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Research Article

Existence and Global Exponential Stability of Equilibrium Solution to Reaction-Diffusion Recurrent Neural Networks on Time Scales

Kaihong Zhao[1,2] and Yongkun Li[1]

[1] *Department of Mathematics, Yunnan University, Kunming, Yunnan 650091, China*
[2] *Department of Applied Mathematics, Kunming University of Science and Technology, Kunming, Yunnan 650093, China*

Correspondence should be addressed to Yongkun Li, yklie@ynu.edu.cn

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The existence of equilibrium solutions to reaction-diffusion recurrent neural networks with Dirichlet boundary conditions on time scales is proved by the topological degree theory and M-matrix method. Under some sufficient conditions, we obtain the uniqueness and global exponential stability of equilibrium solution to reaction-diffusion recurrent neural networks with Dirichlet boundary conditions on time scales by constructing suitable Lyapunov functional and inequality skills. One example is given to illustrate the effectiveness of our results.

1. Introduction

In the past few years, various neural network models have been extensively investigated and successfully applied to signal processing, image processing, pattern classification, quadratic optimization, associative memory, moving object speed detection, and so forth. Such applications heavily depend on the dynamical behaviors of the neural networks. Therefore, the analysis of the dynamical behaviors is a necessary step for practical design of neural networks.

As is well known, both in biological and man-made neural networks, strictly speaking, diffusion effects cannot be avoided in the neural network models when electrons are moving in asymmetric electromagnetic fields, so we must consider that the activations vary in spaces as well as in time. References [1–10] have considered the stability of neural networks with diffusion terms, which are expressed by partial differential equations. It is also common to consider the diffusion effects in biological systems (such as immigration, see, e.g., [11–13]).

For more details of the literature related to models of reaction-diffusion neural networks and their applications, the reader is referred to [14–21] and the references cited therein.

In fact, both continuous and discrete systems are very important in implementing and applications. But it is troublesome to study the dynamics behavior for continuous and discrete systems, respectively. Therefore, it is meaningful to study that on time scales which can unify the continuous and discrete situations [22, 23].

To the best of our knowledge, few authors have considered global exponential stability of reaction-diffusion recurrent neural networks with Dirichlet boundary conditions on time scales, which is a very important in theories and applications and also is very challenging problem. Motivated by the above discussion, in this paper, we will investigate the global exponential stability of the following reaction-diffusion recurrent neural network with initial value conditions and Dirichlet boundary conditions on time scales:

$$u_i^{\Delta}(t,x) = \sum_{k=1}^{m} \frac{\partial}{\partial x_k}\left(a_{ik}\frac{\partial u_i}{\partial x_k}\right) - b_i u_i(t,x) + \sum_{j=1}^{n} c_{ij} f_j\left(u_j(t,x)\right) + I_i, \quad (t,x) \in \mathbb{T}\times\Omega, \tag{1.1}$$

$$u_i(0,x) = \phi_i(x), \quad x \in \Omega, \tag{1.2}$$

$$u_i(t,x) = 0, \quad (t,x) \in [0,+\infty)_{\mathbb{T}} \times \partial\Omega, \tag{1.3}$$

where $i = 1,2,\ldots,n, \mathbb{T} \subset \mathbb{R}$ is a time scale and $\mathbb{T} \cap [0,+\infty) = [0,+\infty)_{\mathbb{T}}$ is unbounded, n is the number of neurons in the networks, $x = (x_1,x_2,\ldots,x_m)^T \in \Omega \subset \mathbb{R}^m$, and $\Omega = \{x = (x_1,x_2,\ldots,x_m)^T : |x_i| < l_i, i = 1,2,\ldots,m\}$ is a bounded compact set with smooth boundary $\partial\Omega$ in space $\mathbb{R}^m$, $u(t,x) = (u_1(t,x),u_2(t,x),\ldots,u_n(t,x))^T : \mathbb{T}\times\Omega \to \mathbb{R}^n$, and $u_i(t,x)$ is the state of the ith neurons at time t and in space x. Smooth function $a_{ik} > 0$ corresponds to the transmission diffusion operator along with the ith unit, $b_i > 0$ represents the rate with which the ith unit will reset its potential to the resting state in isolation when disconnected from the network and external inputs, c_{ij} denotes the strength of the jth unit on the ith unit at time t and in space x, d_{ij} is the synaptic connection strength of jth unit on ith unit at time t and in space x, $f_j(\cdot)$ denotes the activation function of the jth unit at time t and in space x, $\phi(x) = (\phi_1(x),\phi_2(x),\ldots,\phi_n(x))^T \in C(\Omega,\mathbb{R}^n)$, and $I = (I_1,I_2,\ldots,I_n)^T \in \mathbb{R}^n$ is a constant input vector.

2. Preliminaries

In this section, we first recall some basic definitions and lemmas on time scales which are used in what follows.

Let $\mathbb{T}$ be a nonempty closed subset (time scale) of $\mathbb{R}$. The forward and backward jump operators $\sigma,\rho : \mathbb{T} \to \mathbb{T}$ and the graininess $\mu : \mathbb{T} \to \mathbb{R}^+$ are defined, respectively, by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \qquad \rho(t) = \sup\{s \in \mathbb{T} : s < t\}, \qquad \mu(t) = \sigma(t) - t. \tag{2.1}$$

A point $t \in \mathbb{T}$ is called left-dense if $t > inf\,\mathbb{T}$ and $\rho(t) = t$, left-scattered if $\rho(t) < t$, right-dense if $t < sup\,\mathbb{T}$ and $\sigma(t) = t$, and right-scattered if $\sigma(t) > t$. If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T} \setminus \{m\}$; otherwise $\mathbb{T}^k = \mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_k = \mathbb{T} \setminus \{m\}$; otherwise $\mathbb{T}^k = \mathbb{T}$.

Definition 2.1 (see [24]). A function $f : \mathbb{T} \to \mathbb{R}$ is called *regulated* provided its right-side limits exist (finite) at all right-side points in $\mathbb{T}$ and its left-side limits exist (finite) at all left-side points in $\mathbb{T}$.

Definition 2.2 (see [24]). A function $f : \mathbb{T} \to \mathbb{R}$ is called rd continuous provided it is continuous at right-dense point in $\mathbb{T}$ and its left-side limits exist (finite) at left-dense points in $\mathbb{T}$. The set of rd continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by $C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R})$.

Definition 2.3 (see [24]). Assume $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$. Then we define $f^{\Delta}(t)$ to be the number (if it exists) with the property that given any $\varepsilon > 0$ there exists a neighborhood U of t (i.e., $U = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$) such that

$$\left| \left[f(\sigma(t)) - f(s) \right] - f^{\Delta}(t) [\sigma(t) - s] \right| < \varepsilon |\sigma(t) - s| \tag{2.2}$$

for all $s \in U$. We call $f^{\Delta}(t)$ the delta (or Hilger) derivative of f at t. The set of functions $f : \mathbb{T} \to \mathbb{R}$ that are differentiable and whose derivative is rd continuous is denoted by $C^1_{rd} = C^1_{rd}(\mathbb{T}) = C^1_{rd}(\mathbb{T}, \mathbb{R})$.

If f is continuous, then f is rd continuous. If f is rd continuous, the f is regulated. If f is delta differentiable at t, then f is continuous at t.

Lemma 2.4 (see [24]). *Let f be regulated, then there exists a function F which is delta differentiable with region of differentiation D such that*

$$F^{\Delta}(t) = f(t), \quad \forall t \in D. \tag{2.3}$$

Definition 2.5 (see [24]). Assume $f : \mathbb{T} \to \mathbb{R}$ is a regulated function. Any function F as in Lemma 2.4 is called a Δ-antiderivative of f. We define the indefinite integral of a regulated function f by

$$\int f(t) \Delta t = F(t) + C, \tag{2.4}$$

where C is an arbitrary constant and F is a Δ-antiderivative of f. We define the Cauchy integral by

$$\int_a^b f(s) \Delta s = F(b) - F(a), \quad \forall a, b \in \mathbb{T}. \tag{2.5}$$

A function $F : \mathbb{T} \to \mathbb{R}$ is called an antiderivative of $f : \mathbb{T} \to \mathbb{R}$ provided that

$$F^{\Delta}(t) = f(t), \quad \forall t \in \mathbb{T}^k. \tag{2.6}$$

Lemma 2.6 (see [24]). *If $a, b \in \mathbb{T}$, $\alpha, \beta \in \mathbb{R}$, and $f, g \in C(\mathbb{T}, \mathbb{R})$, then*

(i) $\int_a^b [\alpha f(t) + \beta g(t)] \Delta t = \alpha \int_a^b f(t) \Delta t + \beta \int_a^b g(t) \Delta t$;

(ii) *if* $f(t) \geq 0$ *for all* $a \leq t < b$, *then* $\int_a^b f(t)\Delta t \geq 0$;

(iii) *if* $|f(t)| \leq g(t)$ *on* $[a,b) := \{t \in \mathbb{T} : a \leq t < b\}$, *then* $|\int_a^b f(t)\Delta t| \leq \int_a^b g(t)\Delta t$.

A function $p : \mathbb{T} \to \mathbb{R}$ is called regressive if $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}^k$. The set of all regressive and rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T},\mathbb{R})$. We define the set $\mathcal{R}^+$ of all positively regressive elements of $\mathcal{R}$ by $\mathcal{R}^+ = \mathcal{R}^+(\mathbb{T},\mathbb{R}) = \{p \in \mathcal{R} : 1 + \mu(t)p(t) > 0 \text{ for all } t \in \mathbb{T}\}$. If p is a regressive function, then the generalized exponential function e_p is defined by $e_p(t,s) = \exp\{\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\}$ for $s,t \in \mathbb{T}$, with the cylinder transformation

$$\xi_h(z) = \begin{cases} \dfrac{\operatorname{Log}(1+hz)}{h}, & \text{if } h \neq 0, \\ z, & \text{if } h = 0. \end{cases} \tag{2.7}$$

Let $p,q : \mathbb{T} \to \mathbb{R}$ be two regressive functions, then we define

$$p \oplus q = p + q + \mu pq, \qquad \ominus p = -\frac{p}{1+\mu p}, \qquad p \ominus q = p \oplus p(\ominus q). \tag{2.8}$$

The generalized exponential function has the following properties.

Lemma 2.7 (see [24]). *Assume that* $p,q : \mathbb{T} \to \mathbb{R}$ *are two regressive functions, then,*

(i) $e_p(\sigma(t),s) = (1+\mu(t)p(t))e_p(t,s)$;

(ii) $1/e_p(t,s) = e_{\ominus p}(t,s)$;

(iii) $e_p(t,s) = 1/e_p(s,t) = e_{\ominus p}(s,t)$;

(iv) $e_p(t,s)e_p(s,r) = e_p(t,r)$;

(v) $[e_p(t,s)]^\Delta = p(t)e_p(t,s)$;

(vi) $[e_p(c,\cdot)]^\Delta = -p[e_p(c,\cdot)]^\sigma$ *for* $c \in \mathbb{T}$;

(vii) $d/dz[e_z(t,s)] = (\int_s^t (1/(1+\mu(\tau)z))\Delta\tau)e_z(t,s)$.

Lemma 2.8 (see [24]). *Assume that* $f,g : \mathbb{T} \to \mathbb{R}$ *are delta differentiable at* $t \in \mathbb{T}^k$. *Then,*

$$(fg)^\Delta(t) = f^\Delta(t)g(t) + f(\sigma(t))g^\Delta(t) = g^\Delta(t)f(t) + g(\sigma(t))f^\Delta(t). \tag{2.9}$$

Next, let us introduce the Banach space which is suitable for (1.1)–(1.3) and some assumed conditions which are needed in this paper.

Let $\Omega = \{x = (x_1,x_2,\ldots,x_m)^T : |x_i| < l_i,\ i = 1,2,\ldots,m\}$ be an open bounded domain in $\mathbb{R}^m$ with smooth boundary $\partial\Omega$. Let $C_{rd}(\mathbb{T}\times\Omega,\mathbb{R}^n)$ be the set consisting of all the vector functions $u(t,x)$ which are rd-continuous with respect to $t \in \mathbb{T}$ and continuous with respect to $x \in \mathbb{R}^m$, respectively. For every $t \in \mathbb{T}$ and $x \in \Omega$, we define the set $C_t = \{u(t,\cdot) : u \in C(\Omega,\mathbb{R}^n)\}$. Then for every $t \in \mathbb{T}$, C_t is a Banach space with the norm $\|u(t,\cdot)\| = (\sum_i^n \|u_i(t,\cdot)\|_2^2)^{1/2}$, where $u(t,x) = (u_1(t,x),u_2(t,x),\ldots,u_n(t,x))^T$, $\|u_i(t,\cdot)\|_2 = (\int_\Omega |u_i(t,x)|^2 \mathrm{d}x)^{1/2}$, $i = 1,2,\ldots,n$. Obviously, $C(\Omega,\mathbb{R}^n)$ is a Banach space equipped with the norm $\|\varphi\|_0 = (\sum_i^n \|\varphi_i\|_\tau^2)^{1/2}$, where $\varphi(x) = (\varphi_1(x),\varphi_2(x),\ldots,\varphi_n(x))^T \in C(\Omega,\mathbb{R}^n)$, $\|\varphi_i\|_\tau = (\int_\Omega |\varphi_i(x)|^2 \mathrm{d}x)^{1/2}$, $i = 1,2,\ldots,n$.

Definition 2.9. A function $u : \mathbb{T} \times \overline{\Omega} \to \mathbb{R}^n$ is called a solution of (1.1)–(1.3) if and only if u satisfies (1.1), initial value conditions (1.2) and Dirichlet boundary conditions (1.3).

Definition 2.10. A constant vector $u^* = (u_1^*, u_2^*, \ldots, u_n^*)^T$ is said to be an equilibrium solution to (1.1)–(1.3), if it satisfies $-b_i u_i^* + \sum_{j=1}^n c_{ij} f_j(u_j^*) + I_i = 0$, $i = 1, 2, \ldots, n$.

Definition 2.11. The equilibrium solution $u^* = (u_1^*, u_2^*, \ldots, u_n^*)^T$ of recurrent neural network (1.1)–(1.3) is said to be globally exponentially stable if there exists a positive constant $\alpha \in \mathcal{R}^+$ and $M \geq 1$ such that every solution $u(t,x) = (u_1(t,x), u_2(t,x), \ldots, u_n(t,x))^T$ of (1.1)–(1.3) satisfies

$$\|u(t,\cdot) - u^*\| \leq M e_{\ominus\alpha}(t,0), \quad t \in \mathbb{T}^+. \tag{2.10}$$

Definition 2.12 (Lakshmikantham and Vatsala [25]). For each $t \in \mathbb{T}$, let N be a neighborhood of t. Then, for $V \in C_{rd}[\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^+]$, define $D^+V^\Delta(t, x(t))$ to mean that, given $\varepsilon > 0$, there exists a right neighborhood $N_\varepsilon \cap N$ of t such that

$$\frac{1}{\mu(t,s)}\left[V(\sigma(t), x(\sigma(t))) - V(s, x(\sigma(t))) - \mu(s,t) f(t, x(t))\right] < D^+V^\Delta(t, x(t)) + \varepsilon \tag{2.11}$$

for each $s \in N_\varepsilon, s > t$, where $\mu(t,s) \equiv \sigma(t) - s$. If t is rs and $V(t, x(t))$ is continuous at t, this reduces to

$$D^+V^\Delta(t, x(t)) = \frac{V(\sigma(t), x(\sigma(t))) - V(t, x(\sigma(t)))}{\sigma(t) - t}. \tag{2.12}$$

3. Main Results

In this section, we will consider the existence, uniqueness, and global exponential stability of equilibrium of (1.1)–(1.3). To proceed, we need the following lemma.

Lemma 3.1 (see [14]). *Let Ω be a cube $|x_i| < l_i$ $(i = 1, 2, \ldots, m)$ and let $h(x)$ be a real-valued function belonging to $C^1(\Omega)$ which vanish on the boundary $\partial\Omega$ of Ω, that is, $h(x)|_{\partial\Omega} = 0$. Then,*

$$\int_\Omega h^2(x)dx \leq l_i^2 \int_\Omega \left|\frac{\partial h}{\partial x_i}\right|^2 dx. \tag{3.1}$$

Throughout this paper, we always assume that

(H_1) $f_j(\cdot)$ is Lipschitz continuous, that is, there exists constant $F_j > 0$ such that $|f_j(\xi) - f_j(\eta)| \leq F_j|\xi - \eta|$, for any $\xi, \eta \in \mathbb{R}$, $j = 1, 2, \ldots, n$;

(H_2) $W = B - C^+F$ is an M-matrix, where $B = \operatorname{diag}(b_1, b_2, \ldots, b_n)$, $C^+ = (|c_{ij}|)_{n\times n}$, $F = \operatorname{diag}(F_1, F_2, \ldots, F_n)$,

(H_3) $-2b_i - \sum_{k=1}^m (2a_{ik}/l_k^2) + \sum_{j=1}^n (|c_{ij}|F_j + |c_{ji}|F_i) < 0$, $i = 1, 2, \ldots, n$.

Theorem 3.2. *Assume that (H_1) and (H_2) hold, then* (1.1)–(1.3) *has at least one equilibrium solution $u^* = (u_1^*, u_2^*, \ldots, u_n^*)^T$.*

Proof. By (H_1), it follows that

$$\left|f_j(s)\right| \le F_j|s| + \left|f_j(0)\right|, \quad j = 1,2,\ldots,n, \ \forall s \in R. \tag{3.2}$$

Let

$$h(u,I) = Bu - Cf(u) - I = 0, \tag{3.3}$$

where $f(u) = (f_1(u_1), f_2(u_2), \ldots, f_n(u_n))^T$. It is obvious that solutions to (3.3) are equilibria of (1.1)–(1.3). Let us define homotopic mapping

$$H(u,\lambda) = \lambda h(u,I) + (1-\lambda)u, \quad \lambda \in J = [0,1]. \tag{3.4}$$

We have

$$\begin{aligned} |H_i(u,\lambda)| &= \left|\lambda\left[b_i u_i - \left(\sum_{j=1}^{n} c_{ij} f_j(u_j) + I_i\right)\right] + (1-\lambda)u_i\right| \\ &\ge [1+\lambda(b_i-1)]|u_i| - \lambda\sum_{j=1}^{n}\left|c_{ij}\right|F_j\left|u_j\right| \\ &\quad - \lambda\left[|I_i| + \sum_{j=1}^{n}\left|c_{ij}\right|\left|f_j(0)\right|\right], \quad i = 1,2,\ldots,n. \end{aligned} \tag{3.5}$$

That is,

$$\begin{aligned} H^+ &\ge [E+\lambda(B-E),u]^+ - \lambda C^+F[u]^+ - \lambda\left(I^+ + C^+f^+(0)\right) \\ &= (1-\lambda)[u]^+ + \lambda(B-C^+F)[u]^+ - \lambda\left(I^+ + C^+f^+(0)\right), \end{aligned} \tag{3.6}$$

where $H^+ = (|H_1|,|H_2|,\ldots,|H_n|)^T$, $[u]^+ = (|u_1|,|u_2|,\ldots,|u_n|)^T$, $I^+ = (|I_1|,|I_2|,\ldots,|I_n|)^T$, $f^+(0) = (|f_1(0)|,|f_2(0)|,\ldots,|f_n(0)|)^T$, and E is an identity matrix.

Since $W = B - C^+F$ is an M-matrix, we have $W^{-1} = (B-C^+F)^{-1} \ge 0$ (nonnegative matrix) and there exists a $Q = (Q_1,Q_2,\ldots,Q_n)^T > 0$ $(Q_i > 0,\ i = 1,2,\ldots,n)$ such that $(B - C^+F)Q > 0$. Let

$$U(R_0) = \left\{u : [u]^+ \le R_0 = Q + (B-C^+F)^{-1}\left(I^+ + C^+f^+(0)\right)\right\}. \tag{3.7}$$

Then, $U(R_0)$ is not empty and it follows from (3.7) that for any $u \in \partial U(R_0)$ (boundary of $U(R_0)$),

$$\begin{aligned} H^+ &\ge (1-\lambda)[u]^+ + \lambda(B-C^+F)\left[[u]^+ - (B-C^+F)^{-1}\left(I^+ + C^+f^+(0)\right)\right] \\ &= (1-\lambda)[u]^+\lambda(B-C^+F)Q > 0, \quad \lambda \in [0,1], \end{aligned} \tag{3.8}$$

which implies that $H(u,\lambda)\neq 0$, for $u\in\partial U(R_0)$, $\lambda\in[0,1]$. So, from homotopy invariance theorem, we have

$$\deg(h,U(R_0),0)=\deg(H(u,1),U(R_0),0)=\deg(H(u,0),U(R_0),0)=1, \tag{3.9}$$

where $\deg(h,U(R_0),0)$ denotes topological degree. By topological degree theory, we can conclude that (3.3) has at least one solution in $U(R_0)$. That is, (1.1)–(1.3) has at least an equilibrium solution $u^*=(u_1^*,u_2^*,\ldots,u_n^*)^T$. This completes the proof. □

Theorem 3.3. *Assume that* (H_1)–(H_3) *hold, then the reaction-diffusion recurrent neural network* (1.1)–(1.3) *has a unique equilibrium solution* $u^*=(u_1^*,u_2^*,\ldots,u_n^*)^T$ *which is globally exponentially stable.*

Proof. The existence of equilibrium solutions for (1.1)–(1.3) follows from Theorem 3.2. Now we only need to prove the uniqueness and global exponential stability of equilibrium solutions for (1.1)–(1.3).

Suppose that $u(t,x)$ and $v(t,x)$ are two arbitrary solutions of (1.1)–(1.3) with conditions $\phi^u(x),\phi^v(x)\in C(\Omega,\mathbb{R}^n)$, and define $z(t,x)=u(t,x)-v(t,x)$, $\phi^z(x)=\phi^u(x)-\phi^v(x)$, then $z(t,x)$ is governed by the following equations:

$$z_i^{\Delta}(t,x)=\sum_{k=1}^{m}\frac{\partial}{\partial x_k}\left(a_{ik}\frac{\partial z_i(t,x)}{\partial x_k}\right)-b_iz_i(t,x)+\sum_{j=1}^{n}c_{ij}\big(f_j\big(u_j(t,x)\big)-f_j\big(v_j(t,x)\big)\big), \tag{3.10}$$

where $i=1,2,\ldots,n$. Calculating the delta derivation of $\|z_i(t,\cdot)\|_2^2$ along the solution of (3.10), we have, for $i=1,2,\ldots,n$,

$$\begin{aligned}
\left(\|z_i(t,\cdot)\|_2^2\right)^{\Delta}&=\int_\Omega\left((z_i(t,x))^2\right)^{\Delta}\mathrm{d}x=\int_\Omega(z_i(t,x)+z_i(\sigma(t),x))(z_i(t,x))^{\Delta}\mathrm{d}x\\
&=\int_\Omega\left(2z_i(t,x)+\mu(t)(z_i(t,x))^{\Delta}\right)(z_i(t,x))^{\Delta}\mathrm{d}x\\
&=2\int_\Omega z_i(t,x)(z_i(t,x))^{\Delta}\mathrm{d}x+\mu(t)\int_\Omega\left((z_i(t,x))^{\Delta}\right)^2\mathrm{d}x\\
&=2\sum_{k=1}^{m}\int_\Omega z_i(t,x)\frac{\partial}{\partial x_k}\left(a_{ik}\frac{\partial z_i(t,x)}{\partial x_k}\right)\mathrm{d}x-2b_i\int_\Omega(z_i(t,x))^2\mathrm{d}x\\
&\quad+2\sum_{j=1}^{n}c_{ij}\int_\Omega z_i(t,x)\big(f_j\big(u_j(t,x)\big)-f_j\big(v_j(t,x)\big)\big)\mathrm{d}x\\
&\quad+\mu(t)\left\|(z_i(t,\cdot))^{\Delta}\right\|_2^2.
\end{aligned} \tag{3.11}$$

From Green formula [26], Dirichlet boundary condition, and Lemma 3.1, we have, for $i = 1, 2, \ldots, n$,

$$\begin{aligned}\sum_{k=1}^{m}\int_{\Omega} z_i(t,x)\frac{\partial}{\partial x_k}\left(a_{ik}\frac{\partial z_i(t,x)}{\partial x_k}\right)\mathrm{d}x &= -\sum_{k=1}^{m}\int_{\Omega} a_{ik}\left(\frac{\partial z_i(t,x)}{\partial x_k}\right)^2\mathrm{d}x \\ &\leq -\sum_{k=1}^{m}\int_{\Omega}\frac{a_{ik}}{l_k^2}(z_i(t,x))^2\mathrm{d}x.\end{aligned} \tag{3.12}$$

By (3.11), (3.12), condition (H_1), and Holder inequality, we get

$$\begin{aligned}\left(\|z_i(t,\cdot)\|_2^2\right)^{\Delta} &\leq -\sum_{k=1}^{m}\frac{2a_{ik}}{l_k^2}\|z_i(t,\cdot)\|_2^2 - 2b_i\|z_i(t,\cdot)\|_2^2 \\ &\quad + 2\sum_{\mathrm{j}=1}^{n}|c_{ij}|F_j\|z_i(t,\cdot)\|_2\|z_j(t,\cdot)\|_2 + \mu(t)\left\|(z_i(t,\cdot))^{\Delta}\right\|_2^2 \\ &= -\sum_{k=1}^{m}\frac{2a_{ik}}{l_k^2}\|z_i(t,\cdot)\|_2^2 - 2b_i\|z_i(t,\cdot)\|_2^2 \\ &\quad + 2\sum_{j=1}^{n}|c_{ij}|F_j\|z_i(t,\cdot)\|_2\|z_j(t,\cdot)\|_2 + \mu(t)q(t)\|z_i(t,\cdot)\|_2^2,\end{aligned} \tag{3.13}$$

where $\|(z_i(t,\cdot))^{\Delta}\|_2^2 = q(t)\|z_i(t,\cdot)\|_2^2$, $q(t) \geq 0$, $i = 1, 2, \ldots, n$.

If condition (H_3) holds, we can always choose a positive number $\sigma > 0$ (may be very small) such that for $i = 1, 2, \ldots, n$,

$$-2b_i - \sum_{k=1}^{m}\frac{2a_{ik}}{l_k^2} + \sum_{j=1}^{n}\left(|c_{ij}|F_j + |c_{ji}|F_i\right) + \sigma < 0. \tag{3.14}$$

Let us consider functions

$$\begin{aligned}q_i(y_i) &= y_i \oplus y_i - 2b_i - \sum_{k=1}^{m}\frac{2a_{ik}}{l_k^2} + \sum_{j=1}^{n}\left(|c_{ij}|F_j + |c_{ji}|F_i\right) \\ &\quad + \frac{w(y_i)\mu(t)q(t)\max\left\{e_{y_i\oplus y_i}(\sigma(t),0), e_{(w(y_i)-1)\mu(t)q(t)\|z_i(t,\cdot)\|_2^2}(t,0)\right\}}{e_{y_i\oplus y_i}(\sigma(t),0)},\end{aligned} \tag{3.15}$$

where $w(y_i) = \int_0^{y_i}(e^{y_i-s}/(y_i-s)^2)\mathrm{d}s$, $i = 1, 2, \ldots, n$. From (3.15), we obtain that $q_i(0) < -\sigma < 0$ and $q_i(y_i)$ is continuous for $y_i \in [0, +\infty)$, furthermore, $q_i(y_i) \to +\infty$ as $y_i \to +\infty$, thus there

exist constant $\varepsilon_i \in (0,+\infty)$ such that $q_i(\varepsilon_i^*) = 0$ and $q_i(\varepsilon_i) < 0$, for $\varepsilon_i \in (0,\varepsilon_i^*)$. By choosing $\varepsilon = \min_{1\le i\le n}\{\varepsilon_i\}$, obviously $\varepsilon > 0$, we have, for $i = 1,2,\ldots,n$,

$$\begin{aligned} q_i(\varepsilon) &= \varepsilon \oplus \varepsilon - 2b_i - \sum_{k=1}^{m}\frac{2a_{ik}}{l_k^2} + \sum_{j=1}^{n}\left(\left|c_{ij}\right|F_j + \left|c_{ji}\right|F_i\right) \\ &\quad + \frac{w(\varepsilon)\mu(t)q(t)\max\left\{e_{\varepsilon\oplus\varepsilon}(\sigma(t),0), e_{(w(\varepsilon)-1)\mu(t)q(t)\|z_i(t,\cdot)\|_2^2}(t,0)\right\}}{e_{\varepsilon\oplus\varepsilon}(\sigma(t),0)} \le 0. \end{aligned} \tag{3.16}$$

Now consider the Lyapunov functional

$$V(t,z(t)) = \sum_{i=1}^{n}\left\{e_{\varepsilon\oplus\varepsilon}(t,0)\|z_i(t,\cdot)\|_2^2 + e_{(w(\varepsilon)-1)\mu(t)q(t)\|z_i(t,\cdot)\|_2^2}(t,0)\right\}. \tag{3.17}$$

Calculating the delta derivatives of $V(t,z(t))$ along the solution of (3.10) and noting that $d/dz[e_z(t,s)] = (\int_s^t (1/(1+\mu(\tau)z))\Delta\tau)e_z(t,s) > 0$ if and only if $z \in \mathcal{R}^+$(i.e., $e_z(t,s)$ is increasing with respect to z if and only if $z \in \mathcal{R}^+$), we have

$$\begin{aligned} &D^+V^\Delta(t,x(t)) \\ &= \sum_{i=1}^{n}\left\{(\varepsilon\oplus\varepsilon)e_{\varepsilon\oplus\varepsilon}(t,0)\|z_i(t,\cdot)\|_2^2 + e_{\varepsilon\oplus\varepsilon}(\sigma(t),0)\left(\|z_i(t,\cdot)\|_2^2\right)^\Delta\right. \\ &\qquad \left. +(w(\varepsilon)-1)\mu(t)q(t)\|z_i(t,\cdot)\|_2^2 e_{(w(\varepsilon)-1)\mu(t)q(t)\|z_i(t,\cdot)\|_2^2}(t,0)\right\} \\ &\le \sum_{i=1}^{n}\left\{(\varepsilon\oplus\varepsilon)e_{\varepsilon\oplus\varepsilon}(t,0)\|z_i(t,\cdot)\|_2^2 + e_{\varepsilon\oplus\varepsilon}(\sigma(t),0)\right. \\ &\qquad \times\left(-\sum_{k=1}^{m}\frac{2a_{ik}}{l_k^2}\|z_i(t,\cdot)\|_2^2 - 2b_i\|z_i(t,\cdot)\|_2^2 + 2\sum_{j=1}^{n}\left|c_{ij}\right|F_j\|z_i(t,\cdot)\|_2\left\|z_j(t,\cdot)\right\|_2 + \mu(t)q(t)\|z_i(t,\cdot)\|_2^2\right) \\ &\qquad \left. +(w(\varepsilon)-1)\mu(t)q(t)\|z_i(t,\cdot)\|_2^2 e_{(w(\varepsilon)-1)\mu(t)q(t)\|z_i(t,\cdot)\|_2^2}(t,0)\right\} \\ &\le e_{\varepsilon\oplus\varepsilon}(\sigma(t),0)\sum_{i=1}^{n}\left\{(\varepsilon\oplus\varepsilon)\|z_i(t,\cdot)\|_2^2 - \sum_{k=1}^{m}\frac{2a_{ik}}{l_k^2}\|z_i(t,\cdot)\|_2^2 - 2b_i\|z_i(t,\cdot)\|_2^2\right. \\ &\qquad + \sum_{j=1}^{n}\left|c_{ij}\right|F_j\left(\|z_i(t,\cdot)\|_2^2 + \left\|z_j(t,\cdot)\right\|_2^2\right) \\ &\qquad \left. + \frac{\max\left\{e_{\varepsilon\oplus\varepsilon}(\sigma(t),0), e_{(w(\varepsilon)-1)\mu(t)q(t)\|z_i(t,\cdot)\|_2^2}(t,0)\right\}w(\varepsilon)\mu(t)q(t)}{e_{\varepsilon\oplus\varepsilon}(\sigma(t),0)}\|z_i(t,\cdot)\|_2^2)\right\} \end{aligned}$$

$$\leq e_{\varepsilon\oplus\varepsilon}(\sigma(t),0)\sum_{i=1}^{n}\|z_i(t,\cdot)\|_2^2\left\{(\varepsilon\oplus\varepsilon)-\sum_{k=1}^{m}\frac{2a_{ik}}{l_k^2}+\sum_{j=1}^{n}\left(|c_{ij}|F_j+|c_{ji}|F_i\right)\right.$$
$$\left.+\frac{\max\left\{e_{\varepsilon\oplus\varepsilon}(\sigma(t),0),e_{(w(\varepsilon)-1)\mu(t)q(t)\|z_i(t,\cdot)\|_2^2}(t,0)\right\}w(\varepsilon)\mu(t)q(t)}{e_{\varepsilon\oplus\varepsilon}(\sigma(t),0)}\right\}\leq 0. \tag{3.18}$$

From (3.17) and (3.18), we have, for $t>0$,

$$\begin{aligned} e_{\varepsilon\oplus\varepsilon}(t,0)\|z(t,\cdot)\|^2 &= e_{\varepsilon\oplus\varepsilon}(t,0)\sum_{i=1}^{n}\|z_i(t,\cdot)\|_2^2\leq V(t,z(\mathrm{t}))\leq V(0,z(0)) \\ &= \sum_{i=1}^{n}\left\{\|z_i(0,\cdot)\|_2^2+1\right\}=\sum_{i=1}^{n}\left\{\|\phi_i^z\|_\tau^2+1\right\}=\|\phi^z\|_0^2+n, \end{aligned} \tag{3.19}$$

which implies that

$$\|z(t,\cdot)\|\leq Me_{\ominus\varepsilon}(t,0), \tag{3.20}$$

where $M=\sqrt{\|\phi^z\|_0^2+n}>1$.

Let $u^*=(u_1^*,u_2^*,\ldots,u_n^*)^T$ and $u^{**}=(u_1^{**},u_2^{**},\ldots,u_n^{**})^T$ be two arbitrary equilibrium solutions of system (1.1)–(1.3). According to (3.20), we get $\|u^*-u^{**}\|\leq Me_{\ominus\varepsilon}(t,0)\to 0$ $(t\to+\infty)$, here $M=\sqrt{\|u^*-u^{**}\|_0^2+n}>1$. It follows that $u^*=u^{**}$, that is, the equilibrium solution of (1.1)–(1.3) is unique.

Let $u(t,x)=(u_1(t,x),u_2(t,x),\ldots,u_n(t,x))^T$ and $u^*=(u_1^*,u_2^*,\ldots,u_n^*)^T$ be arbitrary solutions and an unique equilibrium solution of (1.1)–(1.3), respectively. In the light of (3.20), we obtain $\|u(t,x)-u^*\|\leq Me_{\ominus\varepsilon}(t,0)$, here $M=\sqrt{\|\phi-u^*\|_0^2+n}>1$. Thus, by Definition 2.11, we obtain the global exponential stability of unique equilibrium solution of (1.1)–(1.3). The proof is complete. □

4. An Illustrative Example

Example 4.1. Consider the following reaction-diffusion recurrent neural network with Dirichlet boundary conditions on time scales:

$$\begin{gathered} u_i^{\Delta}(t,x)=\sum_{k=1}^{m}\frac{\partial}{\partial x_k}\left(a_{ik}\frac{\partial u_i}{\partial x_k}\right)-b_iu_i(t,x)+\sum_{j=1}^{n}c_{ij}f_j\left(u_j(\mathrm{t},x)\right)+I_i,\quad (t,x)\in\ \mathbb{T}\times\Omega, \\ u_i(0,x)=\phi_i(x),\qquad x\in\Omega, \\ u_i(t,x)=0,\quad (t,x)\in[0,+\infty)_{\mathbb{T}}\times\partial\Omega, \end{gathered} \tag{4.1}$$

where $\mathbb{T}\subset\mathbb{R}$ is a time scale and $\mathbb{T}\cap[0,+\infty)=[0,+\infty)_{\mathbb{T}}$ is unbounded, $f_1(v)=f_2(v)=(e^v-e^{-v})/(e^v+e^{-v})$, $\Omega=\{x:|x_i|<1,i=1,2\}$, and $I=(I_1,I_2)$ is the constant input vector.

Obviously, $f_j(v)$ satisfies the Lipschitz condition with $F_j = 1$. Let $a_{11} = 0.5$, $a_{12} = 0.5$, $a_{21} = 0.3$, $a_{22} = 0.7$, $b_1 = 1.5$, $b_2 = 0.4$, $c_{11} = 0.5$, $c_{12} = 0.4$, $c_{21} = 0.3$, and $c_{22} = 0.2$. By simple calculation, we have

$$B - C^{+}F = \begin{pmatrix} 1.5 & 0 \\ 0 & 0.4 \end{pmatrix} - \begin{pmatrix} 0.5 & 0.4 \\ 0.3 & 0.2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -0.4 \\ -0.3 & 0.2 \end{pmatrix},$$

$$-2b_1 - \sum_{k=1}^{2} \frac{2a_{1k}}{l_k^2} + \sum_{j=1}^{2} \left(|c_{1j}| F_j + |c_{j1}| F_1 \right) = -3.3 < 0, \tag{4.2}$$

$$-2b_2 - \sum_{k=1}^{2} \frac{2a_{2k}}{l_k^2} + \sum_{j=1}^{2} \left(|c_{2j}| F_j + |c_{j2}| F_2 \right) = -0.9 < 0,$$

that is, (H_1), (H_2), and (H_3) hold. Hence, it follows from Theorem 3.3 that (4.1) has one unique equilibrium solution which is globally exponentially stable.

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