Research Article

Topological Entropy and Special \( \alpha \)-Limit Points of Graph Maps

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Let \( G \) a graph and \( f : G \to G \) be a continuous map. Denote by \( h(f) \), \( R(f) \), and \( SA(f) \) the topological entropy, the set of recurrent points, and the set of special \( \alpha \)-limit points of \( f \), respectively. In this paper, we show that \( h(f) > 0 \) if and only if \( SA(f) \neq R(f) \).

1. Introduction

Let \((X,d)\) be a metric space. For any \( Y \subseteq X \), denote by \( \overset{\circ}{Y} \), \( \partial Y \), and \( \overline{Y} \) the interior, the boundary, and the closure of \( Y \) in \( X \), respectively. For any \( y \in X \) and any \( r > 0 \), write \( B(y, r) = \{ x \in X : d(x, y) < r \} \). Let \( \mathbb{N} \) be the set of all positive integers and \( \mathbb{Z}_+ = \mathbb{N} \cup \{0\} \).

Denote by \( C^0(X) \) the set of all continuous maps from \( X \) to \( X \). For any \( f \in C^0(X) \), let \( f^0 \) be the identity map of \( X \) and \( f^n = f \circ f^{n-1} \) the composition map of \( f \) and \( f^{n-1} \). A point \( x \in X \) is called a periodic point of \( f \) with period \( n \) if \( f^n(x) = x \) and \( f^i(x) \neq x \) for \( 1 \leq i < n \). The orbit of \( x \) under \( f \) is the set \( O(x, f) = \{ f^n(x) : n \in \mathbb{Z}_+ \} \). Write \( \omega(x, f) = \bigcap_{n=1}^{\infty} \overline{O(f^n(x), f)} \), called the \( \omega \)-limit set of \( x \) under \( f \). In fact, \( y \in \omega(x, f) \) if and only if there exists a sequence of positive integers \( n_1 < n_2 < n_3 < \cdots \) such that \( \lim_{i \to \infty} f^{n_i}(x) = y \). \( x \) is called a recurrent point of \( f \) if \( x \in \omega(x, f) \). \( x \) is called a special \( \alpha \)-limit point of \( f \) if there exist a sequence of positive integers \( \{ n_i \}_{i=1}^{\infty} \) and a sequence of points \( \{ y_i \}_{i=0}^{\infty} \) such that \( f^{n_i}(y_i) = y_{i-1} \) for any \( i \in \mathbb{N} \) and \( \lim_{i \to \infty} y_i = x \). Denote by \( P(f) \), \( R(f) \), and \( SA(f) \) the sets of periodic points, recurrent points, and special \( \alpha \)-limit points of \( f \), respectively. From the definitions it is easy to see that \( P(f) \subseteq SA(f) \) and \( P(f) \subseteq R(f) \). Let \( h(f) \) denote the topological entropy of \( f \), for the definition see [1, Chapter VIII].

A metric space \( X \) is called an arc (resp., an open arc, a circle) if it is homeomorphic to the interval \([0,1]\) (resp., the open interval \((0,1)\), the unit circle \(S^1\)). Let \( A \) be an arc and
intervals, and denote the number of elements of a tree map, an interval map.

In this section, we will prove Theorem 1.1. To do this, we need the following lemmas.

2. Proof of Theorem 1.1

In this section, we will prove Theorem 1.1. To do this, we need the following lemmas.
Lemma 2.1 (see [11, Theorem 1]). Let $G$ be a graph and $f \in C^0(G)$. If $x \in SA(f)$, then there exist a sequence of positive integers $n_1 \leq n_2 \leq n_3 \leq \cdots$ and a sequence of points $\{y_i\}_{i=0}^{\infty}$ with $y_0 = x$ such that $f^n(y_i) = y_{i-1}$ for any $i \in \mathbb{N}$ and $\lim_{i \to \infty} y_i = x$.

Remark 2.2. The main idea of the proof of Theorem 1 in [11] is similar to the one of Main Theorem in [2].

Lemma 2.3. Let $G$ be a graph and $f \in C^0(G)$. Then $SA(f) \subset f(SA(f))$.

Proof. Let $x \in SA(f)$. Then there exist a sequence of points $\{x_i\}_{i=0}^{\infty}$ and a sequence of positive integers $2 \leq m_1 \leq m_2 \leq \cdots$ such that $f^{m_i}(x_i) = x_{i-1}$ for every $i \in \mathbb{N}$ and $\lim_{i \to \infty} x_i = x$. Write $y_i = f^{m_i-1}(x_i)$ for $i \in \mathbb{N}$. Let $y_{k_i} = y_{i_1}, y_{i_2}, y_{i_3}, \ldots, y_{i_n}, \ldots$ be a convergence subsequence of $\{y_i\}_{i=1}^{\infty}$, and let $\lim_{i \to \infty} y_{k_i} = y$. Then

$$f(y) = \lim_{i \to \infty} f(y_{k_i}) = \lim_{i \to \infty} f^{m_{k_i}}(x_{k_i}) = \lim_{i \to \infty} x_{k_i-1} = x. \quad (2.1)$$

Write

$$\mu_i = \begin{cases} m_{k_{i-1}} + \cdots + m_i, & \text{if } i = 1, \\ m_{k_{i-1}} + m_{k_{i-2}} + \cdots + m_{k_i}, & \text{if } i \geq 2. \end{cases} \quad (2.2)$$

Then $f^{\mu_i}(y_{k_i}) = f^{\mu_{k_{i-1}}-1}(y_{k_i}) = f^{m_{k_{i-1}}-1}(x_{k_{i-1}}) = y_{k_{i-1}}$ for any $i \in \mathbb{N}$, which implies that $y \in SA(f)$ and $SA(f) \subset f(SA(f))$. The proof is completed.

Lemma 2.4 (see [3, Lemma 2.4]). Let $G$ be a graph and $f \in C^0(G)$. Suppose that $I$ and $L = [a,b]$ are intervals of $G$. If there exist $x \in (a,b)$ and $y \notin (a,b)$ such that $\{x,y\} \subset f(I)$, then $f(I) \supseteq [a,x]$ or $f(I) \supseteq [x,b]$.

Theorem 2.5. Let $G$ be a graph and $f \in C^0(G)$. Then $h(f) > 0$ if and only if $SA(f) - R(f) \neq \emptyset$.

Proof Necessity

If $SA(f) - R(f) \neq \emptyset$, then take a point $x_0 \in SA(f) - R(f)$. By Lemma 2.3 and $f(R(f)) = R(f)$, for every $i = 1, 2, \ldots$, there exists a point $x_i \in SA(f) - R(f)$ such that $f(x_i) = x_{i-1}$. Note that $x_0, x_1, x_2, \ldots$ are mutually different. Since the numbers of vertexes and edges of $G$ are finite, there exists an edge $I$ of $G$ such that $I \cap \{x_0, x_1, x_2, \ldots\}$ is an infinite set. We can choose integers $1 < i_1 < i_2 < \cdots$ such that $\{x_{i_k} : k \in \mathbb{N}\} \subset I$ and $x_{i_k} \in (\{x_{i_{k-1}}, x_{i_{k+1}}\})$ for every $k \geq 2$. Take points $\{y, x, z\} \subset I \cap (SA(f) - R(f))$ with $x \in (y, z)$ such that $f^m(y) = x$ and $f^n(x) = z$ for some $m, n \in \mathbb{N}$. Without loss of generality we may assume that $I = [0,1]$ and $0 < y < x < z < 1$. Since $y \in SA(f) - R(f)$, we can take points $\{y_i : i \in \mathbb{N}\} \subset (0,1)$ and positive integers $m + n < m_1 < m_2 < m_3 < \cdots$ satisfying the following conditions:

1. the sequence $(y_1, y_2, y_3, \ldots)$ is strictly monotonic with $f^m(y_i) = y_{i-1}$ for any $i \in \mathbb{N}$ and $y_0 = y$ (see Lemma 2.1) and $\lim_{i \to \infty} y_i = y$;
2. $m_i > m_1 + m_2 + \cdots + m_{i-1}$ for any $i \geq 2$. 


Let $x_i = f^m(y_i)$ and $z_i = f^n(x_i)$ for any $i \in \mathbb{Z}_+$. Then $\lim_{i \to \infty} x_i = x$ and $\lim_{i \to \infty} z_i = z$. Noting that $x, z \in \text{SA}(f) - R(f)$, we can assume that $\{x_i, z_i : i \in \mathbb{N}\} \subset (0, 1)$, and there exists $\varepsilon > 0$ such that the following conditions hold:

(3) $f^i(x) \notin [x - \varepsilon, x + \varepsilon]$ for any $i \in \mathbb{N}$;

(4) the sequences $(x_1, x_2, x_3, \ldots)$ and $(z_1, z_2, z_3, \ldots)$ are strictly monotonic, and $\{x_i : i \in \mathbb{N}\} \subset [x - \varepsilon, x + \varepsilon] \subset (y, z)$.

In the following we may consider only the case that $(x_1, x_2, x_3, \ldots)$ is strictly decreasing since the other case that $(x_1, x_2, x_3, \ldots)$ is strictly increasing is similar.

Write $\mu_i = m_i + m_{i-1} + \cdots + m_1$ for any $i \in \mathbb{N}$. Put $I_i = [x_i, x_{i-1}]$ and $A_i = f^{\mu_i}(I_i)$ for any $i \geq 2$. Then $A_i$ is a connected set, and

$$\{f^{\mu_i}(x_{i-1}), f^{\mu_i}(x_i)\} = \{x, f^{\mu_i}(x_i)\} \subset A_i. \quad (2.3)$$

Noting that $f^m(f^{\mu_i}(x_i)) = f^\mu(x_i) = x$, we have $x \in f^m(A_i) \cap A_i$. Write $S_i = \bigcup_{i=0}^{\infty} f^m(A_i)$. Then $S_i$ is a connected set containing $x$ and $f^m(S_i) \subset S_i$ for every $i \geq 2$.

Since $f^m(x_{i-1}) = f^{m_{i-1}+\mu_i}(x)$ and $f^m(x_i) = x_{i-1}$ for any $i \geq 2$, by Lemma 2.4 it follows that $f^m(I_i) \supset [x - \varepsilon, x_{i-1}]$ or $f^m(I_i) \supset [x_{i-1}, x + \varepsilon]$. There are two cases to consider.

Case 1. There exist $2 \leq a < \beta < \lambda$ such that $f^m(I_i) \supset [x - \varepsilon, x_{i-1}]$ for every $i \in \{a, \beta, \lambda\}$.

Subcase 1.1. There exists $\lambda \leq \tau$ such that $S_r \supset (0, 1)$. Then $S_r \cap \{y_\alpha, z_{a+1}\} \neq \emptyset$, and there exist $r \geq \mu_r - 1$ and $u \in I_{\tau}$ such that $f^r(u) \in \{y_\alpha, z_{a+1}\}$, from which and $m_{a+1} > m + n$ it follows

$$f^{m_r}(u) = f^m(y_\alpha) = x_\alpha \quad \text{or} \quad f^{m_{a+1}+m_r}(u) = f^{m_{a+1}+n}(z_{a+1}) = x_\alpha. \quad (2.4)$$

Noting $f^{m_r}(x_{r-1}) = f^{m_r+\mu_{r-1}}(x)$ and $f^{m_{a+1}+m_r}(x_{r-1}) = f^{m_{a+1}+n+\mu_{r-1}}(x)$, we have

$$\{f^{m_r+\mu_{r-1}}(x), f^{m_{a+1}+n+\mu_{r-1}}(x)\} \cap [x - \varepsilon, x + \varepsilon] = \emptyset. \quad (2.5)$$

There exists $s \in \{m + r, m_{a+1} + n + r\}$ such that $f^s(I_{\tau}) \supset I_\beta \cup I_\lambda$ or $f^s(I_{\tau}) \supset I_\alpha$, which implies

$$f^{s+m_1}(I_\lambda) \supset f^s(I_{\tau}) \supset I_\beta \cup I_1 \quad \text{or} \quad f^{s+m_{a+1}}(I_\lambda) \supset f^{s+m_1}(I_\tau) \supset f^s(I_\tau) \supset I_\beta \cup I_1. \quad (2.6)$$

On the other hand, $f^{m_{a+1}}(I_\beta) \supset I_\beta \cup I_1$. Thus we can obtain $f^l(I_\beta) \supset I_\beta \cup I_1$ and $f^l(I_\beta) \supset I_\beta \cup I_1$ for some $l \in \{(s + m_1)m_\beta, (s + m_\alpha + m_1)m_\beta\}$. By Theorem B it follows that $h(f) > 0$.

Subcase 1.2. $S_i \subset (0, 1)$ for all $i \geq \lambda$, and there exists $\tau \geq \lambda$ such that $x < \sup S_\tau$. Then we can take $a \geq \tau$ such that $[x, x_a] \subset S_\tau$. Thus there exist $r \geq \mu_r - 1$ and $u \in I_\tau$ such that $f^r(u) = x_a$, which implies $f^{m_r+\mu_{r-1}}(u) = x_a$. Write $s = r + m_{a+1} \cdots + m_1$. Then $f^s(I_\tau) \supset I_\beta \cup I_\lambda$ or $f^s(I_\tau) \supset I_\alpha$ since $f^s(x_{r-1}) = f^{s+\mu_{r-1}}(x) \notin [x - \varepsilon, x + \varepsilon]$, which implies

$$f^{s+m_1}(I_\lambda) \supset f^s(I_{\tau}) \supset I_\beta \cup I_1 \quad \text{or} \quad f^{s+m_{a+1}}(I_\lambda) \supset f^{s+m_1}(I_\tau) \supset f^s(I_\tau) \supset I_\beta \cup I_1. \quad (2.7)$$

On the other hand, $f^{m_{a+1}}(I_\beta) \supset I_\beta \cup I_1$. Thus we can obtain $f^l(I_\beta) \supset I_\beta \cup I_1$ and $f^l(I_\beta) \supset I_\beta \cup I_1$ for some $l \in \{(s + m_1)m_\beta, (s + m_\alpha + m_1)m_\beta\}$. By Theorem B it follows that $h(f) > 0$. 

Subcase 1.3. One has $S_i \subset (0, 1)$ and $x = \sup S_i$ for all $i \geq \lambda$.

If $f^{m_i}(x) < f^{2m_i}(x) < x$ for some $r \geq \lambda$, then there exist $j \geq r + 2$ and $u \in I_r$ such that

\[ f^{m_i}(u) = f^{2m_i}(x) \]

since $\lim_{i \to \infty} f^{2m_i}(x_i) = f^{m_i}(x)$ and $\{ f^{m_i}(x), x \} \subset f^{m_i}(I_r)$, which implies $f^{m_j+1,m_i+1,\ldots+m_r-2m_i}(u) = x_r$. Using arguments similar to ones developed in the proof of Subcase 2.1, we can obtain $f^j(I_r) \supset I_\beta \cup I_\lambda$ and $f^j(I_\beta) \supset I_\beta \cup I_\lambda$ for some $l \in \mathbb{N}$. By Theorem B it follows that $h(f) > 0$. Now we assume $f^{m_i}(x) \leq f^{r_m}(x) < x$ for all $r \geq \lambda$. Note $f^{r_m-1}(x_r) \not\in O(f^{m_i}(x), x)$ since $x \notin R(f)$.

If $f^{2m_i}(x) \leq f^{m_i}(x) < f^{r_m-1}(x_r) < x$ for some $r \geq \lambda$, then $f^{m_i}([f^{m_i}(x), f^{r_m-1}(x_r)]) \not\supset [f^{m_i}(x), x]$ and $f^{m_i}([f^{r_m-1}(x_r), x]) \not\supset [f^{m_i}(x), x]$. By Theorem B it follows that $h(f) > 0$.

Case 2. There exists $\kappa \geq 2$ such that $f^{m_i}(I_i) \supset [x_{i-1}, x + \epsilon]$ for all $i \geq \kappa$.

Subcase 2.1. There exist $\kappa \leq \alpha < \beta$ such that $S_i \not\subset (0, 1)$ for every $i \in [\alpha, \beta]$ and $S_i \not\subset [x, z_{\beta+1}] = \Phi$. Thus there exist $r \geq \mu_{\beta-1}$ and $u \in I_\beta$ such that $f^r(u) \in [y_t, z_{\beta+1} \setminus y_{t+1}]$, from which it follows that $f^{m_{r+1}}(u) = x_\beta$ or $f^{\mu_{\beta-1}+m_r}(u) = x_\beta$. Since $f^{m_{r+1}}(x_{\beta-1}) = f^{m_{r+1}+m_\beta-1}(x)$, $f^{m_{r+1}+m_\beta-1}(x_{\beta-1}) = f^{m_{r+1}+m_\beta-1}(x)$ and

\[ \{ f^{m_{r+1}+m_\beta-1}(x), f^{m_{r+1}+m_\beta-1}(x) \} \cap [x - \epsilon, x + \epsilon] = \emptyset, \tag{2.8} \]

there exists $s \in [m + r, m_{\beta+1} - n + r]$ such that $f^s(I_\beta) \supset I_\beta \cup I_\beta$ or $f^s(I_\beta) \supset I_\beta_{s+1}$, which implies $f^s(I_\beta) \supset I_\beta \cup I_\beta$ or $f^{s_{\beta+1}+m_\beta-1}(I_\beta_{s+1}) \supset I_\beta \cup I_\beta$. In similar fashion, we can show $f^s(I_\beta) \supset I_\beta \cup I_\beta$ for some $t \in \mathbb{N}$. Thus we get $f^s(I_\beta) \supset I_\beta \cup I_\beta$ and $f^s(I_\beta) \supset I_\beta \cup I_\beta$ for some $l \in [st, (s + m_{\beta+1})t]$. It follows from Theorem B that $h(f) > 0$.

Subcase 2.2. There exists $\theta \geq \kappa$ such that $S_i \subset (0, 1)$ for all $i \geq \theta$ and there exists $\tau \geq \lambda \geq \theta$ such that $x \subset S_i$ and $x \subset S_1$ for all $i \in [\lambda, \tau]$. Take $j \geq r + 2$ and $u \in I_{\tau}$, $v \in I_{r}$ such that $f^r(v) = x_\beta$. Using arguments similar to ones developed in the proof of Subcase 2.1, we can obtain $f^j(I_\tau) \supset I_\tau \cup I_\lambda$ and $f^j(I_\tau) \supset I_\tau \cup I_\lambda$ for some $l \in \mathbb{N}$. By Theorem B it follows that $h(f) > 0$.

Subcase 2.3. There exists $\theta \geq \kappa$ such that $S_i \subset (0, 1)$ and $x = \sup S_i$ for all $i \geq \theta$.

If there exist $\tau \geq \lambda \geq \theta$ such that $f^{m}(x) < f^{2m}(x) < x$ for all $i \in [\tau, \lambda]$, then there exist $j \geq r + 2$, $u \in I_{\tau}$, $v \in I_{r}$ such that $f^{m}(u) = f^{2m}(x)$ and $f^{m}(v) = f^{2m}(x)$. Using arguments similar to ones developed in the proof of Subcase 2.1, we can obtain $f^j(I_{\tau}) \supset I_{\tau} \cup I_\lambda$ and $f^j(I_{\tau}) \supset I_{\tau} \cup I_\lambda$ for some $l \in \mathbb{N}$. By Theorem B it follows that $h(f) > 0$. Now we assume that there exists $\theta \geq \lambda$ such that $f^{2m}(x) \leq f^{m}(x) < x$ for all $i \geq \theta$.

If $f^{m}(x) < f^{m}(x) < x$ for all $i \geq \theta$, then using arguments similar to ones developed in the above proof, we can obtain $h(f) > 0$.
Sufficiency

If $h(f) > 0$, then it follows from Theorem B that there exist $n \in \mathbb{N}$ and closed intervals $L, J, K \subset G$ with $J, K \subset L$ and $|K \cap J| \leq 1$ such that $f^n(J) = L$ and $f^n(K) = L$. Without loss of generality we may assume that $L = [0, 1]$ and $J = [a, b]$ and $K = [c, d]$ with $0 \leq a < b \leq c < d \leq 1$ such that $f^n([a, b]) = [0, 1]$ and $f^n([c, d]) = [0, 1]$. By [1, Chapter II, Lemma 2] we can choose $u, v, w \in [0, 1]$ with $u < v < w$ such that one of the following statements holds:

(i) $f^n(u) = f^n(w) = u$, $f^n(v) = w$, $f^n(x) > u$ for $u < x < w$ and $x < f^n(x) < w$ for $u < x < v$.

(ii) $f^n(u) = f^n(w) = w$, $f^n(v) = u$, $f^n(x) < w$ for $u < x < w$ and $u < f^n(x) < x$ for $v < x < w$.

We may consider only case (i) since case (ii) is similar. We claim that, for any $x \in (v, w)$ and any $0 < \varepsilon < w - x$, there exist $y \in (w - \varepsilon, w)$ and $s \in \mathbb{N}$ such that $f^s(y) = x$. In fact, we can choose $u < \cdots < x_i < x_{i-1} < \cdots < x_1 \leq v < x_0 = x$ such that $\lim_{i \to \infty} x_i = u$ and $f^n(x_i) = x_{i-1}$ for any $i \in \mathbb{N}$. Thus there exists some $x_N \in f^n([w - \varepsilon, w))$. That is, we can choose $y \in (w - \varepsilon, w)$ satisfying $f^n(y) = x_N$, which implies $f^{(N+1)}(y) = x$. The claim is proven.

By the above claim we can choose a sequence of positive integers $\{s_i\}_{i=1}^\infty$ and a sequence of points $v \prec y_0 \prec y_1 \prec y_2 \prec \cdots \prec w$ such that $f^{s_i}(y_i) = y_{i-1}$ for any $i \in \mathbb{N}$ and $\lim_{i \to \infty} y_i = w$. Note that $f^n(w) = f^n(u) = u$; then $w \in \text{SA}(f^n) - \text{R}(f^n) \subset \text{SA}(f) - \text{R}(f)$. The proof is completed.

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References


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