Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2010, Article ID 610467, 17 pages
doi:10.1155/2010/610467

Research Article

Boundedness and Global Attractivity of a Higher-Order Nonlinear Difference Equation

Xiu-Mei Jia[1,2] and Wan-Tong Li[2]

[1] *Department of Mathematics, Hexi University, Zhangye, Gansu 734000, China*
[2] *School of Mathematics and Statistics, Lanzhou University, Lanzhou, Gansu 730000, China*

Correspondence should be addressed to Wan-Tong Li, wtli@lzu.edu.cn

Received 5 November 2009; Accepted 4 February 2010

Academic Editor: Guang Zhang

We investigate the local stability, prime period-two solutions, boundedness, invariant intervals, and global attractivity of all positive solutions of the following difference equation: $y_{n+1} = (r + py_n + y_{n-k})/(qy_n + y_{n-k})$, $n \in \mathbb{N}_0$, where the parameters $p, q, r \in (0, \infty)$, $k \in \{1, 2, 3, \ldots\}$ and the initial conditions $y_{-k}, \ldots, y_0 \in (0, \infty)$. We show that the unique positive equilibrium of this equation is a global attractor under certain conditions.

1. Introduction and Preliminaries

Our aim in this paper is to study the dynamical behavior of the following rational difference equation

$$y_{n+1} = \frac{r + py_n + y_{n-k}}{qy_n + y_{n-k}}, \quad n \in \mathbb{N}_0, \tag{1.1}$$

where $p, q, r \in (0, \infty)$, $\mathbb{N}_0 \in \{0, 1, \ldots\}$, $k \in \{1, 2, 3, \ldots\}$ and the initial conditions $y_{-k}, \ldots, y_0 \in (0, \infty)$.

When $k = 1$, (1.1) reduces to

$$y_{n+1} = \frac{r + py_n + y_{n-1}}{qy_n + y_{n-1}}, \quad n \in \mathbb{N}_0. \tag{1.2}$$

In [1] (see also [2]), the authors investigated the global convergence of solutions to (1.2) and they obtained the following result.

Theorem 1.1. *Let p, q and r be positive numbers. Then every solution of* (1.2) *converges to the unique equilibrium or to a prime-two solution.*

The main purpose of this paper is to further consider the global attractivity of all positive solutions of (1.1). That is to say, we will prove that the unique positive equilibrium of (1.1) is a global attractor under certain conditions (see Theorem 4.10).

For the general theory of difference equations, one can refer to the monographes [3] and [2]. For other related results on nonlinear difference equations, see, for example, [1–18].

For the sake of convenience, we firstly present some definitions and known results which will be useful in the sequel.

Let I be some interval of real numbers and let $f : I \times I \to I$ be a continuously differentiable function. Then for initial conditions $x_{-k}, \ldots, x_0 \in I$, the difference equation

$$x_{n+1} = f(x_n, x_{n-k}), \quad n \in \mathbb{N}_0 \tag{1.3}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$.

A point $\overline{x}$ is called an equilibrium of (1.3) if

$$\overline{x} = f(\overline{x}, \overline{x}). \tag{1.4}$$

That is, $x_n = \overline{x}$ for $n \geq 0$ is a solution of (1.3), or equivalently, $\overline{x}$ is a fixed point of f.

An interval $J \subseteq I$ is called an invariant interval of (1.3) if

$$x_{-k}, \ldots, x_0 \in J \Longrightarrow x_n \in J \quad \forall n \in \mathbb{N}_0. \tag{1.5}$$

That is, every solution of (1.3) with initial conditions in J remains in J.

Let

$$P = \frac{\partial f}{\partial u}(\overline{x}, \overline{x}), \qquad Q = \frac{\partial f}{\partial v}(\overline{x}, \overline{x}) \tag{1.6}$$

denote the partial derivatives of $f(u, v)$ evaluated at an equilibrium $\overline{x}$ of (1.3). Then the linearized equation associated with (1.3) about the equilibrium $\overline{x}$ is

$$z_{n+1} = Pz_n + Qz_{n-k}, \quad n = 0, 1, \ldots, \tag{1.7}$$

and its characteristic equation is

$$\lambda^{k+1} - P\lambda^k - Q = 0. \tag{1.8}$$

Lemma 1.2 (see [3]). *Assume that $P, Q \in R$ and $k \in \{1, 2, \ldots\}$. Then*

$$|P| + |Q| < 1 \tag{1.9}$$

is a sufficient condition for asymptotic stability of the difference equation (1.7). *Suppose in addition that one of the following two cases holds:*

(i) *k odd and $Q > 0$,*

(ii) *k even and $PQ > 0$.*

Then (1.9) *is also a necessary condition for the asymptotic stability of the difference equation* (1.7).

The following result will be useful in establishing the global attractivity character of the equilibrium of (1.1), and it is a reformulation of [2, 7].

Lemma 1.3. *Suppose that a continuous function $f : [a,b] \times [a,b] \to [a,b]$ satisfies one of (i)–(iii):*

(i) $f(x,y)$ is nonincreasing in x, y, and

$$\forall (m,M) \in [a,b] \times [a,b], \qquad \left(f(m,m) = M, f(M,M) = m\right) \Longrightarrow m = M, \tag{1.10}$$

(ii) $f(x,y)$ is nondecreasing in x and nonincreasing in y, and

$$\forall (m,M) \in [a,b] \times [a,b], \quad \left(f(m,M) = m, f(M,m) = M\right) \Longrightarrow m = M, \tag{1.11}$$

(iii) $f(x,y)$ is nonincreasing in x and nondecreasing in y, and

$$\forall (m,M) \in [a,b] \times [a,b], \qquad \left(f(M,m) = m, f(m,M) = M\right) \Longrightarrow m = M. \tag{1.12}$$

(Note that for k odd this is equivalent to (1.3) having no prime period-two solution)
Then (1.3) has a unique equilibrium in $[a,b]$ and every solution with initial values in $[a,b]$ converges to the equilibrium.

This work is organized as follows. In Section 2, the local stability and periodic character are discussed. In Section 3, the boundedness, invariant intervals of (1.1) are presented. Our main results are formulated and proved in Section 4, where the global attractivity of (1.1) is investigated.

2. Local Stability and Period-Two Solutions

The unique positive equilibrium of (1.1) is

$$\overline{y} = \frac{(1+p) + \sqrt{(1+p)^2 + 4r(1+q)}}{2(1+q)}. \tag{2.1}$$

The linearized equation associated with (1.1) about $\overline{y}$ is

$$z_{n+1} - \frac{(p-q)\overline{y} - qr}{(q+1)\left[r + (p+1)\overline{y}\right]} z_n + \frac{(p-q)\overline{y} + r}{(q+1)\left[r + (p+1)\overline{y}\right]} z_{n-k} = 0, \tag{2.2}$$

and its characteristic equation is

$$\lambda^{k+1} - \frac{(p-q)\overline{y} - qr}{(q+1)\left[r + (p+1)\overline{y}\right]}\lambda^{k} + \frac{(p-q)\overline{y} + r}{(q+1)\left[r + (p+1)\overline{y}\right]} = 0. \tag{2.3}$$

From this and Lemma 1.2, we have the following result.

Theorem 2.1. *Assume that $p, q, r \in (0, \infty)$ and initial conditions $y_{-k}, \ldots, y_0 \in (0, \infty)$. Then the following stataments are true.*

(i) *If*

$$(p-q)\overline{y} - qr \geq 0, \qquad (p - 3q - pq - 1)\overline{y} < 2qr, \tag{2.4}$$

then the unique positive equilibrium $\overline{y}$ of (1.1) *is locally asymptotically stable;*

(ii) *If*

$$(p-q)\overline{y} - qr < 0 < (p-q)\overline{y} + r, \tag{2.5}$$

then the unique positive equilibrium $\overline{y}$ of (1.1) *is locally asymptotically stable. In particular, if k is even, then the equilibrium $\overline{y}$ is locally asymptotically stable if and only if* (2.5) *holds;*

(iii) *If*

$$(p-q)\overline{y} + r \leq 0, \tag{2.6}$$

then the unique positive equilibrium $\overline{y}$ of (1.1) *is locally asymptotically stable.*

In the following, we will consider the period-two solutions of (1.1).

Let

$$\ldots, \phi, \psi, \phi, \psi, \ldots \tag{2.7}$$

be a period-two solution of (1.1), where ϕ and ψ are two arbitrary positive real numbers.

If k is even, then $y_n = y_{n-k}$, and ϕ and ψ satisfy the following system:

$$\phi = \frac{r + p\psi + \psi}{q\psi + \psi}, \qquad \psi = \frac{r + p\phi + \phi}{q\phi + \phi}, \tag{2.8}$$

then $(\phi - \psi)(p+1) = 0$, we have $\phi = \psi$, which is a contradiction.

If k is odd, then $y_{n+1} = y_{n-k}$, and ϕ and ψ satisfy the following system:

$$\phi = \frac{r + p\psi + \phi}{q\psi + \phi}, \qquad \psi = \frac{r + p\phi + \psi}{q\phi + \psi}, \tag{2.9}$$

then $\phi + \psi = 1 - p$, $\phi\psi = p(1-p)/(q-1)$. By calculating, (1.1) has prime period-two solution if and only if

$$p < 1, q > 1, \qquad 4r < (1-p)(q-1-pq-3p). \tag{2.10}$$

From the above discussion, we have the following result.

Theorem 2.2. *Equation* (1.1) *has a positive prime period-two solution*

$$\ldots, \phi, \psi, \phi, \psi, \ldots \tag{2.11}$$

if and only if

$$k \text{ is odd}, \quad p < 1, q > 1, \quad 4r < (1-p)(q-pq-3p-1). \tag{2.12}$$

Furthermore, if (2.12) *holds, then the prime period-two solution of* (1.1) *is "unique" and the values of* ϕ *and* ψ *are the positive roots of the quadratic equation*

$$t^2 - (1-p)t + \frac{r + p(1-p)}{q-1} = 0. \tag{2.13}$$

3. Boundedness and Invariant Intervals

In this section, we discuss the boundedness, invariant intervals of (1.1).

3.1. Boundedness

Theorem 3.1. *All positive solutions of* (1.1) *are bounded.*

Proof. Equation (1.1) can be written as

$$y_{n+1} = \frac{r + py_n + y_{n-k}}{qy_n + y_{n-k}} \geq \frac{py_n + y_{n-k}}{qy_n + y_{n-k}} \geq \frac{\min\{(p/q), 1\}(qy_n + y_{n-k})}{qy_n + y_{n-k}} = \min\left\{\frac{p}{q}, 1\right\} \tag{3.1}$$

for all $n \geq 0$. We denote

$$K = \min\left\{\frac{p}{q}, 1\right\}. \tag{3.2}$$

Then

$$\begin{aligned} y_{n+1} &= \frac{r+py_n+y_{n-k}}{qy_n+y_{n-k}} \le \frac{r+py_n+y_{n-k}}{(q/2)K+(K/2)+(q/2)y_n+(1/2)y_{n-k}} \\ &\le \frac{\max\{r,p,1\}(1+y_n+y_{n-k})}{\min\{(q/2)K+(K/2),(q/2),(1/2)\}(1+y_n+y_{n-k})} \\ &= \frac{\max\{r,p,1\}}{\min\{(q/2)K+(K/2),(q/2),(1/2)\}} \end{aligned} \tag{3.3}$$

for all $n > k$. The proof is complete. □

Let $\{y_n\}_{n=-k}^{\infty}$ be a positive solution of (1.1). Then the following identities are easily established:

$$y_{n+1}-1 = \frac{(q-p)((r/(q-p))-y_n)}{qy_n+y_{n-k}}, \quad n \in \mathbb{N}_0, \tag{3.4}$$

$$y_{n+1}-\frac{p}{q} = \frac{((p-q)/q)((qr/(p-q))-y_{n-k})}{qy_n+y_{n-k}}, \quad n \in \mathbb{N}_0, \tag{3.5}$$

$$\begin{aligned} y_{n+1}-\frac{qr}{p-q} &= \frac{((p^2-pq-q^2r)/(p-q))(y_n+(1/q))}{qy_n+y_{n-k}} \\ &\quad + \frac{((p-q-qr)/(p-q))(y_{n-k}-(p/q))}{qy_n+y_{n-k}}, \quad n \in \mathbb{N}_0, \end{aligned} \tag{3.6}$$

$$y_{n+1}-\frac{p+r}{q} = \frac{r(1-y_n)+((q-p-r)/q)y_{n-k}}{qy_n+y_{n-k}}, \quad n \in \mathbb{N}_0, \tag{3.7}$$

$$\begin{aligned} y_n-y_{n+2(k+1)} &= \frac{(y_n-(p/q))(q^2y_{n+k}y_{n+2k+1}+qy_ny_{n+2k+1})}{qy_{n+2k+1}(qy_{n+k}+y_n)+(r+py_{n+k}+y_n)} \\ &\quad + \frac{py_{n+k}(y_n-((p+qr)/p))+(y_n^2-y_n-r)}{qy_{n+2k+1}(qy_{n+k}+y_n)+(r+py_{n+k}+y_n)}, \quad n \in \mathbb{N}_0. \end{aligned} \tag{3.8}$$

When $q = p + r$, the unique positive equilibrium of (1.1) is $\overline{y} = 1$, (3.4) becomes

$$y_{n+1}-1 = \frac{r(1-y_n)}{qy_n+y_{n-k}}, \quad n \in \mathbb{N}_0. \tag{3.9}$$

When $p = q(1+\sqrt{1+4r})/2$, the unique positive equilibrium is $\overline{y} = p/q$, (3.5) becomes

$$y_{n+1}-\frac{p}{q} = \frac{(p-q)/q((p/q)-y_{n-k})}{qy_n+y_{n-k}}, \quad n \in \mathbb{N}_0, \tag{3.10}$$

and (3.8) becomes

$$y_n - y_{n+2(k+1)} = \frac{(y_n - (p/q))(q^2 y_{n+k} y_{n+2k+1} + q y_n y_{n+2k+1} + p y_{n+k} + y_n + (p-q)/q)}{q y_{n+2k+1}(q y_{n+k} + y_n) + (r + p y_{n+k} + y_n)}, \quad n \in \mathbb{N}_0. \tag{3.11}$$

Set

$$f(x,y) = \frac{r + px + y}{qx + y}. \tag{3.12}$$

Lemma 3.2. *Assume that $f(x,y)$ is defined in* (3.12). *Then the following statements are true:*

(i) *assume $p < q$. Then $f(x,y)$ is strictly decreasing in x and increasing in y for $x \geq r/(q-p)$; and it is strictly decreasing in each of its arguments for $x < r/(q-p)$;*

(ii) *assume $p > q$. Then $f(x,y)$ is increasing in x and strictly decreasing in y for $y \geq qr/(p-q)$; and it is strictly decreasing in each of its arguments for $y < qr/(p-q)$.*

Proof. By calculating the partial derivatives of the function $f(x,y)$, we have

$$f'_x(x,y) = \frac{(p-q)y - qr}{(qx+y)^2}, \qquad f'_y(x,y) = \frac{(q-p)x - r}{(qx+y)^2}, \tag{3.13}$$

from which these statements easily follow. □

3.2. Invariant Interval

In this subsection, we discuss the invariant interval of (1.1).

3.2.1. The Case $p < q$

Lemma 3.3. *Assume that $p < q$, and $\{y_n\}_{n=-k}^{\infty}$ is a positive solution of* (1.1). *Then the following statements are true:*

(i) *$y_n > p/q$ for all $n \geq 1$;*

(ii) *If for some $N \geq 0$, $y_N > r/(q-p)$, then $y_{N+1} < 1$;*

(iii) *If for some $N \geq 0$, $y_N = r/(q-p)$, then $y_{N+1} = 1$;*

(iv) *If for some $N \geq 0$, $y_N < r/(q-p)$, then $y_{N+1} > 1$;*

(v) *If $p < q < p+r$, then* (1.1) *possesses an invariant interval $[1, r/(q-p)]$ and $\overline{y} \in (1, r/(q-p))$;*

(vi) *If $p + r < q < p + qr/p$, then* (1.1) *possesses an invariant interval $[r/(q-p), 1]$ and $\overline{y} \in (r/(q-p), 1)$;*

(vii) *If $q \geq p + qr/p$, then* (1.1) *possesses an invariant interval $[p/q, 1]$ and $\overline{y} \in (p/q, 1)$.*

Proof. The proofs of (i)–(iv) are straightforward consequences of the identities (3.5) and (3.4). So we only prove (v)–(vii). By the condition (i) of Lemma 3.2, the function $f(x,y)$ is strictly decreasing in x and increasing in y for $x \geq r/(q-p)$; and it is strictly decreasing in both arguments for $x < r/(q-p)$.

(v) Using the decreasing character of f, we obtain

$$1 = f\left(\frac{r}{q-p}, \frac{r}{q-p}\right) < f(x,y) < f(1,1) = \frac{r+p+1}{q+1} < \frac{r}{q-p}. \tag{3.14}$$

The inequalities

$$1 < \frac{r}{q-p}, \qquad \frac{r+p+1}{q+1} < \frac{r}{q-p} \tag{3.15}$$

are equivalent to the inequality $q < p + r$.

On the other hand, $\overline{y}$ is the unique positive root of quadratic equation

$$(q+1)y^2 - (p+1)y - r = 0. \tag{3.16}$$

Since

$$\begin{gathered}(q+1)\left(\frac{r}{q-p}\right)^2 - (p+1)\frac{r}{q-p} - r = \frac{r(q+1)(p+r-q)}{(q-p)^2} > 0, \\ (q+1) - (p+1) - r = q - p - r < 0,\end{gathered} \tag{3.17}$$

then we have that $\overline{y} \in (1, r/(q-p))$.

(vi) By using the monotonic character of f, we obtain

$$\frac{(q-p)(p+r)+r}{q^2-pq+r} = f\left(1, \frac{r}{q-p}\right) \leq f(x,y) \leq f\left(\frac{r}{q-p}, 1\right) = 1. \tag{3.18}$$

The inequalities

$$\frac{(q-p)(p+r)+r}{q^2-pq+r} > \frac{r}{q-p}, \qquad \frac{r}{q-p} < 1 \tag{3.19}$$

follow from the inequality $q > p + r$.

On the other hand, similar to (v) it can be proved that $\overline{y} \in (r/(q-p), 1)$.

(vii) In this case note that $r/(q-p) \le p/q < 1$ holds, and using the monotonic character of f, we obtain

$$\frac{p}{q} < \frac{qr+pq+p}{q^2+p} = f\left(1, \frac{p}{q}\right) \le f(x,y) \le f\left(\frac{p}{q}, 1\right) = \frac{qr+p^2+q}{q(p+1)} \le 1. \tag{3.20}$$

Furthermore, similar to (v) it follows $\overline{y} \in (p/q, 1)$. The proof is complete. □

When $q = p + r$, (3.9) implies that the following result holds.

Lemma 3.4. *Assume that $q = p + r$, and $\{y_n\}_{n=-k}^{\infty}$ is a positive solution of* (1.1). *Then the following statements are true:*

(i) *If for some $N \ge 0$, $y_N > 1$, then $y_{N+1} < 1$;*

(ii) *If for some $N \ge 0$, $y_N = 1$, then $y_{N+1} = 1$;*

(iii) *If for some $N \ge 0$, $y_N < 1$, then $y_{N+1} > 1$.*

3.2.2. The Case $p > q$

Lemma 3.5. *Assume that $p > q$, and $\{y_n\}_{n=-k}^{\infty}$ is a positive solution of* (1.1). *Then the following statements are true:*

(i) *$y_n > 1$ for all $n \ge 1$;*

(ii) *If for some $N \ge 0$, $y_N < qr/(p-q)$, then $y_{N+k+1} > p/q$;*

(iii) *If for some $N \ge 0$, $y_N = qr/(p-q)$, then $y_{N+k+1} = p/q$;*

(iv) *If for some $N \ge 0$, $y_N > qr/(p-q)$, then $y_{N+k+1} < p/q$;*

(v) *If $q < p < q(1+\sqrt{1+4r})/2$, then* (1.1) *possesses an invariant interval $[p/q, qr/(p-q)]$ and $\overline{y} \in (p/q, qr/(p-q))$;*

(vi) *If $q(1+\sqrt{1+4r})/2 < p < q + qr$, then* (1.1) *possesses an invariant interval $[qr/(p-q), p/q]$ and $\overline{y} \in (qr/(p-q), p/q)$;*

(vii) *If $p \ge q + qr$, then* (1.1) *possesses an invariant interval $[1, p/q]$ and $\overline{y} \in (1, p/q)$.*

Proof. The proofs of (i)–(iv) are direct consequences of the identities (3.4) and (3.5). So we only give the proofs (v)–(vii). By Lemma 3.2 (ii), the function $f(x,y)$ is increasing in x and strictly decreasing in y for $y \ge qr/(p-q)$; and it is strictly decreasing in each of its arguments for $y < qr/(p-q)$.

(v) Using the decreasing character of f, we obtain

$$\frac{p}{q} = f\left(\frac{qr}{p-q}, \frac{qr}{p-q}\right) \le f(x,y) \le f\left(\frac{p}{q}, \frac{p}{q}\right) = \frac{qr+p(p+1)}{p(q+1)} \le \frac{qr}{p-q}. \tag{3.21}$$

The inequalities

$$\frac{qr+p(p+1)}{p(q+1)} \le \frac{qr}{p-q}, \qquad \frac{p}{q} < \frac{qr}{p-q} \tag{3.22}$$

are equivalent to the inequality $p < q(1+\sqrt{1+4r})/2$. That is, $[p/q, qr/(p-q)]$ is an invariant interval of (1.1).

On the other hand, similar to Lemma 3.3 (v), it can be proved that $\overline{y} \in (p/q, qr/(p-q))$.

(vi) By using the monotonic character of f, we obtain

$$\frac{qr}{p-q} \le \frac{(qr+p)(p-q)+pq^2r}{q^3r+p(p-q)} = f\left(\frac{qr}{p-q}, \frac{p}{q}\right) \le f(x,y) \le f\left(\frac{p}{q}, \frac{qr}{p-q}\right) = \frac{p}{q}. \tag{3.23}$$

The inequalities

$$\frac{(qr+p)(p-q)+pq^2r}{q^3r+p(p-q)} \ge \frac{qr}{p-q}, \qquad \frac{qr}{p-q} \le \frac{p}{q} \tag{3.24}$$

are equivalent to the inequality $p > q(1+\sqrt{1+4r})/2$.

On the other hand, similar to Lemma 3.3 (v) it can be proved that $\overline{y} \in (qr/(p-q), p/q)$.

(vii) In this case note that $qr/(p-q) \le 1 < p/q$ holds. By the monotonic character of f, we have

$$1 < \frac{qr+pq+p}{q^2+p} = f\left(1, \frac{p}{q}\right) \le f(x,y) \le f\left(\frac{p}{q}, 1\right) = \frac{qr+p^2+q}{q(p+1)} \le \frac{p}{q}. \tag{3.25}$$

The inequalities

$$\frac{qr+pq+p}{q^2+p} > 1, \qquad \frac{qr+p^2+q}{q(p+1)} \le \frac{p}{q} \tag{3.26}$$

are equivalent to the inequality $p \ge q + qr$.

Furthermore, similar to Lemma 3.3 (v), it follows $\overline{y} \in (1, p/q)$. The proof is complete. □

Lemma 3.6. *Assume that $p = q(1+\sqrt{1+4r})/2$, and $\{y_n\}_{n=-k}^{\infty}$ is a positive solution of* (1.1). *Then the following statements are true:*

(i) *If for some $N \ge 0$, $y_N < p/q$, then $y_{N+k+1} > p/q$;*

(ii) *If for some $N \ge 0$, $y_N = p/q$, then $y_{N+k+1} = p/q$;*

(iii) *If for some $N \ge 0$, $y_N > p/q$, then $y_{N+k+1} < p/q$;*

(iv) *If for some $N \ge 0$, $y_N > p/q$, then $y_N > y_{N+2(k+1)} > p/q$;*

(v) *If for some $N \ge 0$, $y_N < p/q$, then $y_N < y_{N+2(k+1)} < p/q$.*

Proof. In this case, we have that $qr/(p-q) = p/q$. These results follow from the identities (3.10) and (3.11) and the details are omitted. □

4. Global Attractivity

In this section, we discuss the global attractivity of the positive equilibrium of (1.1). We show that the unique positive equilibrium $\overline{y}$ of (1.1) is a global attractor when $p = q$ or $p < 1$ and $p < q \leq pq + 1 + 3p$ or $q < p \leq 1$.

4.1. The Case $p = q$

In this subsection, we discuss the behavior of positive solutions of (1.1) when $p = q$.

Theorem 4.1. *Assume that $p = q$ holds, and $\{y_n\}_{n=-k}^{\infty}$ is a positive solution of* (1.1). *Then the unique positive equilibrium $\overline{y}$ of* (1.1) *is a global attractor.*

Proof. By the change of variables

$$y_n = 1 + \frac{r}{p+1} u_n, \tag{4.1}$$

Equation (1.1) reduces to the difference equation

$$u_{n+1} = \frac{1}{1 + \left(pr/(p+1)^2\right) u_n + \left(r/(p+1)^2\right) u_{n-k}}, \quad n \in \mathbb{N}_0. \tag{4.2}$$

The unique positive equilibrium $\overline{u}$ of (4.2) is

$$\overline{u} = \frac{-(p+1) + \sqrt{(p+1)^2 + 4r(p+1)}}{2r}. \tag{4.3}$$

Applying Lemma 1.3 in interval $[0,1]$, then every positive solution of (1.1) converges to $\overline{u}$. That is, $\overline{u}$ is a global attractor. So, $\overline{y}$ is a global attractor. □

4.2. The Case $p < q$

In this subsection, we present global attractivity of (1.1) when $p < q$.

The following result is straightforward consequence of the identity (3.7).

Lemma 4.2. *Assume that $p < q$ holds, and $\{y_n\}_{n=-k}^{\infty}$ is a positive solution of* (1.1). *Then the following statements are true:*

(i) *Suppose that $q < p + r$. If for some $N \geq 0$, $y_N > 1$, then $y_{N+1} < (p + r)/q$;*

(ii) *Suppose that $q > p + r$. If for some $N \geq 0$, $y_N < 1$, then $y_{N+1} > (p + r)/q$.*

Theorem 4.3. *Assume that $p < q$, $p < 1$ and $q \leq pq + 1 + 3p$ hold. Let $\{y_n\}_{n=-k}^{\infty}$ be a positive solution of* (1.1). *Then the following statements hold true:*

(i) *Suppose $q < p + r$. If $y_0 \in [1, r/(q-p)]$, then $y_n \in [1, r/(q-p)]$ for $n \geq 1$. Furthermore, every positive solution of* (1.1) *lies eventually in the interval* $[1, r/(q-p)]$.

(ii) *Suppose* $q > p + r$. *If* $y_0 \in [r/(q-p), 1]$, *then* $y_n \in [r/(q-p), 1]$ *for* $n \geq 1$. *Furthermore, every positive solution of* (1.1) *lies eventually in the interval* $[r/(q-p), 1]$.

Proof. We only give the proof of (i), the proof of (ii) is similar and will be omitted. First, note that in this case $p/q < 1 < (p+r)/q < r/(q-p)$ holds.

If $y_0 \in [1, r/(q-p)]$, then by Lemma 3.3 (iv), we have that $y_1 > 1$, and by Lemma 4.2 (i), we obtain that $y_1 < (p+r)/q < r/(q-p)$, which implies that $y_1 \in [1, r/(q-p)]$, by induction, we have $y_n \in [1, r/(q-p)]$, for $n \geq 1$.

Now, to complete the proof it remains to show that when $y_0 \notin [1, r/(q-p)]$, there exists $N > 0$ such that $y_N \in [1, r/(q-p)]$.

If $y_0 \notin [1, r/(q-p)]$, then we have the following two cases to be considered:

(a) $y_0 > r/(q-p)$;

(b) $y_0 < 1$.

Case (a). From Lemma 3.3 (ii), we see that $y_1 < 1$. Thus, in the sequel, we only consider case (b).

If $y_0 < 1$, then by Lemma 3.3 (iv), we have $y_1 > 1$, and from Lemma 4.2 (i), we have $y_2 < (p+r)/q < r/(q-p)$. So $y_3 > 1$ and $y_4 < r/(q-p)$. By induction, there exists exactly one term greater than 1, which is followed by exactly one term less than $r/(q-p)$, which is followed by exactly one term greater than 1, and so on. If for some $N > 0$, $1 \leq y_N \leq r/(q-p)$, then the former assertion implies that the result is true.

So assume for the sake of contradiction, that for all $n \geq 1$, y_n never enter the interval $[1, r/(q-p)]$, then the sequence $\{y_n\}_{n=1}^{\infty}$ will oscillate relative to the interval $[1, r/(q-p)]$ with semicycles of length one. Consider the subsequence $\{y_{2n}\}_{n=1}^{\infty}$ and $\{y_{2n+1}\}_{n=1}^{\infty}$ of solution $\{y_n\}_{n=-k}^{\infty}$, we have

$$y_{2n} < 1, \quad y_{2n+1} > \frac{r}{q-p} \quad \text{for } n \geq 1. \tag{4.4}$$

Let

$$\begin{aligned} L &= \lim_{n\to\infty} \sup y_{2n}, \qquad l = \lim_{n\to\infty} \inf y_{2n}, \\ M &= \lim_{n\to\infty} \sup y_{2n+1}, \qquad m = \lim_{n\to\infty} \inf y_{2n+1}, \end{aligned} \tag{4.5}$$

which in view of Theorem 3.1 exist as finite numbers, such that

$$L \leq \frac{r + pm + l}{qm + l}, \qquad l \geq \frac{r + pM + L}{qM + L}, \tag{4.6}$$

$$M \leq \frac{r + pl + m}{ql + m}, \qquad m \geq \frac{r + pL + M}{qL + M}. \tag{4.7}$$

From (4.6), we have $q(Lm - lM) \leq p(m - M) + (l - L)$, which implies that $Lm - lM \leq 0$. Also, from (4.7), we have $q(lM - Lm) \leq p(l - L) + (m - M)$, which implies that $lM - Lm \leq 0$. Thus $lM - Lm = 0$ and $L = l$, $M = m$ hold, from which it follows that $\lim_{n\to\infty} y_{2n}$ and $\lim_{n\to\infty} y_{2n+1}$ exist.

Set

$$\lim_{n\to\infty} y_{2n} = L, \qquad \lim_{n\to\infty} y_{2n+1} = M, \tag{4.8}$$

then $L \le 1$, $M \ge r/(q-p)$ and L, M satisfies the system

$$L = \frac{r + pM + L}{qM + L}, \qquad M = \frac{r + pL + M}{qL + M}, \tag{4.9}$$

which implies that L, M is a period-two solution of (1.1). Furthermore, in view of Theorem 2.2, (1.1) has no period-two solution when $p < 1$ and $q \le pq + 1 + 3p$ hold. This is a contradiction, as desired. The proof is complete. □

Theorem 4.4. *Assume that $p < q$, $p < 1$ and $q \le pq + 1 + 3p$ hold. Then the unique positive equilibrium $\overline{y}$ of* (1.1) *is a global attractor.*

Proof. To complete the proof, there are four cases to be considered.

Case (i). $q < p + r$.

By Theorem 4.3 (i), we know that all solutions of (1.1) lies eventually in the invariant interval $[1, r/(q-p)]$. Furthermore, the function $f(x,y)$ is non-increasing in each of its arguments in the interval $[1, r/(q-p)]$. Thus, applying Lemma 1.3, every solution of (1.1) converges to $\overline{y}$, that is, $\overline{y}$ is a global attractor.

Case (ii). $q = p + r$.

In this case, the only positive equilibrium is $\overline{y} = 1$. In view of Lemma 3.4, we see that, after the first semicycle, the nontrivial solution oscillates about $\overline{y}$ with semicycles of length one. Consider the subsequences $\{y_{2n}\}_{n=1}^{\infty}$ and $\{y_{2n+1}\}_{n=1}^{\infty}$ of any nontrivial solution $\{y_n\}_{n=-k}^{\infty}$ of (1.1). We have

$$y_{2n} < 1, \quad y_{2n+1} > 1, \quad \text{for } n \ge 1, \tag{4.10}$$

or vice versa. Here, we may assume, without loss of generality, that $y_{2n} < 1$ and $y_{2n+1} > 1$, for $n \ge 1$.

Let

$$\begin{aligned} L &= \lim_{n\to\infty} \sup y_{2n}, \qquad l = \lim_{n\to\infty} \inf y_{2n}, \\ M &= \lim_{n\to\infty} \sup y_{2n+1}, \qquad m = \lim_{n\to\infty} \inf y_{2n+1}, \end{aligned} \tag{4.11}$$

which, in view of Theorem 3.1, exist. Then as the same argument in Theorem 4.3, we can see that $\lim_{n\to\infty} y_{2n}$ and $\lim_{n\to\infty} y_{2n+1}$ exist.

Set

$$\lim_{n\to\infty} y_{2n} = L, \qquad \lim_{n\to\infty} y_{2n+1} = M, \tag{4.12}$$

then $L \le 1$, $M \ge 1$. If $L \ne M$, then, also as the same argument in Theorem 4.3, we can see that L, M is a period-two solution of (1.1), which contradicts Theorem 2.2. Thus $L = M$, from which it follows that $\lim_{n\to\infty} y_n = 1$, which implies that $\overline{y} = 1$ is a global attractor.

Case (iii). $p + r < q < p + (q/p)r$.

By Theorem 4.3 (ii), we know that all solutions of (1.1) lies eventually in the invariant interval $[r/(q-p), 1]$. Furthermore, the function $f(x, y)$ decreases in x and increases in y in the interval $[r/(q-p), 1]$. Thus, applying Lemma 1.3, every solution of converges to $\overline{y}$, that is, $\overline{y}$ is a global attractor.

Case (iv). $q \geq p + (q/p)r$.

In this case, we note that $r/(q-p) \leq p/q < 1$ holds. From Theorem 4.3 (ii) and Lemma 3.3 (i), we know that all solutions of (1.1) eventually enter the invariant interval $[p/q, 1]$. Hence, by using the same argument in (iii), $\overline{y}$ is a global attractor. The proof is complete. □

4.3. The Case $p > q$

In this subsection, we discuss the global behavior of (1.1) when $p > q$.

The following three results are the direct consequences of equations (3.4), (3.5), (3.6), and (3.8).

Lemma 4.5. *Assume that $q < p < q(1+\sqrt{1+4r})/2$, and $\{y_n\}_{n=-k}^{\infty}$ is a positive solution of* (1.1). *Then the following statements are true:*

(i) *If for some $N \geq 0$, $y_N < p/q$, then $y_N < y_{N+2(k+1)} < qr/(p-q)$;*

(ii) *If for some $N \geq 0$, $y_N > qr/(p-q)$, then $p/q < y_{N+2(k+1)} < y_N$;*

(iii) *If for some $N \geq 0$, $p/q \leq y_N \leq qr/(p-q)$, then $p/q \leq y_{N+2(k+1)} \leq qr/(p-q)$.*

Lemma 4.6. *Assume that $q(1+\sqrt{1+4r})/2 < p < q + qr$, and $\{y_n\}_{n=-k}^{\infty}$ is a positive solution of* (1.1). *Then the following statements are true:*

(i) *If for some $N \geq 0$, $y_N < qr/(p-q)$, then $y_N < y_{N+2(k+1)} < p/q$;*

(ii) *If for some $N \geq 0$, $y_N > p/q$, then $qr/(p-q) < y_{N+2(k+1)} < y_N$;*

(iii) *If for some $N \geq 0$, $qr/(p-q) \leq y_N \leq p/q$, then $qr/(p-q) \leq y_{N+2(k+1)} \leq p/q$.*

Lemma 4.7. *Assume that $p \geq q + qr$, and $\{y_n\}_{n=-k}^{\infty}$ is a positive solution of* (1.1). *Then the following statements are true:*

(i) *$y_n > 1$ for $n \geq 1$;*

(ii) *If for some $N \geq 0$, $y_N > p/q$, then $1 < y_{N+2(k+1)} < p/q$ and $y_{N+2(k+1)} < y_N$;*

(iii) *If for some $N \geq 0$, $1 < y_N \leq p/q$, then $1 < y_{N+2(k+1)} \leq p/q$.*

Theorem 4.8. *Assume that $p > q$ holds, and let $\{y_n\}_{n=-k}^{\infty}$ be a positive solution of* (1.1). *Then the following statements hold true:*

(i) *If $p < q(1+\sqrt{1+4r})/2$, then every positive solution of* (1.1) *lies eventually in the interval $[p/q, qr/(p-q)]$.*

(ii) *If $q(1+\sqrt{1+4r})/2 < p < q + qr$, then every positive solution of* (1.1) *lies eventually in the interval $[qr/(p-q), p/q]$.*

(iii) *If $p \geq q + qr$, then every positive solution of* (1.1) *lies eventually in the interval $[1, p/q]$.*

Proof. We only give the proof of (i), the proofs of (ii) and (iii) are similar and will be omitted.

When $q < p < q(1+\sqrt{1+4r})/2$, recall that from Lemma 3.5, $[p/q, qr/(p-q)]$ is an invariant interval and so it follows that every solution of (1.1) with $k+1$ consecutive values in $[p/q, qr/(p-q)]$, lies eventually in this interval. If the solution is not eventually in $[p/q, qr/(p-q)]$, there are three cases to be considered.

Case (i). If for some $N \geq 0$, $y_N > qr/(p-q)$, then there are two cases to be considered. If $y_{N+2(k+1)n} \geq qr/(p-q)$ for every $n \in N$, then by Lemma 4.5, we have

$$y_{N+2(k+1)(n-1)} > y_{N+2(k+1)n} > \frac{p}{q}, \tag{4.13}$$

hence, the subsequence $\{y_{N+2(k+1)n}\}$ is strictly monotonically decreasing convergent and its limit S satisfies $S \geq qr/(p-q)$. Taking limit on both sides of (3.8), we obtain a contradiction. If for some n_0, $y_{N+2(k+1)n_0} < qr/(p-q)$, then by Lemma 4.5 we obtain that $\{y_{N+2(k+1)n}\}$ is eventually in the interval $[p/q, qr/(p-q)]$.

Case (ii). If for some $N \geq 0$, $y_N < p/q$, then there are two cases to be considered. If $y_{N+2(k+1)n} < p/q$ for every $n \in N$, then by Lemma 4.5 we obtain

$$y_{N+2(k+1)(n-1)} < y_{N+2(k+1)n} < \frac{qr}{p-q}, \tag{4.14}$$

which implies that the subsequence $\{y_{N+2(k+1)n}\}$ is convergent. Then as the same argument in case (i), obtain a contradiction. If for some n_0, $y_{N+2(k+1)n_0} > p/q$, then by Lemma 4.5 we have that $\{y_{N+2(k+1)n}\}$ is eventually in the interval $[p/q, qr/(p-q)]$.

Case (iii). If for some $N \geq 0, p/q \leq y_N \leq qr/(p-q)$, then by Lemma 4.5 it follows that $p/q \leq y_{N+2(k+1)n} \leq qr/(p-q)$. Assume that there is a subsequence $\{y_{N_0+2(k+1)n}\}$ such that $y_{N_0+2(k+1)n} \geq qr/(p-q)$, or $y_{N_0+2(k+1)n} \leq p/q$, for every $n \in N$. Then its limit S satisfies $S \geq qr/(p-q)$, or $S \leq p/q$. Taking limit on both sides of (3.8), obtain a contradiction. Hence, for all $N \in \{1, 2, \ldots, 2(k+1)\}$ the subsequences $\{y_{N+2(k+1)n}\}$ are eventually in the interval $[p/q, qr/(p-q)]$. □

Theorem 4.9. *Assume that $p > q$ and $p \leq 1$ hold. Then the unique positive equilibrium $\overline{y}$ of* (1.1) *is a global attractor.*

Proof. The proof will be accomplished by considering the following four cases.

Case (i). $p < q(1+\sqrt{1+4r})/2$.

By part (i) of Theorem 4.8, we know that all positive solutions of (1.1) lie eventually in the invariant interval $[p/q, qr/(p-q)]$. Furthermore, the function $f(x,y)$ is nonincreasing in each of its arguments in the interval $[p/q, qr/(p-q)]$. Thus, applying Lemma 1.3, every solution of (1.1) converges to $\overline{y}$, that is, $\overline{y}$ is a global attractor.

Case (ii). $p = q(1+\sqrt{1+4r})/2$.

In this case, the only positive equilibrium of (1.1) is $\overline{y} = p/q$. From Lemma 3.6 and (3.11), we know that each of the $2(k+1)$ subsequences

$$\left\{y_{2(k+1)n+i}\right\}_{n=0}^{\infty} \quad \text{for } i = 1, 2, \ldots, 2(k+1) \tag{4.15}$$

of any solution $\{y_n\}_{n=-k}^{\infty}$ of (1.1) is either identically equal to p/q or strictly monotonically convergent and its limit is greater than zero. Set

$$L_i = \lim_{n\to\infty} y_{2(k+1)n+i} \quad \text{for } i = 1, 2, \ldots, 2(k+1). \tag{4.16}$$

Then, clearly,

$$\ldots, L_1, L_2, \ldots, L_{2(k+1)}, \ldots \tag{4.17}$$

is a period solution of (1.1) with period $2(k+1)$. By applying (3.11) to the solution (4.17) and using the fact $L_i > 0$ for $i = 1, 2, \ldots, 2(k+1)$, we see that

$$L_i = \frac{p}{q} \quad \text{for } i = 1, 2, \ldots, 2(k+1), \tag{4.18}$$

and so

$$\lim_{n\to\infty} y_n = \frac{p}{q}, \tag{4.19}$$

which implies that $\overline{y} = p/q$ is a global attractor.

Case (iii). $q(1+\sqrt{1+4r})/2 < p < q + qr$.

By Theorem 4.8 (ii), all positive solutions of (1.1) eventually enter the invariant interval $[qr/(p-q), p/q]$. Furthermore, the function $f(x,y)$ increases in x and decreases in y in the interval $[qr/(p-q), p/q]$. Thus, applying Lemma 1.3 and assumption $p \le 1$, every solution of (1.1) converges to $\overline{y}$. So, $\overline{y}$ is a global attractor.

Case (iv). $p \ge q + qr$.

In this case, we note that $qr/(p-q) \le 1 < p/q$ holds. In view of Theorem 4.8 (iii), we obtain that all solutions of (1.1) eventually enter the invariant interval $[1, p/q]$. Furthermore, the function $f(x,y)$ increases in x and decreases in y in the interval $[1, p/q]$. Then using the same argument in case (iii), every solution of (1.1) converges to $\overline{y}$. Thus the equilibrium $\overline{y}$ is a global attractor. The proof is complete. □

Finally, we summarize our results and obtain the following theorem, which shows that $\overline{y}$ is a global attractor in three cases.

Theorem 4.10. *The unique positive equilibrium $\overline{y}$ of* (1.1) *is a global attractor, when one of the following three cases holds:*

(i) $p = q$;

(ii) $p < 1$ *and* $p < q \le pq + 1 + 3p$;

(iii) $q < p \le 1$.

References

[1] S. Basu and O. Merino, "Global behavior of solutions to two classes of second-order rational difference equations," *Advances in Difference Equations*, vol. 2009, Article ID 128602, 27 pages, 2009.

[2] M. R. S. Kulenović and G. Ladas, *Dynamics of Second Order Rational Difference Equations with Open Problem and Conjectures*, Chapman & Hall/CRC, Boca Raton, Fla, USA, 2002.

[3] V. L. Kocić and G. Ladas, *Global Behavior of Nonlinear Difference Equations of Higher Order with Applications*, vol. 256 of *Mathematics and Its Applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1993.

[4] R. P. Agarwal, W.-T. Li, and P. Y. H. Pang, "Asymptotic behavior of a class of nonlinear delay difference equations," *Journal of Difference Equations and Applications*, vol. 8, no. 8, pp. 719–728, 2002.

[5] A. M. Amleh, E. Camouzis, and G. Ladas, "On the boundedness character of rational equations. II," *Journal of Difference Equations and Applications*, vol. 12, no. 6, pp. 637–650, 2006.

[6] K. Cunningham, M. R. S. Kulenović, G. Ladas, and S. V. Valicenti, "On the recursive sequence $x_{n+1} = (\alpha + \beta x_n)/(Bx_n + Cx_{n-1})$," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 47, no. 7, pp. 4603–4614, 2001.

[7] R. DeVault, W. Kosmala, G. Ladas, and S. W. Schultz, "Global behavior of $y_{n+1} = (p + y_{n-k})/(qy_n + y_{n-k})$," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 47, no. 7, pp. 4743–4751, 2001.

[8] M. M. El-Afifi, "On the recursive sequence $x_{n+1} = (\alpha+\beta x_n+\gamma x_{n-1})/(Bx_n+Cx_{n-1})$," *Applied Mathematics and Computation*, vol. 147, no. 3, pp. 617–628, 2004.

[9] G. H. Gibbons, M. R. S. Kulenović, and G. Ladas, "On the dynamics of $x_{n+1} = (\alpha + \beta x_n + \gamma x_{n-1})/(A + Bx_n)$," in *New Trends in Difference Equaations*, G. Ladas and M. Pinto, Eds., pp. 141–158, Taylor & Francis, London, UK, 2002.

[10] L.-X. Hu, W.-T. Li, and S. Stević, "Global asymptotic stability of a second order rational difference equation," *Journal of Difference Equations and Applications*, vol. 14, no. 8, pp. 779–797, 2008.

[11] Y. S. Huang and P. M. Knopf, "Boundedness of positive solutions of second-order rational difference equations," *Journal of Difference Equations and Applications*, vol. 10, no. 11, pp. 935–940, 2004.

[12] V. L. Kocić, G. Ladas, and I. W. Rodrigues, "On rational recursive sequences," *Journal of Mathematical Analysis and Applications*, vol. 173, no. 1, pp. 127–157, 1993.

[13] M. R. S. Kulenović, G. Ladas, L. F. Martins, and I. W. Rodrigues, "The dynamics of $x_{n+1} = (\alpha + \beta x_n)/(A + Bx_n + Cx_{n-1})$ facts and conjectures," *Computers and Mathematics with Applications*, vol. 45, no. 6–9, pp. 1087–1099, 2003.

[14] M. R. S. Kulenović and O. Merino, "A note on unbounded solutions of a class of second order rational difference equations," *Journal of Difference Equations and Applications*, vol. 12, no. 7, pp. 777–781, 2006.

[15] M. R. S. Kulenović and O. Merino, "Global attractivity of the equilibrium of $x_{n+1} = (px_n+x_{n-1})/(qx_n+x_{n-1})$ for $q < p$," *Journal of Difference Equations and Applications*, vol. 12, no. 1, pp. 101–108, 2006.

[16] R. Mazrooei-Sebdani and M. Dehghan, "Dynamics of a non-linear difference equation," *Applied Mathematics and Computation*, vol. 178, no. 2, pp. 250–261, 2006.

[17] D. Simsek, B. Demir, and C. Cinar, "On the solutions of the system of difference equations $x_{n+1} = \max\{A/x_n, y_n/x_n\}, y_{n+1} = max\{A/y_n, x_n/y_n\}$," *Discrete Dynamics in Nature and Society*, vol. 2009, Article ID 325296, 11 pages, 2009.

[18] Y.-H. Su, W.-T. Li, and S. Stević, "Dynamics of a higher order nonlinear rational difference equation," *Journal of Difference Equations and Applications*, vol. 11, no. 2, pp. 133–150, 2005.