Review Article

A Note on the Modified q -Bernstein Polynomials

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We propose the modified q -Bernstein polynomials of degree n which are different q -Bernstein polynomials of Phillips (1997). From these modified q -Bernstein polynomials of degree n , we derive some recurrence formulae for the modified q -Bernstein polynomials.

1. Introduction

Let $C[0,1]$ denote the set of continuous function on $[0,1]$. For $f \in C[0,1]$, Bernstein introduced the following well-known linear positive operators in [1]:

$$B_n(f : x) := \sum_{k=0}^n f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k} = \sum_{k=0}^n f\left(\frac{k}{n}\right) B_{k,n}(x), \quad (1.1)$$

where $\binom{n}{k} = n(n-1) \cdots (n-k+1)/k!$. Here $B_n(f : x)$ is called the *Bernstein operator of order n for f* . For $k, n \in \mathbb{Z}_+$, the *Bernstein polynomial of degree n* is defined by

$$B_{k,n}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad (1.2)$$

where $x \in [0, 1]$. For example,

$$\begin{aligned} B_{0,1}(x) &= 1 - x, & B_{1,1}(x) &= x, \\ B_{0,2}(x) &= (1 - x)^2, & B_{1,2}(x) &= 2x(1 - x), & B_{2,2}(x) &= x^2, \dots \end{aligned} \quad (1.3)$$

Also, $B_{k,n}(x) = 0$, for $k > n$, because $\binom{n}{k} = 0$.

Some people have studied the Bernstein polynomials in the area of approximation theory (see [2] through [3]). Note that for $k \in \mathbb{Z}_+$ and $x \in [0, 1]$,

$$\begin{aligned} \frac{t^k e^{(1-x)t} x^k}{k!} &= \frac{x^k}{k!} \left(t^k \sum_{n=0}^{\infty} \frac{(1-x)^n t^n}{n!} \right) \\ &= \frac{x^k}{k!} \sum_{n=0}^{\infty} \frac{(1-x)^n (n+1) \cdots (n+k)}{(n+k)!} t^{n+k} \\ &= \sum_{n=k}^{\infty} \left(\binom{n}{k} x^k (1-x)^{n-k} \right) \frac{t^n}{n!} \\ &= \sum_{n=k}^{\infty} B_{k,n}(x) \frac{t^n}{n!}. \end{aligned} \quad (1.4)$$

Because $B_{k,0}(x) = B_{k,1}(x) = \cdots = B_{k,k-1}(x) = 0$, we obtain the generating function for $B_{k,n}(x)$ as follows:

$$F^{(k)}(t, x) := \frac{t^k e^{(1-x)t} x^k}{k!} = \sum_{n=0}^{\infty} B_{k,n}(x) \frac{t^n}{n!} \quad (1.5)$$

(see [4, 5]), where $k \in \mathbb{Z}_+$ and $x \in [0, 1]$. Notice that

$$B_{k,n}(x) = \begin{cases} \binom{n}{k} x^k (1-x)^{n-k} & \text{if } n \geq k, \\ 0 & \text{if } n < k, \end{cases} \quad (1.6)$$

for $n, k \in \mathbb{Z}_+$ (see [2]).

Let $0 < q < 1$. Define the q -number of x by

$$[x]_q := \frac{1 - q^x}{1 - q}. \quad (1.7)$$

See [2] through [3] for details and related facts. Note that $\lim_{q \rightarrow 1} [x]_q = x$. In [6], Phillips proposed a generalization of the classical Bernstein polynomials based on q -integers. In the last decade some new generalizations of well-known positive linear operators, based on q -integers were introduced and studied by several authors (see [1–13]). Recently, Simsek

and Acikgoz have also studied the q -extension of Bernstein-type polynomials [5]. Their q -Bernstein-type polynomials are given by

$$Y_n(k; x : q) = \binom{n}{k} \frac{(-1)^k k!}{(1-q)^{n-k}} \sum_{m,l=0}^{\infty} \sum_{j=0}^{n-k} \binom{k+l-1}{l} \binom{n-k}{k} \times \left(\frac{(-1)^j q^{l+j(1-x)} S(m, k) (x \ln q)^m}{m!} \right), \quad (1.8)$$

where $S(m, k)$ are the second-kind stirling number. In [5], we can find some interesting formulae related to q -extension of Bernstein polynomials which are different q -Bernstein polynomials of Phillips. In the conference of Jangjeon Mathematical Society which was held in IRAN (on Feb.2010), Acikgoz and Arci has introduced several-type Bernstein polynomials (see [2]). The Acikgoz paper [2] announced in the conference is actually what motivated us to write this paper. In this paper, we considered the q -extension of Bernstein polynomials which were introduced by Acikgoz at the conference of Jangjeon Mathematical Society on Feb. 2010. First, we consider the q -extension of the generating function of Bernstein polynomials in (1.5). Indeed, this generating function is also treated by Simsek and Acikgoz in a previous paper (see [5]). From this q -extension of the generating function for the Bernstein polynomials, we propose the modified q -Bernstein polynomials of degree n which are different q -Bernstein polynomials of Phillips. By using the properties of the modified q -Bernstein polynomials, we obtain some recurrence formulae for the modified q -Bernstein polynomials of degree n .

2. The Modified q -Bernstein Polynomials

For $0 < q < 1$, consider the q -extension of (1.5) as follows:

$$\begin{aligned} F_q^{(k)}(t, x) &:= \frac{t^k e^{[1-x]_q t} [x]_q^k}{k!} \\ &= \frac{[x]_q^k}{k!} \sum_{n=0}^{\infty} \frac{[1-x]_q^n}{n!} t^{n+k} \\ &= \sum_{n=k}^{\infty} \binom{n}{k} [x]_q^k [1-x]_q^{n-k} \frac{t^n}{n!}, \end{aligned} \quad (2.1)$$

where $k, n \in \mathbb{Z}_+$ and $x \in [0, 1]$. Note that $\lim_{q \rightarrow 1} F_q^{(k)}(t, x) = F^{(k)}(t, x)$. We define the *modified q -Bernstein polynomials* as follows:

$$F_q^{(k)}(t, x) = \frac{t^k e^{[1-x]_q t} [x]_q^k}{k!} = \sum_{n=0}^{\infty} B_{k,n}(x, q) \frac{t^n}{n!}, \quad (2.2)$$

where $k, n \in \mathbb{Z}_+$ and $x \in [0, 1]$.

Remark. This generating function is also introduced by Simsek and Acikgoz in a previous paper (see [5]).

By comparing the coefficients of (2.1) and (2.2), we obtain the following theorem.

Theorem 2.1. For $k, n \in \mathbb{Z}_+$ and $x \in [0, 1]$,

$$B_{k,n}(x, q) = \begin{cases} \binom{n}{k} [x]_q^k [1-x]_q^{n-k}, & \text{if } n \geq k \\ 0, & \text{if } n < k. \end{cases} \quad (2.3)$$

For $0 \leq k \leq n$, we have

$$\begin{aligned} & [1-x]_q B_{k,n-1}(x, q) + [x]_q B_{k-1,n-1}(x, q) \\ &= [1-x]_q \binom{n-1}{k} [x]_q^k [1-x]_q^{n-1-k} + [x]_q \binom{n-1}{k-1} [x]_q^{k-1} [1-x]_q^{n-k} \\ &= \binom{n-1}{k} [x]_q^k [1-x]_q^{n-k} + \binom{n-1}{k-1} [x]_q^k [1-x]_q^{n-k} \\ &= \binom{n}{k} [x]_q^k [1-x]_q^{n-k}, \end{aligned} \quad (2.4)$$

and the derivatives of the modified q -Bernstein polynomials of degree n are also polynomials of degree $n-1$, that is,

$$\begin{aligned} \frac{d}{dx} B_{k,n}(x, q) &= \binom{n}{k} k [x]_q^{k-1} [1-x]_q^{n-k} \frac{\ln q}{q-1} q^x + \binom{n}{k} [x]_q^k (n-k) [1-x]_q^{n-k-1} \left(\frac{-\ln q}{q-1} \right) q^{1-x} \\ &= \frac{\ln q}{q-1} \left\{ \binom{n}{k} k [x]_q^{k-1} [1-x]_q^{n-k} q^x - \binom{n}{k} [x]_q^k (n-k) [1-x]_q^{n-k-1} q^{1-x} \right\} \\ &= n \left(q^x B_{k-1,n-1}(x, q) - q^{1-x} B_{k,n-1}(x, q) \right) \frac{\ln q}{q-1}. \end{aligned} \quad (2.5)$$

Therefore, we obtain the following recurrence formulae.

Theorem 2.2 (recurrence formulae for $B_{k,n}(x, q)$). For $k, n \in \mathbb{Z}_+$ and for $x \in [0, 1]$,

$$\begin{aligned} & [1-x]_q B_{k,n-1}(x, q) + [x]_q B_{k-1,n-1}(x, q) = B_{k,n}(x, q), \\ & \frac{d}{dx} B_{k,n}(x, q) = n \left(q^x B_{k-1,n-1}(x, q) - q^{1-x} B_{k,n-1}(x, q) \right) \frac{\ln q}{q-1}. \end{aligned} \quad (2.6)$$

Let f be a continuous function on $[0, 1]$. Then the *modified q -Bernstein operator of order n for f* is defined by

$$B_{n,q}(f : x) := \sum_{k=0}^n f\left(\frac{k}{n}\right) B_{k,n}(x, q), \quad (2.7)$$

where $0 \leq x \leq 1$, $n \in \mathbb{Z}_+$. We get from Theorem 2.1 and (2.7) that for $f(x) = x$,

$$\begin{aligned} B_{n,q}(f : x) &= \sum_{k=0}^n f\left(\frac{k}{n}\right) \binom{n}{k} [x]_q^k [1-x]_q^{n-k} \\ &= [x]_q \left(1 - [1-x]_q [x]_q (q-1)\right)^{n-1} \\ &= f\left([x]_q\right) \left(1 + (1-q)[x]_q [1-x]_q\right)^{n-1}. \end{aligned} \quad (2.8)$$

We also see from Theorem 2.1 that

$$\begin{aligned} B_{n,q}(1 : x) &= \sum_{k=0}^n B_{k,n}(x, q) \\ &= \sum_{k=0}^n \binom{n}{k} [x]_q^k [1-x]_q^{n-k} \\ &= \sum_{k=0}^n \binom{n}{k} [x]_q^k \left(1 - q^{1-x} [x]_q\right)^{n-k} \\ &= \left(1 + (1-q)[x]_q [1-x]_q\right)^n. \end{aligned} \quad (2.9)$$

The modified q -Bernstein polynomials are symmetric polynomials in the following sense:

$$B_{n-k,n}(1-x, q) = \binom{n}{n-k} [1-x]_q^{n-k} [x]_q^k = B_{k,n}(x, q). \quad (2.10)$$

Therefore, we get the following theorem.

Theorem 2.3. For $k, n \in \mathbb{Z}_+$ and $x \in [0, 1]$,

$$\begin{aligned} B_{n-k,n}(1-x, q) &= B_{k,n}(x, q), \\ B_{n,q}(1 : x) &= \left(1 + (1-q)[x]_q [1-x]_q\right)^n. \end{aligned} \quad (2.11)$$

For $\zeta \in \mathbb{C}$, $x \in [0, 1]$ and for $n \in \mathbb{Z}_+$, consider

$$\frac{n!}{2\pi i} \oint_C \frac{([x]_q \zeta)^k}{k!} e^{([1-x]_q \zeta)} \frac{d\zeta}{\zeta^{n+1}}, \quad (2.12)$$

where C is a circle around the origin and integration is in the positive direction. We see from the definition of the modified q -Bernstein polynomials and the basic theory of complex analysis including Laurent series that

$$\oint_C \frac{([x]_q \zeta)^k}{k!} e^{[1-x]_q \zeta} \frac{d\zeta}{\zeta^{n+1}} = \sum_{m=0}^{\infty} \oint_C \frac{B_{k,m}(x, q) \zeta^m}{m!} \frac{d\zeta}{\zeta^{n+1}} = 2\pi i \left(\frac{B_{k,n}(x, q)}{n!} \right). \quad (2.13)$$

We get from (2.12) and (2.13) that

$$\frac{n!}{2\pi i} \oint_C \frac{([x]_q \zeta)^k}{k!} e^{[1-x]_q \zeta} \frac{d\zeta}{\zeta^{n+1}} = B_{k,n}(x, q), \quad (2.14)$$

$$\begin{aligned} \oint_C \frac{([x]_q \zeta)^k}{k!} e^{[1-x]_q \zeta} \frac{d\zeta}{\zeta^{n+1}} &= \frac{[x]_q^k}{k!} \sum_{m=0}^{\infty} \left(\frac{[1-x]_q^m}{m!} \oint_C \zeta^{m-n-1+k} d\zeta \right) \\ &= 2\pi i \left(\frac{[x]_q^k [1-x]_q^{n-k}}{k!(n-k)!} \right). \end{aligned} \quad (2.15)$$

We also get from (2.12) and (2.15) that

$$\frac{n!}{2\pi i} \oint_C \frac{([x]_q \zeta)^k}{k!} e^{([1-x]_q \zeta)} \frac{d\zeta}{\zeta^{n+1}} = \binom{n}{k} [x]_q^k [1-x]_q^{n-k}. \quad (2.16)$$

Therefore, we see from (2.14) and (2.16) that

$$B_{k,n}(x, q) = \binom{n}{k} [x]_q^k [1-x]_q^{n-k}. \quad (2.17)$$

Note that

$$\begin{aligned}
& \left(\frac{n-k}{n} \right) B_{k,n}(x, q) + \left(\frac{k+1}{n} \right) B_{k+1,n}(x, q) \\
&= \frac{(n-1)!}{k!(n-k-1)!} [x]_q^k [1-x]_q^{n-k} + \frac{(n-1)!}{k!(n-k-1)!} [x]_q^{k+1} [1-x]_q^{n-k-1} \\
&= \left([1-x]_q + [x]_q \right) B_{k,n-1}(x, q) \\
&= \left(1 + [x]_q \left(1 - q^{1-x} \right) \right) B_{k,n-1}(x, q) \\
&= \left(1 + (1-q)[x]_q [1-x]_q \right) B_{k,n-1}(x, q).
\end{aligned} \tag{2.18}$$

Therefore, we can write the modified q -Bernstein polynomials as a linear combination of polynomials of higher order as follows.

Theorem 2.4. For $k, n \in \mathbb{Z}_+$ and $x \in [0, 1]$,

$$\left(\frac{n+1-k}{n+1} \right) B_{k,n+1}(x, q) + \left(\frac{k+1}{n+1} \right) B_{k+1,n+1}(x, q) = \left(1 + (1-q)[x]_q [1-x]_q \right) B_{k,n}(x, q). \tag{2.19}$$

We easily see from (2.17) that for $n, k \in \mathbb{N}$,

$$\begin{aligned}
\left(\frac{n-k+1}{k} \right) \left(\frac{[x]_q}{[1-x]_q} \right) B_{k-1,n}(x, q) &= \left(\frac{n-k+1}{k} \right) \left(\frac{[x]_q}{[1-x]_q} \right) \binom{n}{k-1} [x]_q^{k-1} [1-x]_q^{n-k+1} \\
&= \frac{n!}{k!(n-k)!} [x]_q^k [1-x]_q^{n-k} \\
&= B_{k,n}(x, q).
\end{aligned} \tag{2.20}$$

Thus, the following corollary holds.

Corollary 2.5. For $n, k \in \mathbb{N}$ and $x \in [0, 1]$,

$$\left(\frac{n-k+1}{k} \right) \left(\frac{[x]_q}{[1-x]_q} \right) B_{k-1,n}(x, q) = B_{k,n}(x, q). \tag{2.21}$$

Note from the definition of the modified q -Bernstein polynomials and the binomial theorem that for $k, n \in \mathbb{Z}_+$,

$$\begin{aligned}
 B_{k,n}(x, q) &= \binom{n}{k} [x]_q^k [1-x]_q^{n-k} \\
 &= \binom{n}{k} [x]_q^k \left(1 - q^{1-x} [x]_q\right)^{n-k} \\
 &= \binom{n}{k} [x]_q^k \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l q^{l(1-x)} [x]_q^l \\
 &= \sum_{l=0}^{n-k} \binom{k+l}{k} \binom{n}{k+l} (-1)^l q^{l(1-x)} [x]_q^{l+k} \\
 &= \sum_{j=k}^n \binom{n}{k} \binom{n}{j} (-1)^{j-k} q^{(1-x)(j-k)} [x]_q^j.
 \end{aligned} \tag{2.22}$$

Therefore, we showed that the following theorem holds.

Theorem 2.6. For $k, n \in \mathbb{Z}_+$ and $x \in [0, 1]$,

$$B_{k,n}(x, q) = \sum_{j=k}^n \binom{j}{k} \binom{n}{j} (-1)^{j-k} q^{(1-x)(j-k)} [x]_q^j. \tag{2.23}$$

It is possible to write $[x]_q^k$ as a linear combination of the modified q -Bernstein polynomials by using the degree evaluation formulae and mathematical induction. We easily see from the property of the modified q -Bernstein polynomials that

$$\begin{aligned}
 \sum_{k=1}^n \left(\frac{k}{n}\right) B_{k,n}(x, q) &= \sum_{k=1}^n \binom{n-1}{k-1} [x]_q^k [1-x]_q^{n-k} \\
 &= \sum_{k=0}^{n-1} \binom{n-1}{k} [x]_q^{k+1} [1-x]_q^{n-1-k} \\
 &= [x]_q \left([x]_q + [1-x]_q\right)^{n-1},
 \end{aligned} \tag{2.24}$$

and that

$$\begin{aligned}
 \sum_{k=2}^n \left(\frac{\binom{k}{2}}{\binom{n}{2}}\right) B_{k,n}(x, q) &= \sum_{k=2}^n \binom{n-2}{k-2} [x]_q^k [1-x]_q^{n-k} \\
 &= \sum_{k=0}^{n-2} \binom{n-2}{k} [x]_q^{k+2} [1-x]_q^{n-2-k} \\
 &= [x]_q^2 \left([x]_q + [1-x]_q\right)^{n-2}.
 \end{aligned} \tag{2.25}$$

Continuing this process, we obtain

$$\sum_{k=j}^n \frac{\binom{k}{j}}{\binom{n}{j}} B_{k,n}(x, q) = [x]_q^j ([x]_q + [1-x]_q)^{n-j}, \quad (2.26)$$

for $j \in \mathbb{N}$. Therefore, we obtain the following theorem.

Theorem 2.7. For $n, j \in \mathbb{Z}_+$ and $x \in [0, 1]$,

$$\frac{1}{([1-x]_q + [x]_q)^{n-j}} \sum_{k=j}^n \frac{\binom{k}{j}}{\binom{n}{j}} B_{k,n}(x, q) = [x]_q^j. \quad (2.27)$$

For $k \in \mathbb{N}$, the *Bernoulli polynomial of order k* is defined by

$$\left(\frac{t}{e^t - 1}\right)^k e^{xt} = \underbrace{\left(\frac{t}{e^t - 1}\right) \times \cdots \times \left(\frac{t}{e^t - 1}\right)}_{k\text{-times}} e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x) \frac{t^n}{n!}, \quad (2.28)$$

and $B_n^{(k)} = B_n^{(k)}(0)$ are called the n th *Bernoulli numbers of order k* . It is well known that the *second kind stirling number* is defined by

$$\frac{(e^t - 1)^k}{k!} := \sum_{n=0}^{\infty} S(n, k) \frac{t^n}{n!}, \quad (2.29)$$

for $k \in \mathbb{N}$. We note from (2.2) that

$$\begin{aligned} \frac{([x]_q t)^k e^{[1-x]_q t}}{k!} &= \frac{[x]_q^k (e^t - 1)^k}{k!} \left(\frac{t}{e^t - 1}\right)^k e^{[1-x]_q t} \\ &= [x]_q^k \left(\sum_{m=0}^{\infty} S(m, k) \frac{t^m}{m!}\right) \left(\sum_{n=0}^{\infty} B_n^{(k)}([1-x]_q) \frac{t^n}{n!}\right) \\ &= [x]_q^k \sum_{l=0}^{\infty} \left(\sum_{n=0}^l \frac{B_n^{(k)}([1-x]_q) S(l-n, k) l!}{n! (l-n)!}\right) \frac{t^l}{l!}. \end{aligned} \quad (2.30)$$

We have from (2.2) and (2.30) that

$$B_{k,l}(x, q) = [x]_q^k \sum_{n=0}^l \binom{l}{n} B_n^{(k)}([1-x]_q) S(l-n, k), \quad (2.31)$$

and $B_{k,0}(x, q) = B_{k,1}(x, q) = \cdots = B_{k,k-1}(x, q) = 0$.

Remark. The Equations (2.30) and (2.31) are already known by Simsek and Acikgoz in a previous paper [5, page 7].

Let Δ be the *shift difference operator* defined by $\Delta f(x) = f(x+1) - f(x)$. We see from the iterative method that

$$\Delta^n f(0) = \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} f(k), \quad (2.32)$$

for $n \in \mathbb{N}$. We get from (2.29) and (2.32) that

$$\begin{aligned} \sum_{n=0}^{\infty} S(n, k) \frac{t^n}{n!} &= \frac{1}{k!} \sum_{l=0}^k \binom{k}{l} (-1)^{k-l} e^{lt} \\ &= \sum_{n=0}^{\infty} \left\{ \frac{1}{k!} \sum_{l=0}^k \binom{k}{l} (-1)^{k-l} l^n \right\} \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{\Delta^k 0^n}{k!} \frac{t^n}{n!}. \end{aligned} \quad (2.33)$$

By comparing the coefficients on both sides above, we have

$$S(n, k) = \frac{\Delta^k 0^n}{k!}, \quad (2.34)$$

for $n, k \in \mathbb{Z}_+$. Thus, we get from (2.31) and (2.34) that

$$B_{k,l}(x, q) = [x]_q^k \sum_{n=0}^l \binom{l}{n} B_n^{(k)}([1-x]_q) \frac{\Delta^k 0^{l-n}}{k!}. \quad (2.35)$$

Let $(Eh)(x) = h(x+1)$ be the *shift operator*. Then the *q-difference operator* is defined by

$$\Delta_q^n = \prod_{j=0}^{n-1} (E - q^j I), \quad (2.36)$$

where I is an identity operator (see [7] through [11]). For $f \in C[0, 1]$ and $n \in \mathbb{N}$, we have

$$\Delta_q^n f(0) = \sum_{k=0}^n \binom{n}{k}_q (-1)^k q^{\binom{n}{2}} f(n-k), \quad (2.37)$$

where $\binom{n}{k}_q$ is the *Gaussian binomial coefficient* defined by

$$\binom{x}{k}_q = \frac{[x]_q [x-1]_q \cdots [x-k+1]_q}{[k]_q!}. \quad (2.38)$$

Let $F_q(t)$ be the generating function of the q -extension of the second kind stirling number as follows:

$$F_q(t) := \frac{q^{-\binom{k}{2}}}{[k]_q!} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j}_q q^{\binom{k-j}{2}} e^{[j]_q t} = \sum_{n=0}^{\infty} S(n, k : q) \frac{t^n}{n!}. \quad (2.39)$$

We have from (2.39) that

$$S(n, k : q) = \frac{q^{-\binom{k}{2}}}{[k]_q!} \sum_{j=0}^k (-1)^j q^{\binom{j}{2}} \binom{k}{j}_q [k-j]_q^n = \frac{q^{-\binom{k}{2}}}{[k]_q!} \Delta_q^k 0^n, \quad (2.40)$$

where $[k]_q! = [k]_q [k-1]_q \cdots [2]_q [1]_q$. It is not difficult to see that

$$[x]_q^n = \sum_{k=0}^n q^{\binom{k}{2}} \binom{x}{k}_q [k]_q! S(n, k : q). \quad (2.41)$$

See also [7] through [11] for details and related facts for above. Then, we get from (2.41) and Theorem 2.7 that

$$\sum_{k=0}^j q^{\binom{k}{2}} \binom{x}{k}_q [k]_q! S(j, k : q) = \frac{1}{([1-x]_q + [x]_q)^{n-j}} \sum_{k=j}^n \frac{\binom{k}{j}}{\binom{n}{j}} B_{k,n}(x, q). \quad (2.42)$$

Therefore, this completes the proof of the following theorem.

Theorem 2.8. For $n, j \in \mathbb{Z}_+$ and $x \in [0, 1]$,

$$\frac{1}{([1-x]_q + [x]_q)^{n-j}} \sum_{k=j}^n \frac{\binom{k}{j}}{\binom{n}{j}} B_{k,n}(x, q) = \sum_{k=0}^j q^{\binom{k}{2}} \binom{x}{k}_q [k]_q! S(j, k : q). \quad (2.43)$$

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