Research Article

Synchronization Analysis of Two Coupled Complex Networks with Time Delays

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This paper studies the synchronized motions between two complex networks with time delays, which include individual inner synchronization in each network and outer synchronization between two networks. Based on the Lyapunov stability theory and the linear matrix inequality (LMI), a synchronous criterion for inner synchronization inside each network is derived. Numerical examples are given which fit the theoretical analysis. In addition, the involved numerical results show that the delays between two networks have little effect on inner synchronization. It is also shown that synchronous motions within each network or between two networks are not enhanced if individual intranetwork connections are allowed.

1. Introduction

We refer to network synchronization, that is, synchronizing all the nodes inside a network, as “inner synchronization”, which is widely studied in the science world based on the appearance of small-world [1] and scale-free [2] network models. Afterwards, the improved and expanded work in this respect—that is, introducing weighted connections, time varying coupling matrices, nonlinear coupling function, time delays, and so forth—can be found in the literature [3–10] and many references cited therein. The main approaches on studying synchronization inside a network is decoupling network systems, and studying the low-dimensional systems through the master stability function, or using the tool in equality (LMI toolbox).

Recently “outer synchronization” was proposed, which aimed at studying synchronization between coupled networks. In [11], the authors theoretically and numerically demonstrated the possibility of synchronization between two networks. By the open-plus-closed-loop (OPCL) method [12], synchronization between two networks can be realized with the same topological structures. In reality, if network nodes are of similar properties
discussions are included in the last section. and synchronization analysis are presented, and numerical examples are shown in Section 3, and outer synchronization between two coupled networks. In Section 2, network models including two examples for the same or different dimension of node dynamics. Finally, the discussions are included in the last section.

**2. Model Presentation and Synchronization Analysis**

Consider the following network models: the node dynamics in network $\mathbb{X}$ is $\dot{x}_i = f(x_i(t))$, $i = 1, \ldots, N_x$, and $\dot{y}_j = g(y_j(t))$, $j = 1, \ldots, N_y$, for the nodes in network $\mathbb{Y}$, where $f(x) : \mathbb{R}^{n_x} \to \mathbb{R}^{n_x}$, $g(y) : \mathbb{R}^{n_y} \to \mathbb{R}^{n_y}$ are continuously differential functions and $x_i(y_j)$ is an $n_x$-dimensional $(n_y$-dimensional) state vector. The dynamical equations of the network systems with time delays are as follows:

$$
\dot{x}_i(t) = f(x_i(t)) + \sum_{j=1}^{N_y} A_{ij} \Gamma_1 y_j(t - \tau_y), \quad i = 1, 2, \ldots, N_x,
$$

$$
\dot{y}_j(t) = g(y_j(t)) + \sum_{i=1}^{N_x} B_{ji} \Gamma_2 x_i(t - \tau_x), \quad j = 1, 2, \ldots, N_y,
$$

where $A$ is an $N_x \times N_y$ dimensional coupling matrix, whose entries $(A_{ij})$ represent the intensity of the direct interaction from $i$ in network $\mathbb{X}$ to $j$ in network $\mathbb{Y}$, analogously the entries of $(B_{ji})$. Matrix $\Gamma_1(\Gamma_2) \in \mathbb{R}^{n_x \times n_y}(\mathbb{R}^{n_y \times n_x})$ is the inner-coupling matrix. $\tau_x, \tau_y$ are the time delays between networks. The action sketch between two networks with time delays is shown in Figure 1.
For the more general models, we allow connections in each network, the networked system then reads as

\[
\begin{align*}
\dot{x}_i(t) &= f(x_i(t)) + \sum_{j=1}^{N_y} A_{ij} \Gamma_1 y_j(t - \tau_y) + \sum_{m=1}^{N_x} C_{im} \Gamma_3 x_m(t), \quad i = 1, 2, \ldots, N_x, \\
\dot{y}_j(t) &= g(y_j(t)) + \sum_{i=1}^{N_x} B_{ji} \Gamma_2 x_i(t - \tau_x) + \sum_{k=1}^{N_y} D_{jk} \Gamma_4 y_k(t), \quad j = 1, 2, \ldots, N_y,
\end{align*}
\]

(2.2)

where \( C = (C_{im}) \in R^{N_x \times N_x} \), \( D = (D_{jk}) \in R^{N_y \times N_y} \) are the coupling matrices, satisfying the sum of every row being zero. \( \Gamma_3 \in R^{n_x \times n_x} \), \( \Gamma_4 \in R^{n_y \times n_y} \) are also inner-coupling matrices.

Let us now consider the possibility that the individual networks achieve synchronization; that is, \( x_1(t) = \cdots = x_{N_x}(t) = x_s(t) \) and \( y_1(t) = \cdots = y_{N_y}(t) = y_s(t) \). If there exist such synchronous states, satisfying

\[
\sum_{j=1}^{N_y} A_{ij} = \mu_1, \quad \forall i \in X, \quad \sum_{i=1}^{N_x} B_{ji} = \mu_2, \quad \forall j \in Y,
\]

(2.3)

without loss of generality, we assume that \( \mu_1 = \mu_2 = 1 \). Thus, the synchronized state equations are

\[
\begin{align*}
\dot{x}_s(t) &= f(x_s(t)) + \Gamma_1 y_s(t - \tau_y), \\
\dot{y}_s(t) &= g(y_s(t)) + \Gamma_2 x_s(t - \tau_x).
\end{align*}
\]

(2.4)

Linearizing the synchronous states around \( x_s \) and \( y_s \), we get

\[
\begin{align*}
\delta \dot{x}_i(t) &= J(t) \delta x_i(t) + \sum_{j=1}^{N_y} A_{ij} \Gamma_1 \delta y_j(t - \tau_y), \quad i = 1, 2, \ldots, N_x, \\
\delta \dot{y}_j(t) &= W(t) \delta y_j(t) + \sum_{i=1}^{N_x} B_{ji} \Gamma_2 \delta x_i(t - \tau_x), \quad j = 1, 2, \ldots, N_y.
\end{align*}
\]

(2.5)
where \( J(t) = Df(x_s(t)) \) and \( W(t) = Dg(y_s(t)) \) are the Jacobians of \( f(x(t)) \) and \( g(y(t)) \) at \( x_s \) and \( y_s \), respectively.

We assume that the \((N_x + N_y)\)-independent solutions of (2.5) can be expressed in the form \( \delta x_i = \Phi_x \delta \bar{x}, \ i = 1, 2, \ldots, N_x, \delta y_j = \Phi_y \delta \bar{y}, \ j = 1, 2, \ldots, N_y, \) where \( \{\Phi_x\}, \{\Phi_y\} \) are the suitable time-independent scalars and \( \delta \bar{x}, \delta \bar{y} \) are the appropriate variables. If the dimension of the space vectors given by the values of \( \Phi_x, \Phi_y (i = 1, 2, \ldots, N_x, j = 1, 2, \ldots, N_y) \) is \( N_x + N_y \), we can see that this assumed form includes all possible linear solutions of (2.5). Substituting this assumption into (2.5), they become

\[
\Phi_x \delta \bar{x}(t) = \Phi_x J(t) \delta \bar{x}(t) + \sum_{i=1}^{N_x} A_{ij} \Phi_y \Gamma_1 \delta \bar{y}(t - \tau_y), \quad i = 1, 2, \ldots, N_x, \tag{2.6}
\]

\[
\Phi_y \delta \bar{y}(t) = \Phi_y W(t) \delta \bar{y}(t) + \sum_{i=1}^{N_x} B_{ji} \Phi_x \Gamma_2 \delta \bar{x}(t - \tau_x), \quad j = 1, 2, \ldots, N_y. \tag{2.7}
\]

Thus, in order that (2.6) (resp., (2.7)) is satisfied for all \( i \) (resp., \( j \)), we require that

\[
\Phi^{-1}_x \sum_i A_{ij} \Phi_y = \eta_1, \quad \text{where} \quad \eta_1 \text{ is independent of } i, \quad \text{and} \quad \Phi^{-1}_y \sum_j B_{ji} \Phi_x = \eta_2, \quad \text{where} \quad \eta_2 \text{ is independent of } j. \quad \text{Set} \quad \Phi_x = (\Phi_{x1}, \ldots, \Phi_{xN_x}) \quad \text{and} \quad \Phi_y = (\Phi_{y1}, \ldots, \Phi_{yN_y}). \quad \text{From this, we get} \quad A \Phi_y = \eta_1 \Phi_x, \quad B \Phi_x = \eta_2 \Phi_y, \quad \text{that is,}
\]

\[
\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \begin{pmatrix} \Phi_x \\ \Phi_y \end{pmatrix} = \begin{pmatrix} \eta_1 \Phi_x \\ \eta_2 \Phi_y \end{pmatrix}. \tag{2.8}
\]

Substituting (2.8) in (2.6) and (2.7), we obtain

\[
\delta \bar{x}(t) = J(t) \delta \bar{x}(t) + \eta_1 \Gamma_1 \delta \bar{y}(t - \tau_y), \tag{2.9}
\]

\[
\delta \bar{y}(t) = W(t) \delta \bar{y}(t) + \eta_2 \Gamma_2 \delta \bar{x}(t - \tau_x). \tag{2.10}
\]

One particular solution of (2.8) is derived when \( \eta_1 = \eta_2 = \lambda \), then

\[
\Theta \begin{pmatrix} \Phi_x' \\ \Phi_y' \end{pmatrix} = \lambda \begin{pmatrix} \Phi_x' \\ \Phi_y' \end{pmatrix}, \quad \Theta = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}, \tag{2.10}
\]

where \( \lambda \) is the (possibly complex) eigenvalues of the matrix \( \Theta \).

Set \( \eta_1 = \lambda \xi, \eta_2 = \lambda / \xi \), \( \Phi_x = \Phi_x^* \), \( \Phi_y = \xi \Phi_y^* \), where \( \xi \) is a free parameter. Equation (2.10) becomes

\[
\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \begin{pmatrix} \Phi_x' \\ \xi \Phi_y' \end{pmatrix} = \begin{pmatrix} (\lambda \xi) \Phi_x^* \\ \frac{\lambda}{\xi} \Phi_y^* \end{pmatrix}, \tag{2.11}
\]
which shows that the solution of (2.11) contains all the possible solutions of (2.8). We rewrite (2.9) as

\[
\begin{align*}
\delta \dot{x}(t) &= J(t) \delta x(t) + \lambda \Gamma_1 \delta y(t - \tau_y), \\
\delta \dot{y}(t) &= W(t) \delta y(t) + \lambda \Gamma_2 \delta x(t - \tau_x),
\end{align*}
\] (2.12)

where \( \lambda = \lambda_1, \lambda_2, \ldots, \lambda_{N_x+N_y} \).

In [14], the authors gave the explicit analysis on the spectrum of \( \Theta \), and two forms of the constructed matrix \( \Theta \) were shown. Here, we adopt one that matrix \( \Theta \) has real eigenvalues and only study network (2.1). In the sequel, we utilize the LMI to derive a synchronous theorem.

**Theorem 2.1.** Consider network model (2.1). If there exist two positive matrices \( P, Q > 0 \), satisfying

\[
\Xi = \begin{bmatrix}
\Psi_1 & 0 & 0 & \lambda P \Gamma_1 \\
0 & \Psi_2 & \lambda Q \Gamma_2 & 0 \\
0 & \lambda \Gamma_2 Q & -I_{n_y} & 0 \\
\lambda \Gamma_1 P & 0 & 0 & -I_{n_y}
\end{bmatrix} < 0,
\] (2.13)

where \( \Psi_1 = J(t)^T P + P J(t) + I_{n_x}, \Psi_2 = W(t)^T Q + Q W(t) + I_{n_y} \), then the network (2.1) asymptotically synchronizes to \( x_s, y_s \) defined by (2.4) for the fixed delays \( \tau_x, \tau_y > 0 \), respectively.

**Proof.** Consider (2.12). Choose a Lyapunov-Krasovskii functional as

\[
V(t) = \delta \dot{x}(t) P \delta x(t) + \delta \dot{y}(t) Q \delta y(t) + \int_{t-\tau_x}^{t} \delta \dot{x}(\alpha) \delta x(\alpha) d\alpha + \int_{t-\tau_y}^{t} \delta \dot{y}(\alpha) \delta y(\alpha) d\alpha.
\] (2.14)

Therefore,

\[
\begin{align*}
V(t) &= \delta \dot{x}(t) P \delta x(t) + \delta \dot{y}(t) Q \delta y(t) + \delta \dot{x}(t) P \delta x(t) + \delta \dot{y}(t) Q \delta y(t) + \delta \dot{y}(t) Q \delta y(t) \\
&= \delta \dot{x}(t) P \delta x(t) + \delta \dot{y}(t) Q \delta y(t) + \delta \dot{x}(t) P \delta x(t) + \delta \dot{y}(t) Q \delta y(t) \\
&= \delta \dot{x}(t) P \delta x(t) + \delta \dot{y}(t) Q \delta y(t) + \delta \dot{x}(t) P \delta x(t) + \delta \dot{y}(t) Q \delta y(t) \\
&= \left( \begin{array}{c} \delta x(t) \\ \delta y(t) \\ \delta x(t - \tau_x) \\ \delta y(t - \tau_y) \end{array} \right)^T \Xi \left( \begin{array}{c} \delta x(t) \\ \delta y(t) \\ \delta x(t - \tau_x) \\ \delta y(t - \tau_y) \end{array} \right).
\] (2.15)
According to the known condition $\Xi < 0$ and Lyapunov-Krasovskii stability Theorem [6], the zero solutions of (2.12) are asymptotically stable, which shows that synchronization inside network $\mathcal{X}$ and network $\mathcal{Y}$ happens when $t$ tends to $+\infty$.

Remark 2.2. Matrix $\Xi$ relies upon $t$ and $\tau_x, \tau_y$. The condition can be changed into $\Xi < 0$ when $t > T$ for enough big $T > 0$. The linear matrix equality method is inadequate for assessing the stability of the synchronous solution when both intragroup and extragroup connections are allowed in the network (2.2).

### 3. Numerical Examples

In this section, we will give two examples to illustrate our obtained results, which includes two cases: $n_x \neq n_y$ and $n_x = n_y$. For the case $n_x \neq n_y$, we only consider the inner synchronization inside each network. On the other hand, $n_x = n_y$, we discuss the inner synchronization within each network and investigate the outer synchronization between two networks.

Example 3.1. Consider the coupled network dynamical equations [14] below, which are in the form (2.1):

$$
\dot{x}_i(t) = -a [x_i(t) + s(x_i(t))] + \sum_{j=1}^{N_y} A_{ij} y_j(t - \tau_y), \quad i = 1, \ldots, N_x,
$$

$$
\dot{y}_j(t) = -y_j(t) + y_j(t) + \sum_{i=1}^{N_x} B_{ji} x_i(t - \tau_x),
$$

where $s(x) = m_1 x + 1/2 (m_0 - m_1) (|x + 1| - |x - 1|)$. Here, we take $a = 4.6, b = 6.02, m_0 = -8/7, m_1 = -5/7$, for numerical simulation. We introduce two quantities $E_x = \max_{i=1, \ldots, N_x} \|x_i - x_s\|$ and $E_y = \max_{j=1, \ldots, N_y} \|y_j - y_s\|$ to measure the extent to which synchronization is achieved. The initial values are randomly chosen in $(0, 1)$.

The intercoupling matrices are

$$
A = \begin{pmatrix}
0 & 1 & 1 & 1 \\
1 & 2 & 4 & 2 \\
0 & 3 & 0 & 1 \\
1 & 1 & 1 & 1
\end{pmatrix}, \quad B = \begin{pmatrix}
1 & 0 & 1 \\
2 & 0 & 2 \\
1 & 1 & 2 \\
2 & 2 & 0
\end{pmatrix}.
$$

(3.2)

When $a = 4.6, b = 6.02, m_0 = -8/7, m_1 = -5/7, \tau_x = \tau_y = 0.8$ and $A, B$; by using the Matlab LMI Toolbox, we solve the LMI (2.13) for $P > 0, Q > 0$, and obtain

$$
P = (2.5874); \quad Q = \begin{pmatrix}
10.8749 & -0.9036 \\
-0.9036 & 1.8461
\end{pmatrix}.
$$

(3.3)
which indicates that inner synchronization inside each network appears. The inner synchronization evolution curves are depicted in Figure 2. We plots $E_x + E_y$ with different values of $\tau_x$ or $\tau_y$ in Figure 3, which shows that the delays $\tau_x, \tau_y$ have little influence on inner synchronization with such intercoupling matrices $A, B$. Note that the order of magnitude for $E_x + E_y$ is $10^{-8}$.

Next, we change the connection topology of $A, B$ and consider a special case—globally connected, the elements of $A_{ij} = 1/N_y, B_{ji} = 1/N_x, i = 1, \ldots, N_x, \text{ and } j = 1, \ldots, N_y$ with $N_x = 50, N_y = 100$. The other parameters are the same as in Figure 2. The numerical results are given in Figures 4 and 5. It is shown that the node numbers and connection topology influence the inner synchronization less.
Figure 4: Inner synchronization evolution curves with globally connected topology, where $N_x = 50$, $N_y = 100$, and $\tau_x = \tau_y = 0.8$.

Figure 5: The curves of $E_x + E_y$ at $t = 40$ versus $\tau_y$, with $\tau_x = 0.8$, $N_x = 50$, and $N_y = 100$.

Example 3.2. In this example, we let $n_x = n_y$ and $N_x = N_y = N$. Therefore, we discuss inner synchronization inside network X or Y and study synchronization between two networks, that is, outer synchronization proposed in [11], denoting the quantity $E_{outer} = \max_{i=1,\ldots,N} \|x_i - y_i\|$, to demonstrate that outer synchronization happens. Consider a one-order dynamical system [23] as the dynamical nodes of the complex networks which is described by

$$\dot{x}_i(t) = -x_i^2(t) + \rho_1 x_i(t) + r_x \sum_{j=1}^{N} A_{ij} y_j(t - \tau_y), \quad i = 1, 2, \ldots, N,$$
The elements of $A_{ij} = 1/N$, $B_{ji} = 1/N$, $i, j = 1, \ldots, N$, and $N = 50$, when $\rho_1 = -0.3$, $\rho_2 = 0.6$, $\tau_x = 0.3$, and $\tau_y = 0.5$, by solving the LMI (2.13) for $P > 0$, $Q > 0$, we obtain

$$ P = (2.0946), \quad Q = (1.9328), $$

satisfying the conditions of a synchronous criterion, then the individual network achieves its own inner synchronized state. Figure 6 plots the inner and outer synchronization evolution curves. In addition, we plot the curves of quantities $E_x, E_y, E_{\text{outer}}$ with regard to $r_x$ for $r_y = 0.2$ in Figure 7. We find that the individual network easily achieves inner synchronization for the arbitrary values of $r_x$, while only when $r_x = 1.2$, outer synchronization is realized.

In the following, we numerically discuss the delay effect on inner and outer synchronization. Fix the values of $\rho_1 = -0.3$, $\rho_2 = 0.6$, $r_y = 0.2$, $r_x = 1.2$, and $\tau_y = 0.5$, and let the value of $\tau_x$ vary. We observe that the delay does not influence inner synchronization, and when $\tau_x > 1.5$, outer synchronization disappears. Figure 8 plots the curves of $E_x, E_y, E_{\text{outer}}$ with regard to $\tau_x$.

Now, we add the individual connections in network $X$ and network $Y$, and the following network systems are written as:

$$
\dot{x}_i(t) = -x_i^2(t) + \rho_1 x_i(t) + r_x \sum_{j=1}^{N} A_{ij} y_j(t - \tau_y) + \sum_{m=1}^{N} C_{i,m} x_m(t), \quad i = 1, 2, \ldots, N,
$$

$$
\dot{y}_j(t) = -y_j^2(t) + \rho_2 y_j(t) + r_y \sum_{i=1}^{N} B_{ji} x_i(t - \tau_x) + \sum_{k=1}^{N} D_{j,k} y_k(t), \quad j = 1, 2, \ldots, N,
$$
where \((-C) = -\{C_{im}\}, (-D) = -\{D_{jk}\}\) are the Laplacian matrices: \(\sum_{m=1}^{N} C_{im} = 0\) for all \(i\) and \(\sum_{k=1}^{N} D_{jk} = 0\) for all \(j\); obviously, the diffusive couplings \(\sum_{m=1}^{N} C_{im}, \sum_{k=1}^{N} D_{jk}\) are null in the synchronized manifold. The parameters are chosen as the same as those in Figure 8, and individual couplings \(C, D\) are taken as two random matrices. Numerical results show that the adding individual connections have little effect on inner and outer synchronization.
4. Conclusions

In conclusion, we have studied inner synchronization inside networks $X$ and $Y$ and outer synchronization between them with time delays. A synchronous criterion for the occurrence of inner synchronization has been derived. Numerical examples show that the delays have little effect on inner synchronization, how to estimate the domain of delays $\tau_x$ and $\tau_y$ is an interesting work. It is noted that in Example 3.2, outer synchronization only happens in the coupling strength $rx = 1.2$, and perhaps the node dynamics in networks $X$ and $Y$ plays a central role, or the bidirectional delayed coupling is weak. Because of the various kinds of actions between two networks, how to derive the criteria on inner and outer synchronization simultaneously is an technical challenging. The theoretical understanding is helpful to study the synchronization between two or more neural networks [24] with appropriate couplings. We hope that such work will appear elsewhere.

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References


