Research Article

Permanence of a Discrete Model of Mutualism with Infinite Deviating Arguments

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Received 15 July 2009; Accepted 13 January 2010

Abstract

We propose a discrete model of mutualism with infinite deviating arguments, that is

\[
\begin{align*}
\frac{dN_1(t)}{dt} &= r_1(t)N_1(t) \left[ \frac{K_1(t) + a_1(t) \int_0^\infty J_2(s)N_2(t-s)ds}{1 + \int_0^\infty J_2(s)N_2(t-s)ds} - N_1(t-\sigma_1(t)) \right], \\
\frac{dN_2(t)}{dt} &= r_2(t)N_2(t) \left[ \frac{K_2(t) + a_2(t) \int_0^\infty J_1(s)N_1(t-s)ds}{1 + \int_0^\infty J_1(s)N_1(t-s)ds} - N_2(t-\sigma_2(t)) \right],
\end{align*}
\]

where \( r_i, K_i, a_i, \) and \( \sigma_i, i = 1, 2 \) are continuous functions bounded above and below by positive constants: \( a_i > K_i, i = 1, 2; \) \( J_i \in C([0, +\infty), [0, +\infty)) \) and \( \int_0^\infty J_i(s)ds = 1, i = 1, 2. \) Using the differential inequality theory, they obtained a set of sufficient conditions to ensure the permanence of system (1.1). For more background and biological adjustments of system (1.1), one could refer to [1–4] and the references cited therein.

However, many authors [5–12] have argued that the discrete time models governed by difference equations are more appropriate than the continuous ones when the populations have nonoverlapping generations. Also, since discrete time models can also provide efficient conditions to ensure the permanence of the system.
In this section, we establish permanence results for system\footnote{discrete time models governed by difference equations. Another permanence is one of the most important topics on the study of population dynamics. One of the most interesting questions in mathematical biology concerns the survival of species in ecological models. It is reasonable to ask for conditions under which the system is permanent. Motivated by the above question, we consider the permanence of the following discrete model of mutualism with infinite deviating arguments:

\begin{equation}
\begin{align*}
x_1(n+1) &= x_1(n) \exp \left\{ r_1(n) \left[ \frac{K_1(n) + a_1(n) \sum_{s=0}^{\infty} J_2(s)x_2(n-s)}{1 + \sum_{s=0}^{\infty} J_2(s)x_2(n-s)} - x_1(n - \sigma_1(n)) \right] \right\}, \\
x_2(n+1) &= x_2(n) \exp \left\{ r_2(n) \left[ \frac{K_2(n) + a_2(n) \sum_{s=0}^{\infty} J_1(s)x_1(n-s)}{1 + \sum_{s=0}^{\infty} J_1(s)x_1(n-s)} - x_2(n - \sigma_2(n)) \right] \right\},
\end{align*}
\end{equation}

where \( x_i(n), i = 1, 2 \) is the density of mutualism species \( i \) at the \( n \)th generation. For \( \{r_i(n)\}, \{K_i(n)\}, \{a_i(n)\}, \{J_i(n)\}, \) and \( \{\sigma_i(n)\}, i = 1, 2 \) are bounded nonnegative sequences such that

\begin{equation}
0 < r_i^l \leq r_i^u, \quad 0 < a_i^l \leq a_i^u, \quad 0 < K_i^l \leq K_i^u, \quad 0 < \sigma_i^l \leq \sigma_i^u, \quad \sum_{n=0}^{\infty} I_i(n) = 1.
\end{equation}

Here, for any bounded sequence \( \{a(n)\}, a^u = \sup_{n \in N} a(n), a^l = \inf_{n \in N} a(n) \).

Let \( \sigma = \sup_n \{\sigma_i(n), i = 1, 2\} \), we consider \( (1.2) \) together with the following initial condition:

\begin{equation}
x_i(\theta) = \varphi_i(\theta) \geq 0, \quad \theta \in N[-\tau,0] = \{-\tau, -\tau + 1, \ldots, 0\}, \quad \varphi_i(0) > 0.
\end{equation}

It is not difficult to see that solutions of \( (1.2) \) and \( (1.4) \) are well defined for all \( n \geq 0 \) and satisfy

\begin{equation}
x_i(n) > 0, \quad \text{for} \quad n \in Z, i = 1, 2.
\end{equation}

The aim of this paper is, by applying the comparison theorem of difference equation and some lemmas, to obtain a set of sufficient conditions which guarantee the permanence of system \( (1.2) \).

\section{Permanence}

In this section, we establish permanence results for system \( (1.2) \).

Following Comparison Theorem of difference equation is Theorem 2.6 of [13, page 241].

**Lemma 2.1.** Let \( k \in N_{k_0}^+ = \{k_0, k_0 + 1, \ldots, k_0 + l, \ldots\} \), \( r \geq 0 \). For any fixed \( k \), \( g(k,r) \) is a non-decreasing function with respect to \( r \), and for \( k \geq k_0 \), following inequalities hold: \( y(k + 1) \leq g(k, y(k)), u(k + 1) \geq g(k, u(k)) \). If \( y(k_0) \leq u(k_0) \), then \( y(k) \leq u(k) \) for all \( k \geq k_0 \).
Now let us consider the following single species discrete model:

\[ N(k + 1) = N(k) \exp\{a(k) - b(k)N(k)\}, \quad (2.1) \]

where \{a(k)\} and \{b(k)\} are strictly positive sequences of real numbers defined for \( k \in \mathbb{N} = \{0, 1, 2, \ldots\} \) and \( 0 < a' \leq a''\), \( 0 < b' \leq b''\). Similar to the proof of Propositions 1 and 3 in [6], we can obtain the following.

**Lemma 2.2.** Any solution of system (2.1) with initial condition \( N(0) > 0 \) satisfies

\[ m \leq \lim_{k \to +\infty} \inf N(k) \leq \lim_{k \to +\infty} \sup N(k) \leq M, \quad (2.2) \]

where

\[ M = \frac{1}{b'} \exp\{a'' - 1\}, \quad m = \frac{a'}{b''} \exp\{a' - b'M\}. \quad (2.3) \]

**Lemma 2.3** (see [14]). Let \( x(n) \) and \( b(n) \) be nonnegative sequences defined on \( \mathbb{N} \), and \( c \geq 0 \) is a constant. If

\[ x(n) \leq c + \sum_{s=0}^{n-1} b(s)x(s), \quad \text{for } n \in \mathbb{N}, \quad (2.4) \]

then

\[ x(n) \leq c \prod_{s=0}^{n-1} [1 + b(s)], \quad \text{for } n \in \mathbb{N}. \quad (2.5) \]

**Lemma 2.4** (see [2]). Let \( x : \mathbb{Z} \to \mathbb{R} \) be a nonnegative bounded sequences, and let \( H : \mathbb{N} \to \mathbb{R} \) be a nonnegative sequence such that \( \sum_{i=0}^{\infty} J_i(n) = 1 \). Then

\[ \lim_{n \to +\infty} \inf x(n) \leq \lim_{n \to +\infty} \inf \sum_{s=-\infty}^{n} H(n - s)x(s) \leq \lim_{n \to +\infty} \sup \sum_{s=-\infty}^{n} H(n - s)x(s) \leq \lim_{n \to +\infty} \sup x(n). \quad (2.6) \]

**Proposition 2.5.** Let \( (x_1(n), x_2(n)) \) be any positive solution of system (1.2), then

\[ \lim_{n \to +\infty} \sup x_i(n) \leq M_i, \quad i = 1, 2, \quad (2.7) \]

where

\[ M_i = \exp\{2r_i'' [K_i'' + \alpha_i'']\}, \quad i = 1, 2. \quad (2.8) \]
Proof. Let \((x_1(n), x_2(n))\) be any positive solution of system (1.2), then from the first equation of system (1.2) we have

\[
x_1(n + 1) \leq x_1(n) \exp \left\{ r_1(n) \left[ \frac{K_1(n) + \alpha_1(n) \sum_{s=0}^{\infty} f_2(s) x_2(n-s)}{1 + \sum_{s=0}^{\infty} f_2(s) x_2(n-s)} \right] \right\}
\]

\[
= x_1(n) \exp \left\{ r_1(n) \left[ \frac{K_1(n)}{1 + \sum_{s=0}^{\infty} f_2(s) x_2(n-s)} + \frac{\alpha_1(n) \sum_{s=0}^{\infty} f_2(s) x_2(n-s)}{1 + \sum_{s=0}^{\infty} f_2(s) x_2(n-s)} \right] \right\}
\]

\[
\leq x_1(n) \exp \left\{ r_1(n) \left[ \frac{K_1(n)}{1} + \frac{\alpha_1(n) \sum_{s=0}^{\infty} f_2(s) x_2(n-s)}{\sum_{s=0}^{\infty} f_2(s) x_2(n-s)} \right] \right\}
\]

\[
= x_1(n) \exp \{ r_1(n) [K_1(n) + \alpha_1(n)] \}
\]

\[
\leq x_1(n) \exp \{ r_1^n [K_1^n + \alpha_1^n] \}.
\]

Let \(x_1(n) = \exp\{u_1(n)\}\), then

\[
u_1(n + 1) \leq u_1(n) + r_1^n [K_1^n + \alpha_1^n] = r_1^n [K_1^n + \alpha_1^n] + \sum_{s=0}^{n} b(s)x(s),
\]

where

\[
b(s) = \begin{cases} 
0, & 0 \leq s \leq n - 1, \\
1, & s = n.
\end{cases}
\]

When \(u_1(n)\) is nonnegative sequence, by applying Lemma 2.3, it immediately follows that

\[
u_1(n + 1) \leq 2r_1^n [K_1^n + \alpha_1^n].
\]

When \(u_1(n)\) is negative sequence, (2.12) also holds. From (2.12), we have

\[
\lim_{n \to +\infty} \sup x_1(n) \leq \exp \{2r_1^n [K_1^n + \alpha_1^n] \} := M_1.
\]

By using the second equation of system (1.2), similar to the above analysis, we can obtain

\[
\lim_{n \to +\infty} \sup x_2(n) \leq \exp \{2r_2^n [K_2^n + \alpha_2^n] \} := M_2.
\]

This completes the proof of Proposition 2.5. □

Now we are in the position of stating the permanence of system (1.2).
Theorem 2.6. Under the assumption (1.3), system (1.2) is permanent, that is, there exist positive constants \( m_i, M_i, i = 1, 2 \) which are independent of the solutions of system (1.2) such that, for any positive solution \((x_1(n), x_2(n))\) of system (1.2) with initial condition (1.4), one has

\[
m_i \leq \liminf_{n \to +\infty} x_1(n) \leq \limsup_{n \to +\infty} x_1(n) \leq M_i, \quad i = 1, 2.
\]  

(2.15)

Proof. By applying Proposition 2.5, we see that to end the proof of Theorem 2.6 it is enough to show that under the conditions of Theorem 2.6

\[
\liminf_{n \to +\infty} x_1(n) \geq m_i.
\]

(2.16)

From Proposition 2.5, for all \( \varepsilon > 0 \), there exists a \( N_1 > 0, \quad N_1 \in N \), for all \( n > N_1 \),

\[
x_1(n) \leq M_i + \varepsilon.
\]

(2.17)

According to Lemma 2.4, from (2.13) and (2.14) we have

\[
\lim_{n \to +\infty} \sup_{s=0}^{\infty} \sum_{i=0}^{\infty} J_1(s)x_i(n-s) = \lim_{n \to +\infty} \sup_{k=0}^{n} \sum_{i=0}^{\infty} J_1(n-k)x_i(k) \leq M_i, \quad i = 1, 2.
\]

(2.18)

For above \( \varepsilon > 0 \), according to (2.18), there exists a positive integer \( N_2 \), such that, for all \( n > N_2 \),

\[
\sum_{i=0}^{\infty} J_1(s)x_i(n-s) \leq M_i + \varepsilon, \quad i = 1, 2.
\]

(2.19)

Thus, for all \( n > \max\{N_1, N_2\} + \sigma \), from the first equation of system (1.2), it follows that

\[
x_1(n + 1) \geq x_1(n) \exp \left\{ r_1(n) \left[ \frac{K_1^l}{1 + (M_2 + \varepsilon)} - (M_1 + \varepsilon) \right] \right\}
\]

\[
\geq x_1(n) \exp \left\{ \frac{r_1^l K_1^l}{1 + (M_2 + \varepsilon)} - r_1^m (M_1 + \varepsilon) \right\}.
\]

(2.20)

It follows that, for \( n \geq \sigma_1(n) \),

\[
\prod_{i=n-\sigma_1(n)}^{n-1} x_1(i + 1) \geq \prod_{i=n-\sigma_1(n)}^{n-1} x_1(i) \exp \left\{ \frac{r_1^l K_1^l}{1 + (M_2 + \varepsilon)} - r_1^m (M_1 + \varepsilon) \right\}.
\]

(2.21)

Hence

\[
x_1(n) \geq x_1(n - \sigma_1(n)) \exp \left\{ \frac{r_1^l K_1^l}{1 + (M_2 + \varepsilon)} \sigma_1^l - r_1^m (M_1 + \varepsilon) \sigma_1^m \right\}.
\]

(2.22)
In other words,

\[ x_1(n - \sigma_1(n)) \leq x_1(n) \exp \left\{ -\frac{r_1^l K_1^l}{1 + (M_2 + \varepsilon)} \sigma_1^l + r_1^u (M_1 + \varepsilon) \sigma_1^u \right\}. \] (2.23)

From the first equation of system (1.2) and (2.23), for all \( n > \max\{N_1, N_2\} + \sigma \), it follows that

\[ x_1(n + 1) \geq x_1(n) \exp \left\{ -\frac{r_1^l K_1^l}{1 + (M_2 + \varepsilon)} - r_1^u \exp \left\{ -\frac{r_1^l K_1^l}{1 + (M_2 + \varepsilon)} \sigma_1^l + r_1^u (M_1 + \varepsilon) \sigma_1^u \right\} x_1(n) \right\}. \] (2.24)

By applying Lemmas 2.1 and 2.2 to (2.24), it immediately follows that

\[
\lim_{n \to +\infty} \inf x_1(n) \geq \frac{r_1^l K_1^l}{r_1^u (1 + (M_2 + \varepsilon))} \exp \left\{ \frac{r_1^l K_1^l}{1 + (M_2 + \varepsilon)} \sigma_1^l - r_1^u (M_1 + \varepsilon) \sigma_1^u \right\} \times \exp \left\{ \frac{r_1^l K_1^l}{1 + (M_2 + \varepsilon)} - r_1^u \exp \left\{ -\frac{r_1^l K_1^l}{1 + (M_2 + \varepsilon)} \sigma_1^l + r_1^u (M_1 + \varepsilon) \sigma_1^u \right\} M_1 \right\}. \] (2.25)

Setting \( \varepsilon \to 0 \), it follows that

\[
\lim_{n \to +\infty} \inf x_1(n) \geq \frac{r_1^l K_1^l}{r_1^u (1 + M_2)} \exp \left\{ \frac{r_1^l K_1^l}{1 + M_2} \sigma_1^l - r_1^u M_1 \sigma_1^u \right\} \times \exp \left\{ \frac{r_1^l K_1^l}{1 + M_2} - r_1^u \exp \left\{ -\frac{r_1^l K_1^l}{1 + M_2} \sigma_1^l + r_1^u M_1 \sigma_1^u \right\} M_1 \right\}. \] (2.26)

Similar to the above analysis, from the second equation of system (1.2), we have that

\[
\lim_{n \to +\infty} \inf x_2(n) \geq \frac{r_2^l K_2^l}{r_2^u (1 + M_1)} \exp \left\{ \frac{r_2^l K_2^l}{1 + M_1} \sigma_2^l - r_2^u M_2 \sigma_2^u \right\} \times \exp \left\{ \frac{r_2^l K_2^l}{1 + M_1} - r_2^u \exp \left\{ -\frac{r_2^l K_2^l}{1 + M_1} \sigma_2^l + r_2^u M_2 \sigma_2^u \right\} M_2 \right\}. \] (2.27)

This completes the proof of Theorem 2.6.

\[ \square \]

**References**


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