tag{2.31}$$

it follows that

$$\|u\|_{\infty,1} \le (T+1)^2 \left\|\Delta^2 u\right\|_{\infty,2}. \tag{2.32}$$

(ii) By (2.14) and the fact that $e(t) > 0$ on $\mathbb{T}_2$, it follows that there exists $\gamma > 0$, such that

$$\left|-\Delta^2 u(t-1)\right| \le \gamma e(t), \quad t \in \mathbb{T}_2. \tag{2.33}$$

Let

$$\gamma_0 := \inf\left\{\gamma \mid \left|-\Delta^2 u(t-1)\right| \le \gamma e(t),\ t \in \mathbb{T}_2\right\}. \tag{2.34}$$

Then

$$\left|-\Delta^2 u(t-1)\right| \le \gamma_0 e(t), \quad t \in \mathbb{T}_2. \tag{2.35}$$

This implies $\|\Delta^2 u\|_{\infty,2} \le \rho \cdot (\gamma_0/\rho)$, and accordingly $\|\Delta^2 u\|_{\infty,2} \le \rho \|u\|_X$. □

Proof of Theorem 2.2. For $u \in E$, define a linear operator $K : X \to Y$ and $J : Y \to X$ by

$$Ku(t) := \sum_{s=2}^{T} H(t,s) \sum_{j=2}^{T} H(s,j)a(j)u(j), \quad t \in \mathbb{T}_1, \tag{2.36}$$

$$J(u(1),\ldots,u(T+1)) = (-u(2),u(1),\ldots,u(T+1),-u(T)). \tag{2.37}$$

Then (2.8) can be written as

$$u = \lambda J \circ Ku, \quad u \in X. \tag{2.38}$$

Since X is finite dimensional, we have that $K : X \to Y$ is compact. Obviously, $J \circ K(P) \subseteq P$.

Next, we show that $J \circ K : P \to P$ is strongly positive.

Since $a(t)$ is positive on $\mathbb{T}_2$, there exists a constant $k > 0$ such that $a(t) > k$ on $\mathbb{T}_2$.

For $u \in P \setminus \{0\}$, we have that

$$\sum_{j=2}^{T} H(s,j)a(j)u(j) \geq k \sum_{j=2}^{T} H(s,j)u(j) > 0, \quad s \in \mathbb{T}_1. \tag{2.39}$$

It follows that there exists $k_1 > 0$ such that

$$\sum_{j=2}^{T} H(s,j)a(j)u(j) \geq k_1 e(s). \tag{2.40}$$

Also, for $u \in P \setminus \{0\}$, we have from the fact $J \circ Ku \in P$ and $Ku \neq 0$ in $\mathbb{T}_2$ that

$$\sum_{j=2}^{T} H(s,j)a(j)u(j) \leq \max_{t\in\mathbb{T}_2} a(t) \cdot \sum_{j=2}^{T} H(s,j)u(j) \leq k_2 e(s), \quad s \in \mathbb{T}_1, \tag{2.41}$$

for some constant $k_2 > 0$. By (2.39) and (2.41), we get

$$k_1 e(s) \leq \sum_{j=2}^{T} H(s,j)a(j)u(j) \leq k_2 e(s), \quad s \in \mathbb{T}_1. \tag{2.42}$$

Thus

$$k_1 \sum_{s=2}^{T} H(t,s)e(s) \leq \sum_{s=2}^{T} H(t,s) \sum_{j=2}^{T} H(s,j)a(j)u(j) \leq k_2 \sum_{s=2}^{T} H(t,s)e(s), \quad t \in \mathbb{T}_1. \tag{2.43}$$

Since

$$\sum_{s=2}^{T} H(t,s)e(s) = \frac{1}{\rho} e(t), \quad t \in \mathbb{T}_1. \tag{2.44}$$

Using (2.43) and (2.44), it follows that

$$\frac{k_1}{\rho}e(t) \le (Ku)(t) \le \frac{k_2}{\rho}e(t), \quad t \in \mathbb{T}_1. \tag{2.45}$$

Therefore, $J \circ Ku \in \operatorname{int} P$.

Now, by the Krein-Rutman theorem [21, *Theorem* 7.C; 20, *Theorem* 19.3], K has an algebraically simple eigenvalue $\lambda > 0$ with an eigenvector $\varphi(\cdot) \in \operatorname{int} P$. Moreover, there is no other eigenvalue with an eigenfunction in P. □

3. The Main Result

In this section, we will make the following assumptions:

(H2) $f : [0,\infty) \to [0,\infty)$ is continuous and $f(s) > 0$ for $s > 0$;

(H3) $f_0, f_\infty \in (0,\infty)$, where

$$f_0 = \lim_{u\to 0^+} \frac{f(u)}{u}, \qquad f_\infty = \lim_{u\to +\infty} \frac{f(u)}{u}. \tag{3.1}$$

Remark 3.1. It is not difficult to see that (H2) and (H3) imply that there exists a constant $a_0 \in (0,\infty)$ such that

$$f(u) \ge a_0 u, \quad u \in [0,\infty). \tag{3.2}$$

Theorem 3.2. *Let* (H1), (H2), *and* (H3) *hold. Assume that either*

$$\frac{\lambda_1}{rf_0} < 1 < \frac{\lambda_1}{rf_\infty} \tag{3.3}$$

or

$$\frac{\lambda_1}{rf_\infty} < 1 < \frac{\lambda_1}{rf_0}. \tag{3.4}$$

Then (1.3) *and* (1.9) *have at least one positive solution.*

Remark 3.3. Recently, Ma and Xu [19] considered the nonlinear fourth-order problem

$$\begin{gathered} u^{(4)}(t) = f\big(t, u(t), u''(t)\big), \quad t \in (0,1), \\ u(0) = u(1) = u''(0) = u''(1) = 0 \end{gathered} \tag{3.5}$$

under some conditions involved the generalized eigenvalues of the linear problem

$$\begin{gathered} u'''' = \lambda\big[A(t)u - B(t)u''\big], \quad 0 < t < 1, \\ u(0) = u(1) = u''(0) = u''(1) = 0. \end{gathered} \tag{3.6}$$

Our main result, Theorem 3.2, needs $f_0, f_\infty \in (0,\infty)$; see (H3). However, in [19, Theorem 3.1], some weaker conditions of the form

$$a_0(t)u - b_0 p - \xi_1(t,u,p) \le f(t,u,p) \le a^0(t)u - b^0 p + \xi_2(t,u,p) \tag{3.7}$$

are used.

Remark 3.4. The first eigenvalue μ_1 of the linear problem

$$\begin{gathered} \Delta^2 u(t-1) + \mu u(t) = 0, \qquad t \in \{2,\dots,T\}, \\ u(1) = u(T+1) = 0, \end{gathered} \tag{3.8}$$

is $\mu_1 = 4\sin^2(\pi/2T)$, and the first eigenvalue λ_1 of the linear problem

$$\begin{gathered} \Delta^4 u(t-1) + \lambda u(t) = 0, \quad t \in \{2,\dots,T\}, \\ u(1) = u(T+1) = \Delta^2 u(0) = \Delta^2 u(T) = 0, \end{gathered} \tag{3.9}$$

is

$$\lambda_1 = 16\sin^4 \frac{\pi}{2T}. \tag{3.10}$$

It is easy to check that the function

$$f(s) := \begin{cases} 2\lambda_1 s, & s \in (0,1), \\ \dfrac{\lambda_1 s}{2} + \dfrac{3}{2}\lambda_1, & s \in [1,\infty). \end{cases} \tag{3.11}$$

Then, for each $r \in (1/2, 2)$, the condition (3.3) holds.

Remark 3.5. The condition (3.3) or (3.4) is optimal since for any $\epsilon > 0$, the linear problem

$$\begin{gathered} \Delta^4 u(t-2) - (\lambda_1 - \epsilon)a(t)u(t) = 0, \quad t \in \mathbb{T}_2, \\ u(1) = u(T+1) = \Delta^2 u(0) = \Delta^2 u(T) = 0, \end{gathered} \tag{3.12}$$

has the unique solution $u(t) \equiv 0$. In fact, λ_1 is the least eigenvalue of the linear problem

$$\begin{gathered} \Delta^4 u(t-2) - \mu a(t)u(t) = 0, \quad t \in \mathbb{T}_2, \\ u(1) = u(T+1) = \Delta^2 u(0) = \Delta^2 u(T) = 0. \end{gathered} \tag{3.13}$$

To prove Theorem 3.2, we define $L : D(L) \to Y$ by

$$Lu(t) := -\Delta^4 u(t-2), \quad u \in D(L),\ t \in \mathbb{T}_2 \tag{3.14}$$

where

$$D(L) = \left\{u \in X \mid u(1) = u(T+1) = \Delta^2 u(0) = \Delta^2 u(T) = 0\right\}. \tag{3.15}$$

It is easy to check that $L^{-1} : Y \to X$ is compact.

Let $\zeta, \xi : [0, \infty) \to \mathbb{R}$ be such that

$$\begin{aligned} f(u) &= f_0 u + \zeta(u), \\ f(u) &= f_\infty u + \xi(u). \end{aligned} \tag{3.16}$$

Obviously, (H3) implies

$$\lim_{u \to 0^+} \frac{\zeta(u)}{u} = 0, \qquad \lim_{u \to +\infty} \frac{\xi(u)}{u} = 0. \tag{3.17}$$

Let

$$\widetilde{\xi}(\tau) = \max_{0 \le |s| \le \tau} |\xi(s)|. \tag{3.18}$$

Then $\widetilde{\xi}$ is nondecreasing and

$$\lim_{\tau \to \infty} \frac{\widetilde{\xi}(\tau)}{\tau} = 0. \tag{3.19}$$

Let us consider

$$Lu + \lambda r a(t) f_0 u + \lambda r a(t) \zeta(u) = 0, \quad \lambda > 0, \tag{3.20}$$

as a bifurcation problem for the trivial solution $u \equiv 0$. It is easy to check that (3.20) can be converted to the equivalent equation

$$\begin{aligned} u(t) &= \lambda \left[\sum_{s=2}^{T} \sum_{j=2}^{T} H(t,s) H(s,j) \left(r a(j) f_0 u(j) + r a(s) \zeta(u(s)) \right) \right] \\ &:= \left(\lambda L^{-1} \left[r a(\cdot) f_0 u(\cdot) \right](t) + \lambda L^{-1} \left[r a(\cdot) \zeta(u(\cdot)) \right](t) \right). \end{aligned} \tag{3.21}$$

From the proof process of Theorem 2.2, we have that for each fixed $\lambda > 0$, the operator $K : X \to X$,

$$Ku(t) = \sum_{s=2}^{T} \sum_{j=2}^{T} H(t,s) H(s,j) r a(j) f_0 u(j) \tag{3.22}$$

is compact and strongly positive. Define $F : [0,\infty) \times X \to X$ by

$$F(\lambda, u) := \lambda \left[\sum_{s=2}^{T} \sum_{j=2}^{T} H(t,s) H(s,j) r a(j) \zeta(u(j)) \right]. \tag{3.23}$$

Then we have from (3.17) and Lemma 2.5 that

$$\|F(\lambda, u)\|_X = o(\|u\|_X), \quad \text{as } \|u\|_X \longrightarrow 0, \tag{3.24}$$

locally uniformly in λ. Now, we have from a version of Dancer [18, Theorem 2], see Ma [9, Lemma 2.1] for details, to conclude that there exists an unbounded connected subset $\mathcal{C}$ in the set

$$\{(\lambda, u) \in (0,\infty) \times P : u = \lambda K u + F(\lambda, u),\ u \in \operatorname{int} P\} \cup \left\{ \left(\frac{\lambda_1}{r f_0}, 0 \right) \right\} \tag{3.25}$$

such that $(\lambda_1/(r f_0), 0) \in \mathcal{C}$.

Proof of Theorem 3.2. It is clear that any solution of (3.20) of the form $(1, u)$ yields a solutions u of (1.3) and (1.9). We will show that $\mathcal{C}$ crosses the hyperplane $\{1\} \times X$ in $\mathbb{R} \times X$. To do this, it is enough to show that $\mathcal{C}$ joins $(\lambda_1/r f_0, 0)$ to $(\lambda_1/r f_\infty, \infty)$. Let $(\mu_n, y_n) \in \mathcal{C}$ satisfy

$$\mu_n + \|y_n\|_X \longrightarrow \infty, \quad n \longrightarrow \infty. \tag{3.26}$$

We note that $\mu_n > 0$ for all $n \in \mathbb{N}$, since $(0,0)$ is the only solution of (3.20) for $\lambda = 0$ and $\mathcal{C} \cap (\{0\} \times X) = \emptyset$.

Case 1 $(\lambda_1/r f_\infty < 1 < \lambda_1/r f_0)$. In this case, we show that

$$\left(\frac{\lambda_1}{r f_\infty}, \frac{\lambda_1}{r f_0} \right) \subseteq \{\lambda \in \mathbb{R} \mid \exists (\lambda, u) \in \mathcal{C}\}. \tag{3.27}$$

We divide the proof into two steps.

Step 1. We show that if there exists a constant number $M > 0$ such that

$$\mu_n \subset (0, M], \tag{3.28}$$

then C_k^ν joins $(\lambda_1/r f_0, 0)$ to $(\lambda_1/r f_\infty, \infty)$.

From (3.28), we have that $\|y\|_X \to \infty$. We divide the equation

$$L y_n + \mu_n r a(t) f_\infty y_n + \mu_n r a(t) \xi(y_n) = 0 \tag{3.29}$$

by $\|y_n\|_X$ and set $\overline{y}_n = y_n/\|y\|_X$. Since $\overline{y}_n$ is bounded in X, choosing a subsequence and relabelling if necessary, we see that $\overline{y}_n \to \overline{y}$ for some $\overline{y} \in X$ with $\|\overline{y}\| = 1$. Moreover, from (3.19) and the fact that $\widetilde{\xi}$ is nondecreasing, we have that

$$\lim_{n\to\infty} \frac{|\xi(y_n(t))|}{\|y\|_X} = 0, \tag{3.30}$$

since

$$\frac{\xi(y_n(t))}{\|y_n\|_X} \le \frac{\widetilde{\xi}(|y_n(t)|)}{\|y_n\|_X} \le \frac{\widetilde{\xi}\left(\|y_n\|_{\infty,1}\right)}{\|y_n\|_X} \le \frac{\widetilde{\xi}\left(\rho(T+1)^2\|y_n\|_X\right)}{\|y_n\|_X}. \tag{3.31}$$

Thus,

$$\overline{y}(t) := \sum_{s=2}^{T}\sum_{j=2}^{T} G(t,s)G(s,j)\overline{\mu} r a(j) f_\infty \overline{y}(j), \tag{3.32}$$

where $\overline{\mu} := \lim_{n\to\infty}\mu_n$, choosing a subsequence and relabelling if necessary. Thus,

$$L\overline{y} + \overline{\mu} r a(t) f_\infty \overline{y} = 0. \tag{3.33}$$

By Theorem 2.2, we have

$$\overline{\mu} = \frac{\lambda_1}{r f_\infty}. \tag{3.34}$$

Thus, $\mathcal{C}$ joins $(\lambda_1/rf_0, 0)$ to $(\lambda_1/rf_\infty, \infty)$.

Step 2. We show that there exists a constant M such that $\mu_n \in (0, M]$ for all n.

By [9, Lemma 2.1], we only need to show that A has a linear minorant V and there exists a $(\mu, y) \in (0,\infty) \times P$ such that $\|y\|_X = 1$ and $\mu V y \ge y$.

By Remark 3.1, there exist constants $a_0 \in (0,\infty)$ such that

$$f(u) \ge a_0 u, \quad u \in [0,\infty). \tag{3.35}$$

For $u \in X$, let

$$Vu(t) := \sum_{s=2}^{T}\sum_{j=2}^{T} G(t,s)G(s,j)a_0 u(s). \tag{3.36}$$

Then V is a linear minorant of R. Moreover,

$$\begin{aligned} V\left(\frac{e(t)}{\rho}\right) &= \sum_{s=2}^{T}\sum_{j=2}^{T} G(t,s)G(s,j)a_0\frac{e(j)}{\rho} \\ &= \frac{1}{\rho^2}a_0\sum_{s=2}^{T}G(t,s)e(s) \\ &\geq \frac{c}{\rho}e(t) \end{aligned} \tag{3.37}$$

for some constant $c > 0$, independent of $t \in \mathbb{T}_0$. So,

$$c^{-1}V\left(\frac{e}{\rho}\right) \geq \frac{e}{\rho}. \tag{3.38}$$

Therefore, it follows [9, Lemma 2.1] that

$$|\eta_n| \leq c^{-1}. \tag{3.39}$$

Case 2 ($\lambda_1/rf_0 < 1 < \lambda_1/rf_\infty$). In this case, if $(\eta_n, y_n) \in \mathcal{C}$ is such that

$$\begin{aligned} \lim_{n\to\infty}(\eta_n + \|y_n\|_X) &= \infty, \\ \lim_{n\to\infty}\eta_n &= \infty, \end{aligned} \tag{3.40}$$

then

$$\left(\frac{\lambda_1}{rf_0}, \frac{\lambda_1}{rf_\infty}\right) \subset \{\lambda \in (0,\infty) \mid (\lambda, u) \in \mathcal{C}\} \tag{3.41}$$

and, moreover,

$$(\{1\} \times X) \cap \mathcal{C} \neq \emptyset. \tag{3.42}$$

Assume that there exists $M > 0$, such that for all $n \in \mathbb{N}$,

$$\eta_n \in (0, M]. \tag{3.43}$$

Applying a similar argument to that used in Step 1 of Case 1, after taking a subsequence and relabelling if necessary, if follows that

$$(\eta_n, y_n) \longrightarrow \left(\frac{\lambda_1}{rf_\infty}, \infty\right), \quad n \longrightarrow \infty. \tag{3.44}$$

Again $\mathcal{C}$ joins $(\lambda_1/rf_0, 0)$ to $(\lambda_1/rf_\infty, \infty)$ and the result follows. □

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