Research Article

On a Higher-Order Difference Equation

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We describe in an elegant and short way the behaviour of positive solutions of the higher-order difference equation

\[ x_n = \frac{c x_{n-p} x_{n-p-q} \cdots x_{n-p_k}}{x_{n-q_1} x_{n-q_2} \cdots x_{n-q_l}}, \quad n \in \mathbb{N}_0, \]

where \( c > 0 \), extending some recent results in the literature.

1. Introduction

Studying difference equations has attracted a considerable interest recently, see, for example, [1–39] and the references listed therein. The study of positive solutions of the following higher-order difference equations:

\[ x_n = \max \left\{ A, B \left( \frac{x_{n-p_1} x_{n-p_2} \cdots x_{n-p_k}}{x_{n-q_1} x_{n-q_2} \cdots x_{n-q_l}} \right) \right\}, \quad n \in \mathbb{N}_0, \tag{1.1} \]

and

\[ x_n = A + B \left( \frac{x_{n-p_1} x_{n-p_2} \cdots x_{n-p_k}}{x_{n-q_1} x_{n-q_2} \cdots x_{n-q_l}} \right), \quad n \in \mathbb{N}_0, \tag{1.2} \]

where \( A, B > 0, p_i, q_i \) are natural numbers such that \( p_1 < p_2 < \cdots < p_k, \) \( q_1 < q_2 < \cdots < q_l, \) \( r_i, s_i \in \mathbb{R}_+, \) and \( k \in \mathbb{N} \) was proposed by Stević in several talks, see, for example, [21, 26]. For some results concerning equations related to (1.1) see, for example, [6, 7, 10, 29, 31, 32, 34, 38], while some results on equations related to (1.2) can be found, for example, in [3, 8, 9, 11–14, 18–20, 22, 25, 29, 32, 33, 35] (see also related references cited therein).
Case \( A = 0 \) is of some less interest, since in this case positive solutions of (1.1) and (1.2), by using the change \( y_n = \ln x_n \), become solutions of a linear difference equation with constant coefficients. However, some particular results for the case recently appeared in the literature, see [16, 17, 39].

Nevertheless, motivated by the above-mentioned papers, we will describe the behaviour of positive solutions of the higher-order difference equation

\[
x_n = \frac{c x_{n-p} x_{n-p-q}}{x_{n-q}}, \quad n \in \mathbb{N}_0,
\]

where \( p, q \in \mathbb{N} \) and \( c > 0 \), in, let us say, an elegant and short way.

Let us introduce the following.

**Definition 1.1.** A solution \((x_n)_{n=-(p+q)}^{\infty}\) of (1.3) is said to be **eventually periodic** with period \( \tau \) if there is \( n_0 \in \{-(p + q), \ldots, -1, 0, 1, \ldots\} \) such that \( x_{n+\tau} = x_n \) for all \( n \geq n_0 \). If \( n_0 = -(p + q) \), then we say that the sequence \((x_n)_{n=-}(p+q)\) is **periodic** with period \( \tau \).

For some results on equations all solutions of which are eventually periodic see, for example, [2, 4, 8, 15, 28, 37] and the references therein.

**Definition 1.2.** One says that a solution \((x_n)_{n=n_0}^{\infty}\) of a difference equation converges geometrically to \( x^* \) if there exist \( L \in \mathbb{R}^+ \) and \( \theta \in [0, 1) \) such that

\[
|x_n - x^*| \leq L \theta^n, \quad \forall n \geq n_0.
\]

Now we return to (1.3).

First, note that if \( p = q \), then (1.3) becomes

\[
x_n = c x_{n-2p}, \quad n \in \mathbb{N}_0,
\]

from which easily follow the following results:

(a) if \( c = 1 \), then all positive solutions of (1.5) are periodic with period \( 2p \);

(b) if \( c \in (0, 1) \), then each positive solution of (1.5) converges to zero. Moreover, its subsequences \((x_{2pm-i})_{m \in \mathbb{N}_0}, i = 1, 2, \ldots, 2p \), converges decreasingly to zero as \( m \to \infty \);

(c) if \( c \in (1, \infty) \), then each positive solution of (1.5) tends to infinity as \( n \to \infty \). Moreover, its subsequences \((x_{2pm-i})_{m \in \mathbb{N}_0}, i = 1, 2, \ldots, 2p \), tend increasingly to infinity as \( m \to \infty \).

We may assume that \( p \) and \( q \) are relatively prime integers, that is, \( \gcd(p, q) = 1 \) (the greatest common divisor of numbers \( p \) and \( q \)). Namely, if \( \gcd(p, q) = r > 1 \), then by using the changes \( z^{(1)}_m = x_{mr+i}, i = 0, 1, \ldots, r - 1 \), (1.3) is reduced to \( r \) copies of the following equation:

\[
z_n = \frac{cz_{n-p_1}}{z_{n-q_1}}, \quad n \in \mathbb{N}_0,
\]

where \( p_1 = p/r, q_1 = q/r, c > 0, \) and \( \gcd(p_1, q_1) = 1 \).
By using repeatedly relation (1.7), we have that

\[ x_n x_{n-q} = cx_{n-p} x_{n-p-q}, \quad n \in \mathbb{N}_0, \quad (1.7) \]

which implies that the sequence \( u_n = x_n x_{n-q}, n \geq -p, \) satisfies the following simple difference equation:

\[ u_n = cu_{n-p}, \quad n \in \mathbb{N}_0. \quad (1.8) \]

## 2. Main Results

Here we formulate and prove our main results.

**Theorem 2.1.** Assume that \( c = 1, \gcd (p, q) = 1, \) and \( p \) is odd, then all positive solutions of (1.3) are eventually periodic with period \( \tau = 2pq. \)

**Proof.** By using repeatedly relation (1.7) \( p \)-times, we obtain

\[ x_n = \frac{u_n}{x_{n-q}} = \frac{u_n}{u_{n-q}} x_{n-2q} = \cdots = \frac{u_n}{u_{n-q}} u_{n-2q} \frac{u_{n-2q}}{u_{n-q}} u_{n-3q} \cdots \frac{u_{n-2q(p-1)}}{u_{n-q(2p-1)}} x_{n-2pq}. \quad (2.1) \]

Now, note that from (1.8), it follows that in this case \( u_n \) is periodic with period \( p. \) On the other hand, since \( \gcd (p, q) = 1 \) for each \( i, j \in \{0, 1, \ldots, p-1\}, i \neq j, \) we have that

\[ (n - (2i + 1)q) - (n - (2j + 1)q) = (j - i)2q \equiv 0 \pmod{p}, \]

\[ (n - (2i + 2)q) - (n - (2j + 2)q) = (j - i)2q \equiv 0 \pmod{p}. \quad (2.2) \]

Hence, the indices \((n - (2i + 1)q), i \in \{0, 1, \ldots, p-1\}, \) and \((n - (2i + 2)q), i \in \{0, 1, \ldots, p-1\},\) belong to \( p \) different subsequences. From this and the periodicity of \( u_n, \) it follows that

\[ u_n u_{n-2q} \cdots u_{n-2q(p-1)} = u_{n-q} u_{n-3q} \cdots u_{n-q(2p-1)}, \quad (2.3) \]

from which the theorem follows. \( \square \)

By taking the logarithm of (1.3) and using the change \( v_n = \ln x_n, \) we get

\[ v_n + v_{n-q} - v_{n-p} - v_{n-p-q} = \ln c, \quad n \in \mathbb{N}_0. \quad (2.4) \]

The characteristic polynomial of the homogeneous part of (2.4) is

\[ \lambda^{pq} + \lambda^p - \lambda^q - 1 = (\lambda^q + 1)(\lambda^p - 1) = 0, \quad (2.5) \]
from which it follows that all its roots are expressed by

\[
\exp\left(\frac{(2k + 1)\pi i}{q}\right), \quad k = 0, 1, \ldots, q - 1, \quad \exp\left(\frac{2l\pi i}{p}\right), \quad l = 0, 1, \ldots, p - 1.
\] (2.6)

These roots are simple if and only if

\[
\frac{2k + 1}{q} \neq \frac{2l}{p}, \quad \text{for each } k, l \in \mathbb{N}_0.
\] (2.7)

Clearly, if \( p \) is odd, inequality (2.7) holds. If \( p \) is even, that is, \( p = 2^s r \), for some \( s, r \in \mathbb{N} \), then, since \( \gcd(p, q) = 1 \), \( q \) must be odd. Then, for \( k = (q - 1)/2 \) and \( l = r \), we will get that inequality (2.7) does not hold.

From the above consideration and Theorem 2.1, we get the next corollary.

**Corollary 2.2.** Assume that \( c = 1 \) and \( \gcd(p, q) = 1 \). Then all positive solutions of (1.3) are eventually periodic if and only if \( p \) is odd. Moreover, if \( p \) is odd, then the period is \( \tau = 2pq \).

Since the root \( \lambda = 1 \) of characteristic polynomial (2.5) is a simple one, a particular solution of nonhomogeneous (2.4) has the form

\[
v_n^p = A n,
\] (2.8)

from which, by a direct calculation, we easily get that \( A = \ln c/2p \).

Hence, if \( p \) is odd, the general solution of (1.3) is

\[
x_n = e^{v_n} = c^{n/2p} \exp\left(\sum_{k=0}^{q-1} c_{k,1} \cos \left(\frac{(2k + 1)\pi n}{q}\right) + c_{k,2} \sin \left(\frac{(2k + 1)\pi n}{q}\right) + \sum_{l=1}^{p-1} \left( d_{k,1} \cos \left(\frac{2l\pi n}{p}\right) + d_{k,2} \sin \left(\frac{2l\pi n}{p}\right) \right) \right).
\] (2.9)

Note that from (2.9), it follows that

\[
x_n = c^{n/2p} \hat{x}_n,
\] (2.10)

and that \( \hat{x}_n \) is a positive solution of (1.3) with \( c = 1 \).

From (2.9), (2.10), and Theorem 2.1 the following results directly follow.

**Theorem 2.3.** Assume that \( c \in (0, 1) \), \( \gcd(p, q) = 1 \), and \( p \) is odd, then every positive solution of (1.3) converges geometrically to zero. Moreover, for each \( s \in \{0, 1, \ldots, 2pq - 1\} \), the subsequence \((x_{2^s m})_{m \in \mathbb{N}_0}\) converges monotonically to zero as \( m \to \infty \).

**Theorem 2.4.** Assume that \( c > 1 \), \( \gcd(p, q) = 1 \), and \( p \) is odd, then every positive solution of (1.3) tends to infinity. Moreover, for each \( s \in \{0, 1, \ldots, 2pq - 1\} \), the subsequence \((x_{2^s m})_{m \in \mathbb{N}_0}\) converges increasingly to infinity as \( m \to \infty \).
Finally, there are two concluding interesting remarks.

**Remark 2.5.** Note that, since the functions $\cos((2k+1)\pi n/q)$ and $\sin((2k+1)\pi n/q)$ are periodic with period $2q$ and the functions $\cos(2\pi n/p)$ and $\sin(2\pi n/p)$ are periodic with period $p$, from the representation (2.9) we also obtain Theorem 2.1.

**Remark 2.6.** The results in papers [16, 17, 39], which are obtained in much complicated ways, are particular cases of our results. Namely, in [16] Özban studied a system which is transformed into (1.3) with $p = 1$, $q = m + k + 1$ and $c = 1$, in [17] he studied a system which is transformed into (1.3) with $p = 3$, and $c = b/a$, while in [39] the authors considered a system which is transformed into (1.3) with $c = b/a$, but they only considered the case when $p \leq q$.

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