Research Article

Some Relations between Twisted (h, q) -Euler Numbers with Weight α and q -Bernstein Polynomials with Weight α

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By using fermionic p -adic q -integral on $\mathbb{Z}_p$, we give some interesting relationship between the twisted (h, q) -Euler numbers with weight α and the q -Bernstein polynomials.

1. Introduction

Let p be a fixed odd prime number. Throughout this paper, we always make use of the following notations: $\mathbb{Z}$ denotes the ring of rational integers, $\mathbb{Z}_p$ denotes the ring of p -adic rational integers, $\mathbb{Q}_p$ denotes the field of p -adic rational numbers, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$, respectively. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let $C_{p^n} = \{w \mid w^{p^n} = 1\}$ be the cyclic group of order p^n , and let

$$T_p = \lim_{n \rightarrow \infty} C_{p^n} = C_{p^\infty} = \cup_{n \geq 0} C_{p^n}, \quad (1.1)$$

(see [1–22]), be the locally constant space. For $w \in T_p$, we denote by $\phi_w : \mathbb{Z}_p \rightarrow \mathbb{C}_p$ the locally constant function $x \mapsto w^x$. The p -adic absolute value is defined by $|x|_p = 1/p^r$, where $x = p^r s/t$ ($r \in \mathbb{Q}$ and $s, t \in \mathbb{Z}$ with $(s, t) = (p, s) = (p, t) = 1$). In this paper, we assume that $q \in \mathbb{C}_p$ with $|q - 1|_p < 1$ as an indeterminate. The q -number is defined by

$$[x]_q = \frac{1 - q^x}{1 - q}, \quad (1.2)$$

(see [1–22]). Note that $\lim_{q \rightarrow 1} [x]_q = x$. For

$$f \in UD(\mathbb{Z}_p) = \{f \mid f : \mathbb{Z}_p \longrightarrow \mathbb{C}_p \text{ is uniformly differentiable function}\}, \quad (1.3)$$

the fermionic p -adic q -integral on $\mathbb{Z}_p$ is defined by Kim as follows:

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = \lim_{N \rightarrow \infty} \frac{1+q}{1+q^{p^N}} \sum_{x=0}^{p^N-1} f(x) (-q)^x, \quad (1.4)$$

(see [1–7]). From (1.4), we note that

$$q^n I_{-q}(f_n) = (-1)^n I_{-q}(f) + [2]_q \sum_{l=0}^{n-1} (-1)^{n-1-l} q^l f(l), \quad (1.5)$$

where $f_n(x) = f(x+n)$ for $n \in \mathbb{N}$.

For $k, n \in \mathbb{Z}_+$ and $x \in [0, 1]$, Kim defined q -Bernstein polynomials, which are different q -Bernstein polynomials of Phillips, as follows:

$$B_{k,n}(x, q) = \binom{n}{k}_q [x]_q^k [1-x]_{q^{-1}}^{n-k}, \quad (1.6)$$

(see [5]). In [9], the p -adic extension of (1.6) is given by

$$B_{k,n}(x, q) = \binom{n}{k}_q [x]_q^k [1-x]_{q^{-1}}^{n-k}, \quad \text{where } x \in \mathbb{Z}_p, \ n, k \in \mathbb{Z}_+. \quad (1.7)$$

For $\alpha \in \mathbb{Z}$, $h \in \mathbb{Z}$, $w \in T_p$, and $q \in \mathbb{C}_p$ with $|1-q|_p \leq 1$, twisted (h, q) -Euler numbers $E_{n,q,w}^{(h,\alpha)}$ with weight α are defined by

$$E_{n,q,w}^{(h,\alpha)} = \int_{\mathbb{Z}_p} \phi_w(x) q^{x(h-1)} [x]_{q^\alpha}^n d\mu_{-q}(x). \quad (1.8)$$

In the special case, $x = 0$, $E_{n,q,w}^{(h,\alpha)}(0) = E_{n,q,w}^{(h,\alpha)}$ are called the n -th twisted (h, q) -Euler numbers with weight α .

In this paper, we investigate some relations between the q -Bernstein polynomials and the twisted (h, q) -Euler numbers with weight α . From these relations, we derive some interesting identities on the twisted (h, q) -Euler numbers and polynomials with weight α .

2. Twisted (h, q) -Euler Numbers and Polynomials with Weight α

By using p -adic q -integral and (1.8), we obtain

$$\begin{aligned} \int_{\mathbb{Z}_p} \phi_w(x) q^{x(h-1)} [x]_{q^\alpha}^n d\mu_{-q}(x) &= \lim_{N \rightarrow \infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} [x]_{q^\alpha}^n w^x q^{x(h-1)} (-q)^x \\ &= [2]_q \left(\frac{1}{1-q^\alpha} \right)^n \sum_{l=0}^n \binom{n}{l} (-1)^l \frac{1}{1+wq^{\alpha l+h}}. \end{aligned} \quad (2.1)$$

We set

$$F_{q,w}^{(h,\alpha)}(t) = \sum_{n=0}^{\infty} E_{n,q,w}^{(h,\alpha)} \frac{t^n}{n!}. \quad (2.2)$$

By (2.1) and (2.2), we have

$$\begin{aligned} F_{q,w}^{(h,\alpha)}(t) &= \sum_{n=0}^{\infty} E_{n,q,w}^{(h,\alpha)} \frac{t^n}{n!} \\ &= [2]_q \sum_{n=0}^{\infty} \left(\left(\frac{1}{1-q^\alpha} \right)^n \sum_{l=0}^n \binom{n}{l} (-1)^l \frac{1}{1+wq^{\alpha l+h}} \right) \frac{t^n}{n!} \\ &= [2]_q \sum_{m=0}^{\infty} (-1)^m w^m q^{hm} e^{[m]_{q^\alpha} t}. \end{aligned} \quad (2.3)$$

Since $[x+y]_{q^\alpha} = [x]_{q^\alpha} + q^{\alpha x} [y]_{q^\alpha}$, we obtain

$$\begin{aligned} E_{n,q,w}^{(h,\alpha)}(x) &= \int_{\mathbb{Z}_p} \phi_w(y) q^{y(h-1)} [y+x]_{q^\alpha}^n d\mu_{-q}(y) \\ &= \sum_{l=0}^n \binom{n}{l} q^{\alpha x l} [x]_{q^\alpha}^{n-l} \int_{\mathbb{Z}_p} \phi_w(y) q^{y(h-1)} [y]_{q^\alpha}^l d\mu_{-q}(y) \\ &= \sum_{l=0}^n \binom{n}{l} q^{\alpha x l} [x]_{q^\alpha}^{n-l} E_{l,q,w}^{(h,\alpha)}. \end{aligned} \quad (2.4)$$

Therefore, we obtain the following theorem.

Theorem 2.1. For $n \in \mathbb{Z}_+$ and $w \in T_p$, we have

$$E_{n,q,w}^{(h,\alpha)}(x) = [2]_q \sum_{m=0}^{\infty} (-1)^m w^m q^{hm} [x+m]_{q^\alpha}^n. \quad (2.5)$$

Furthermore,

$$\begin{aligned} E_{n,q,w}^{(h,\alpha)}(x) &= \sum_{l=0}^n \binom{n}{l} q^{\alpha x l} [x]_{q^\alpha}^{n-l} E_{l,q,w}^{(h,\alpha)} \\ &= \left([x]_{q^\alpha} + q^{\alpha x} E_{q,w}^{(h,\alpha)} \right)^n, \end{aligned} \quad (2.6)$$

with usual convention about replacing $(E_{q,w}^{(h,\alpha)})^n$ with $E_{n,q,w}^{(h,\alpha)}$.

Let $F_{q,w}^{(h,\alpha)}(t, x) = \sum_{n=0}^{\infty} E_{n,q,w}^{(h,\alpha)}(x) t^n / n!$. Then we see that

$$F_{q,w}^{(h,\alpha)}(t, x) = [2]_q \sum_{m=0}^{\infty} (-1)^m w^m q^{mh} e^{[x+m]_{q^\alpha} t}. \quad (2.7)$$

In the special case, $x = 0$, let $F_{q,w}^{(h,\alpha)}(t, 0) = F_{q,w}^{(h,\alpha)}(t)$.

By (2.1), we get

$$E_{n,q^{-1},w^{-1}}^{(h,\alpha)}(1-x) = (-1)^n w q^{\alpha n + h - 1} E_{n,q,w}^{(h,\alpha)}(x). \quad (2.8)$$

From (2.3) and (2.7), we note that

$$w q^h F_{q,w}^{(h,\alpha)}(t, 1) + F_{q,w}^{(h,\alpha)}(t) = [2]_q. \quad (2.9)$$

By (2.9), we get the following recurrence formula:

$$E_{0,q,w}^{(h,\alpha)} = \frac{[2]_q}{1 + q^h w}, \quad q^h w E_{n,q,w}^{(h,\alpha)}(1) + E_{n,q,w}^{(h,\alpha)} = 0 \quad \text{if } n > 0. \quad (2.10)$$

By (2.10) and Theorem 2.1, we obtain the following theorem.

Theorem 2.2. For $n \in \mathbb{Z}_+$ and $w \in T_p$, we have

$$E_{0,q,w}^{(h,\alpha)} = \frac{[2]_q}{1 + q^h w}, \quad q^h w \left(q^\alpha E_{q,w}^{(h,\alpha)} + 1 \right)^n + E_{n,q,w}^{(h,\alpha)} = 0 \quad \text{if } n > 0, \quad (2.11)$$

with usual convention about replacing $(E_{q,w}^{(h,\alpha)})^n$ with $E_{n,q,w}^{(h,\alpha)}$.

By (2.4), Theorem 2.1, and Theorem 2.2, we have

$$\begin{aligned} & q^{2h} w^2 E_{n,q,w}^{(h,\alpha)}(2) - \frac{[2]_q}{1 + q^h w} q^{2h} w^2 - \frac{[2]_q}{1 + q^h w} q^h w \\ &= q^{2h} w^2 \sum_{l=0}^n \binom{n}{l} q^{\alpha l} \left(q^\alpha E_{q,w}^{(h,\alpha)} + 1 \right)^l - \frac{[2]_q}{1 + q^h w} q^{2h} w^2 - \frac{[2]_q}{1 + q^h w} q^h w \end{aligned}$$

$$\begin{aligned}
&= q^{2h}w^2 \sum_{l=1}^n \binom{n}{l} q^{\alpha l} \left(q^\alpha E_{q,w}^{(h,\alpha)} + 1 \right)^l - \frac{[2]_q}{1+q^h w} q^h w \\
&= -q^h w \sum_{l=1}^n \binom{n}{l} q^{\alpha l} E_{l,q,w}^{(h,\alpha)} - \frac{[2]_q}{1+q^h w} q^h w \\
&= -q^h w \sum_{l=0}^n \binom{n}{l} q^{\alpha l} E_{l,q,w}^{(h,\alpha)} \\
&= -q^h w E_{n,q,w}^{(h,\alpha)}(1) = E_{n,q,w}^{(h,\alpha)} \quad \text{if } n > 0.
\end{aligned} \tag{2.12}$$

Therefore, we obtain the following theorem.

Theorem 2.3. For $n \in \mathbb{N}$, we have

$$E_{n,q,w}^{(h,\alpha)}(2) = \left(\frac{1}{q^{2h}w^2} \right) E_{n,q,w}^{(h,\alpha)} + \frac{[2]_q}{1+q^h w} + \left(\frac{1}{q^h w} \right) \frac{[2]_q}{1+q^h w}. \tag{2.13}$$

By (2.8), we see that

$$\begin{aligned}
q^{h-1}w \int_{\mathbb{Z}_p} [1-x]_{q^{-\alpha}}^n q^{(h-1)x} w^x d\mu_{-q}(x) &= (-1)^n q^{\alpha n+h-1} w \int_{\mathbb{Z}_p} [x-1]_{q^\alpha}^n q^{(h-1)x} w^x d\mu_{-q}(x) \\
&= (-1)^n q^{\alpha n+h-1} w E_{n,q,w}^{(h,\alpha)}(-1) = E_{n,q^{-1},w^{-1}}^{(h,\alpha)}(2).
\end{aligned} \tag{2.14}$$

Therefore, we obtain the following theorem.

Theorem 2.4. For $n \in \mathbb{Z}_+$, we have

$$q^{h-1}w \int_{\mathbb{Z}_p} [1-x]_{q^{-\alpha}}^n q^{(h-1)x} w^x d\mu_{-q}(x) = E_{n,q^{-1},w^{-1}}^{(h,\alpha)}(2). \tag{2.15}$$

Let $n \in \mathbb{N}$. By Theorems 2.3 and 2.4, we get

$$\begin{aligned}
&q^{h-1}w \int_{\mathbb{Z}_p} [1-x]_{q^{-\alpha}}^n q^{(h-1)x} w^x d\mu_{-q}(x) \\
&= q^{2h}w^2 E_{n,q^{-1},w^{-1}}^{(h,\alpha)} + q^{h-1}w \left(\frac{[2]_q}{1+q^h w} \right) + q^{2h-1}w^2 \left(\frac{[2]_q}{1+q^h w} \right).
\end{aligned} \tag{2.16}$$

From (2.16), we have

$$\int_{\mathbb{Z}_p} [1-x]_{q^{-\alpha}}^n q^{(h-1)x} w^x d\mu_{-q}(x) = q^{h+1}w E_{n,q^{-1},w^{-1}}^{(h,\alpha)} + \left(\frac{[2]_q}{1+q^h w} \right) + q^h w \left(\frac{[2]_q}{1+q^h w} \right). \tag{2.17}$$

Therefore, we obtain the following corollary.

Corollary 2.5. *For $n \in \mathbb{N}$, we have*

$$\int_{\mathbb{Z}_p} [1-x]_{q^{-\alpha}}^n q^{(h-1)x} w^x d\mu_{-q}(x) = q^{h+1} w E_{n,q^{-1},w^{-1}}^{(h,\alpha)} + [2]_q. \quad (2.18)$$

Kim defined the q -Bernstein polynomials with weight α of degree n as below.

For $x \in \mathbb{Z}_p$, the p -adic q -Bernstein polynomials with weight α of degree n are given by

$$B_{k,n}^{(\alpha)}(x, q) = \binom{n}{k}_{q^\alpha} [x]_{q^\alpha}^k [1-x]_{q^\alpha}^{n-k}, \text{ where } n, k \in \mathbb{Z}_+. \quad (2.19)$$

compare [5, 10, 22] By (2.19), we get the symmetry of q -Bernstein polynomials as follows:

$$B_{k,n}^{(\alpha)}(x, q) = B_{n-k,n}^{(\alpha)}(1-x, q^{-1}), \quad (2.20)$$

see [8]. Thus, by Corollary 2.5, (2.19), and (2.20), we see that

$$\begin{aligned} \int_{\mathbb{Z}_p} B_{k,n}^{(\alpha)}(x, q) q^{(h-1)x} w^x d\mu_{-q}(x) &= \int_{\mathbb{Z}_p} B_{n-k,n}^{(\alpha)}(1-x, q^{-1}) q^{(h-1)x} w^x d\mu_{-q}(x) \\ &= \binom{n}{k} \sum_{l=0}^k \binom{k}{l} (-1)^{k+l} \int_{\mathbb{Z}_p} [1-x]_{q^{-\alpha}}^{n-l} q^{(h-1)x} w^x d\mu_{-q}(x) \\ &= \binom{n}{k} \sum_{l=0}^k \binom{k}{l} (-1)^{k+l} (q^{h+1} w E_{n-l,q^{-1},w^{-1}}^{(h,\alpha)} + [2]_q). \end{aligned} \quad (2.21)$$

For $n, k \in \mathbb{Z}_+$ with $n > k$, we have

$$\begin{aligned} \int_{\mathbb{Z}_p} B_{k,n}^{(\alpha)}(x, q) q^{(h-1)x} w^x d\mu_{-q}(x) &= \binom{n}{k} \sum_{l=0}^k \binom{k}{l} (-1)^{k+l} (q^{h+1} w E_{n-l,q^{-1},w^{-1}}^{(h,\alpha)} + [2]_q) \\ &= \begin{cases} q^{h+1} w E_{n,q^{-1},w^{-1}}^{(h,\alpha)} + [2]_q & \text{if } k = 0, \\ q^{h+1} w \binom{n}{k} \sum_{l=0}^k \binom{k}{l} (-1)^{k+l} E_{n-l,q^{-1},w^{-1}}^{(h,\alpha)} & \text{if } k > 0. \end{cases} \end{aligned} \quad (2.22)$$

Let us take the fermionic q -integral on $\mathbb{Z}_p$ for the q -Bernstein polynomials with weight α of degree n as follows:

$$\begin{aligned} \int_{\mathbb{Z}_p} B_{k,n}^{(\alpha)}(x, q) q^{(h-1)x} w^x d\mu_{-q}(x) &= \binom{n}{k} \int_{\mathbb{Z}_p} [x]_q^k [1-x]_{q^{-\alpha}}^{n-k} q^{(h-1)x} w^x d\mu_{-q}(x) \\ &= \binom{n}{k} \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l E_{l+k,q,w}^{(h,\alpha)}. \end{aligned} \quad (2.23)$$

Therefore, by (2.22) and (2.23), we obtain the following theorem.

Theorem 2.6. Let $n, k \in \mathbb{Z}_+$ with $n > k$. Then we have

$$\int_{\mathbb{Z}_p} B_{k,n}^{(\alpha)}(x, q) q^{(h-1)x} w^x d\mu_{-q}(x) = \begin{cases} q^{h+1} w E_{n,q^{-1},w^{-1}}^{(h,\alpha)} + [2]_q & \text{if } k = 0, \\ q^{h+1} w \sum_{l=0}^k \binom{n}{k} \binom{k}{l} (-1)^{k+l} E_{n-l,q^{-1},w^{-1}}^{(h,\alpha)} & \text{if } k > 0. \end{cases} \quad (2.24)$$

Moreover,

$$\sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l E_{l+k,q,w}^{(h,\alpha)} = \begin{cases} q^{h+1} w E_{n,q^{-1},w^{-1}}^{(h,\alpha)} + [2]_q & \text{if } k = 0, \\ q^{h+1} w \sum_{l=0}^k \binom{k}{l} (-1)^{k+l} E_{n-l,q^{-1},w^{-1}}^{(h,\alpha)} & \text{if } k > 0. \end{cases} \quad (2.25)$$

Let $n_1, n_2, k \in \mathbb{Z}_+$ with $n_1 + n_2 > 2k$. Then we get

$$\begin{aligned} \int_{\mathbb{Z}_p} B_{k,n_1}^{(\alpha)}(x, q) B_{k,n_2}^{(\alpha)}(x, q) q^{(h-1)x} w^x d\mu_{-q}(x) \\ = \binom{n_1}{k} \binom{n_2}{k} \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{l+2k} \int_{\mathbb{Z}_p} [1-x]_{q^{-\alpha}}^{n_1+n_2-l} q^{(h-1)x} w^x d\mu_{-q}(x) \\ = \binom{n_1}{k} \binom{n_2}{k} \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{l+2k} \left(q^{h+1} w E_{n_1+n_2-l,q^{-1},w^{-1}}^{(h,\alpha)} + [2]_q \right) \\ = \begin{cases} q^{h+1} w E_{n_1+n_2,q^{-1},w^{-1}}^{(h,\alpha)} + [2]_q & \text{if } k = 0, \\ \binom{n_1}{k} \binom{n_2}{k} \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{2k+l} \left(q^{h+1} w E_{n_1+n_2-l,q^{-1},w^{-1}}^{(h,\alpha)} + [2]_q \right) & \text{if } k \neq 0. \end{cases} \end{aligned} \quad (2.26)$$

Therefore, by (2.26), we obtain the following theorem.

Theorem 2.7. For $n_1, n_2, k \in \mathbb{Z}_+$ with $n_1 + n_2 > 2k$, we have

$$\begin{aligned} & \int_{\mathbb{Z}_p} B_{k,n_1}^{(\alpha)}(x, q) B_{k,n_2}^{(\alpha)}(x, q) q^{(h-1)x} w^x d\mu_{-q}(x) \\ &= \begin{cases} q^{h+1} w E_{n_1+n_2, q^{-1}, w^{-1}}^{(h, \alpha)} + [2]_q & \text{if } k = 0, \\ q^{h+1} w \binom{n_1}{k} \binom{n_2}{k} \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{2k+l} E_{n_1+n_2-l, q^{-1}, w^{-1}}^{(h, \alpha)} & \text{if } k \neq 0. \end{cases} \end{aligned} \quad (2.27)$$

From the binomial theorem, we can derive the following equation:

$$\begin{aligned} & \int_{\mathbb{Z}_p} B_{k,n_1}^{(\alpha)}(x, q) B_{k,n_2}^{(\alpha)}(x, q) q^{(h-1)x} w^x d\mu_{-q}(x) \\ &= \binom{n_1}{k} \binom{n_2}{k} \sum_{l=0}^{n_1+n_2-2k} (-1)^l \binom{n_1+n_2-2k}{l} \int_{\mathbb{Z}_p} [x]_q^{2k+l} q^{(h-1)x} w^x d\mu_{-q}(x) \\ &= \binom{n_1}{k} \binom{n_2}{k} \sum_{l=0}^{n_1+n_2-2k} (-1)^l \binom{n_1+n_2-2k}{l} E_{2k+l, q, w}^{(h, \alpha)}. \end{aligned} \quad (2.28)$$

Thus, by (2.28) and Theorem 2.7, we obtain the following corollary.

Corollary 2.8. Let $n_1, n_2, k \in \mathbb{Z}_+$ with $n_1 + n_2 > 2k$. Then we have

$$\sum_{l=0}^{n_1+n_2-2k} (-1)^l \binom{n_1+n_2-2k}{l} E_{2k+l, q}^{(h, \alpha)} = \begin{cases} q^{h+1} w E_{n_1+n_2, q^{-\alpha}, w^{-1}}^{(h, \alpha)} + [2]_q & \text{if } k = 0, \\ q^{h+1} w \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{2k+l} E_{n_1+n_2-l, q^{-\alpha}, w^{-1}}^{(h, \alpha)} & \text{if } k > 0. \end{cases} \quad (2.29)$$

For $x \in \mathbb{Z}_p$ and $s \in \mathbb{N}$ with $s \geq 2$, let $n_1, n_2, \dots, n_s, k \in \mathbb{Z}_+$ with $n_1 + \dots + n_s > sk$. Then we take the fermionic p -adic q -integral on $\mathbb{Z}_p$ for the q -Bernstein polynomials with weight α of degree n as follows:

$$\begin{aligned} & \int_{\mathbb{Z}_p} \underbrace{B_{k,n_1}^{(\alpha)}(x, q) \cdots B_{k,n_s}^{(\alpha)}(x, q)}_{s\text{-times}} q^{(h-1)x} w^x d\mu_{-q}(x) \\ &= \binom{n_1}{k} \cdots \binom{n_s}{k} \int_{\mathbb{Z}_p} [x]_q^{sk} [1-x]_{q^{-\alpha}}^{n_1+\dots+n_s-sk} q^{(h-1)x} w^x d\mu_{-q}(x) \end{aligned}$$

$$\begin{aligned}
&= \binom{n_1}{k} \cdots \binom{n_s}{k} \sum_{l=0}^{sk} \binom{sk}{l} (-1)^{l+sk} \int_{\mathbb{Z}_p} [1-x]_{q^{-\alpha}}^{n_1+\cdots+n_s-l} q^{(h-1)x} w^x d\mu_{-q}(x) \\
&= \binom{n_1}{k} \cdots \binom{n_s}{k} \sum_{l=0}^{sk} \binom{sk}{l} (-1)^{l+sk} \left(q^{h+1} w E_{n_1+\cdots+n_s-l, q^{-1}, w^{-1}}^{(h, \alpha)} + [2]_q \right) \\
&= \begin{cases} q^{h+1} w E_{n_1+\cdots+n_s, q^{-1}, w^{-1}}^{(h, \alpha)} + [2]_q & \text{if } k = 0, \\ q^{h+1} w \binom{n_1}{k} \cdots \binom{n_s}{k} \sum_{l=0}^{sk} \binom{sk}{l} (-1)^{l+sk} E_{n_1+\cdots+n_s-l, q^{-1}, w^{-1}}^{(h, \alpha)} & \text{if } k > 0. \end{cases}
\end{aligned} \tag{2.30}$$

Therefore, by (2.30), we obtain the following theorem.

Theorem 2.9. For $s \in \mathbb{N}$ with $s \geq 2$, let $n_1, n_2, \dots, n_s, k \in \mathbb{Z}_+$ with $n_1 + \cdots + n_s > sk$. Then we get

$$\begin{aligned}
&\int_{\mathbb{Z}_p} \underbrace{B_{k, n_1}^{(\alpha)}(x, q) \cdots B_{k, n_s}^{(\alpha)}(x, q)}_{s\text{-times}} q^{(h-1)x} w^x d\mu_{-q}(x) \\
&= \begin{cases} q^{h+1} w E_{n_1+\cdots+n_s, q^{-1}, w^{-1}}^{(h, \alpha)} + [2]_q & \text{if } k = 0, \\ q^{h+1} w \binom{n_1}{k} \cdots \binom{n_s}{k} \sum_{l=0}^{sk} \binom{sk}{l} (-1)^{l+sk} E_{n_1+\cdots+n_s-l, q^{-1}, w^{-1}}^{(h, \alpha)} & \text{if } k > 0. \end{cases}
\end{aligned} \tag{2.31}$$

By the definition of q -Bernstein polynomials with weight α and the binomial theorem, we easily get

$$\begin{aligned}
&\int_{\mathbb{Z}_p} \underbrace{B_{k, n_1}^{(\alpha)}(x, q) \cdots B_{k, n_s}^{(\alpha)}(x, q)}_{s\text{-times}} q^{(h-1)x} w^x d\mu_{-q}(x) \\
&= \binom{n_1}{k} \cdots \binom{n_s}{k} \sum_{l=0}^{n_1+\cdots+n_s-sk} (-1)^l \binom{n_1+\cdots+n_s-sk}{l} \int_{\mathbb{Z}_p} [x]_q^{sk+l} q^{(h-1)x} w^x d\mu_{-q}(x) \\
&= \binom{n_1}{k} \cdots \binom{n_s}{k} \sum_{l=0}^{n_1+\cdots+n_s-sk} (-1)^l \binom{n_1+\cdots+n_s-sk}{l} E_{sk+l, q, w}^{(h, \alpha)}.
\end{aligned} \tag{2.32}$$

Therefore, we have the following corollary.

Corollary 2.10. For $w \in T_p, s \in \mathbb{N}$ with $s \geq 2$, let $n_1, n_2, \dots, n_s, k \in \mathbb{Z}_+$ with $n_1 + \dots + n_s > sk$. Then we have

$$\begin{aligned} & \sum_{l=0}^{n_1+\dots+n_s-sk} (-1)^l \binom{n_1+\dots+n_s-sk}{l} E_{sk+l,q,w}^{(h,\alpha)} \\ &= \begin{cases} q^{h+1} w E_{n_1+\dots+n_s, q^{-1}, w^{-1}}^{(h,\alpha)} + [2]_q & \text{if } k = 0, \\ q^{h+1} w \sum_{l=0}^{sk} \binom{sk}{l} (-1)^{l+sk} E_{n_1+\dots+n_s-l, q^{-1}, w^{-1}}^{(h,\alpha)} & \text{if } k > 0. \end{cases} \end{aligned} \quad (2.33)$$

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