Research Article

On the Higher-Order q -Euler Numbers and Polynomials with Weight α

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The main purpose of this paper is to present a systemic study of some families of higher-order q -Euler numbers and polynomials with weight α . In particular, by using the fermionic p -adic q -integral on $\mathbb{Z}_p$, we give a new concept of q -Euler numbers and polynomials with weight α .

1. Introduction

Let p be a fixed odd prime. Throughout this paper $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}$, and $\mathbb{C}_p$ will, respectively, denote the ring of p -adic rational integers, the field, of p -adic rational numbers, the complex number field and the completion of algebraic closure of $\mathbb{Q}_p$. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let v_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = p^{-1}$ (see [1–14]). When one speaks of q -extension, q can be regarded as an indeterminate, complex number $q \in \mathbb{C}$, or p -adic number $q \in \mathbb{C}_p$; it is always clear from context. If $q \in \mathbb{C}$, we assume $|q| < 1$. If $q \in \mathbb{C}_p$, then we assume $|1 - q|_p < 1$ (see [1–14]).

In this paper, we use the notation of q -number as follows:

$$[x]_q = \frac{1 - q^x}{1 - q} \quad (1.1)$$

(see [1–14]). Note that $\lim_{q \rightarrow 1} [x]_q = x$ for any x with $|x|_p \leq 1$ in the p -adic case.

Let $C(\mathbb{Z}_p)$ be the space of continuous functions on $\mathbb{Z}_p$. For $f \in C(\mathbb{Z}_p)$, the fermionic p -adic q -integral on $\mathbb{Z}_p$ is defined by

$$\begin{aligned} I_{-q}(f) &= \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = \lim_{N \rightarrow \infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} f(x) (-q)^x, \\ &= \lim_{N \rightarrow \infty} \frac{[2]_q}{2} \sum_{x=0}^{p^N-1} f(x) (-q)^x \end{aligned} \quad (1.2)$$

(see [4–7]).

From (1.2), we note that

$$qI_{-q}(f_1) + I_{-q}(f) = [2]_q f(0), \quad (1.3)$$

where $f_1(x) = f(x+1)$.

It is well known that the ordinary Euler polynomials are defined by

$$\frac{2}{e^t + 1} e^{xt} = e^{E(x)t} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \quad (1.4)$$

with the usual convention of replacing $E^n(x)$ by $E_n(x)$.

In the special case, $x = 0$ and $E_n(0) = E_n$ are called the n th Euler numbers (see [1–14]).

By (1.5), we get the following recurrence relation as follows:

$$E_0 = 1, \quad (E+1)^n + E = \begin{cases} 2, & \text{if } n = 0, \\ 0, & \text{if } n > 0. \end{cases} \quad (1.5)$$

Recently, (h, q) -Euler numbers are defined by

$$E_{0,q}^{(h)} = \frac{2}{1+q^h}, \quad q^h (qE_q^{(h)} + 1)^n + E_q^{(h)} = \begin{cases} 2, & \text{if } n = 0, \\ 0, & \text{if } n > 0, \end{cases} \quad (1.6)$$

with the usual convention about replacing $(E_q^{(h)})^n$ by $E_{n,q}^{(h)}$ (see [1–12]).

Note that $\lim_{q \rightarrow 1} E_{n,q}^{(h)} = E_n$.

For $\alpha \in \mathbb{N}$, the weight q -Euler numbers are also defined by

$$\tilde{E}_{0,q}^{(\alpha)} = 1, \quad q \left(q^\alpha \tilde{E}_q^{(\alpha)} + 1 \right)^n + \tilde{E}_{n,q}^{(\alpha)} = \begin{cases} [2]_{q^\alpha}, & \text{if } n = 0, \\ 0, & \text{if } n > 0, \end{cases} \quad (1.7)$$

with the usual convention about replacing $(\tilde{E}_q^{(\alpha)})^n$ by $\tilde{E}_{n,q}^{(\alpha)}$ (see [4]).

The purpose of this paper is to present a systemic study of some families of higher-order q -Euler numbers and polynomials with weight α . In particular, by using the fermionic p -adic q -integral on $\mathbb{Z}_p$, we give a new concept of q -Euler numbers and polynomials with weight α .

2. Higher-Order q -Euler Numbers and Polynomials with Weight α

For $h \in \mathbb{Z}$, $\alpha, k \in \mathbb{N}$, and $n \in \mathbb{Z}_+$, let us consider the expansion of higher-order q -Euler polynomials with weight α as follows:

$$\begin{aligned} \tilde{E}_{n,q}^{(\alpha)}(h, k | x) \\ = \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{k\text{-times}} [x_1 + \cdots + x_k + x]_{q^\alpha}^n q^{x_1(h-1) + \cdots + x_k(h-k)} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k). \end{aligned} \quad (2.1)$$

From (1.2) and (2.1), we note that:

$$\tilde{E}_{n,q}^{(\alpha)}(h, k | x) = \frac{[2]_q^k}{(1 - q^\alpha)^n} \sum_{l=0}^n \binom{n}{l} (-1)^l \frac{q^{\alpha l x}}{(1 + q^{\alpha l + h}) \cdots (1 + q^{\alpha l + h - k + 1})}. \quad (2.2)$$

In the special case, $x = 0$, $\tilde{E}_{n,q}^{(\alpha)}(h, k | 0) = \tilde{E}_{n,q}^{(\alpha)}(h, k)$ are called the higher-order q -Euler numbers with weight α .

By (2.1), we get

$$\tilde{E}_{n,q}^{(\alpha)}(h, k) = (q^\alpha - 1) \tilde{E}_{n+1,q}^{(\alpha)}(h - \alpha, k) + \tilde{E}_{n,q}^{(\alpha)}(h - \alpha, k). \quad (2.3)$$

From (2.1) and (2.2), we have

$$\begin{aligned} \tilde{E}_{0,q}^{(\alpha)}(m\alpha, k + 1) \\ = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{\sum_{j=1}^{k+1} (m\alpha - j)x_j} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_{k+1}) \\ = \sum_{l=0}^m \binom{m}{l} (q^\alpha - 1)^l \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x_1 + \cdots + x_{k+1}]_{q^\alpha}^l q^{-\sum_{j=1}^{k+1} jx_j} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_{k+1}) \\ = \sum_{l=0}^m \binom{m}{l} (q^\alpha - 1)^l \tilde{E}_{l,q}^{(\alpha)}(0, k + 1) \\ = \frac{[2]_q^{k+1}}{(1 + q^{\alpha m})(1 + q^{\alpha m - 1}) \cdots (1 + q^{\alpha m - k})}. \end{aligned} \quad (2.4)$$

From (2.1), we can derive the following equation:

$$\begin{aligned}
& \sum_{j=0}^i \binom{i}{j} (q^\alpha - 1)^j \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x_1 + \cdots + x_k]_{q^\alpha}^{n-i+j} q^{(h-\alpha-1)x_1 + \cdots + (h-\alpha-k)x_k} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k) \\
&= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x_1 + \cdots + x_k]_{q^\alpha}^{n-i} q^{(h-1)x_1 + \cdots + (h-k)x_k} q^{\alpha(x_1 + \cdots + x_k)(i-1)} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k) \quad (2.5) \\
&= \sum_{j=0}^{i-1} (q^\alpha - 1)^j \binom{i-1}{j} \tilde{E}_{n-i+j,q}^{(\alpha)}(h, k).
\end{aligned}$$

By (2.1), (2.2), (2.3), and (2.4), we see that

$$\sum_{j=0}^i (q^\alpha - 1)^j \binom{i}{j} \tilde{E}_{n-i+j,q}^{(\alpha)}(h - \alpha, k) = \sum_{j=0}^{i-1} (q^\alpha - 1)^j \binom{i-1}{j} \tilde{E}_{n-i+j,q}^{(\alpha)}(h, k). \quad (2.6)$$

Therefore, we obtain the following theorem.

Theorem 2.1. For $\alpha, k \in \mathbb{N}$ and $n, i \in \mathbb{Z}_+$, one has

$$\sum_{j=0}^i \binom{i}{j} (q^\alpha - 1)^j \tilde{E}_{n-i+j,q}^{(\alpha)}(h - \alpha, k) = \sum_{j=0}^{i-1} (q^\alpha - 1)^j \binom{i-1}{j} \tilde{E}_{n-i+j,q}^{(\alpha)}(h, k). \quad (2.7)$$

By simple calculation, we easily see that

$$\sum_{j=0}^m \binom{m}{j} (q^\alpha - 1)^j \tilde{E}_{j,q}^{(\alpha)}(0, k) = \frac{[2]_q^k}{(1 + q^{\alpha m})(1 + q^{\alpha m-1}) \cdots (1 + q^{\alpha m-k+1})}. \quad (2.8)$$

3. Polynomials $\tilde{E}_{n,q}^{(\alpha)}(0, k \mid x)$

We now consider the polynomials $\tilde{E}_{n,q}^{(\alpha)}(0, k \mid x)$ (in q^x) by

$$\tilde{E}_{n,q}^{(\alpha)}(0, k \mid x) = \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{k\text{-times}} [x + x_1 + \cdots + x_k]_{q^\alpha}^n q^{-\sum_{j=1}^k jx_j} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k). \quad (3.1)$$

By (3.1), we get

$$(q^\alpha - 1)^n \tilde{E}_{n,q}^{(\alpha)}(0, k \mid x) = [2]_q^k \sum_{l=0}^n \binom{n}{l} q^{alx} (-1)^{n-l} \frac{1}{(1 + q^{\alpha l}) \cdots (1 + q^{\alpha l-k+1})}. \quad (3.2)$$

From (3.1) and (3.2), we can derive the following equation:

$$\begin{aligned} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{\sum_{j=1}^k (an-j)x_j + anx} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k) &= \sum_{j=0}^n \binom{n}{j} [\alpha]_q^j (q-1)^j \tilde{E}_{j,q}^{(\alpha)}(0, k \mid x), \\ \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{\sum_{j=1}^k (an-j)x_j + anx} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k) &= \frac{[2]_q^k q^{anx}}{(1+q^{an}) \cdots (1+q^{an-k+1})}. \end{aligned} \quad (3.3)$$

Therefore, by (3.2) and (3.3), we obtain the following theorem.

Theorem 3.1. For $\alpha \in \mathbb{N}$ and $n, k \in \mathbb{Z}_+$, one has

$$\begin{aligned} \tilde{E}_{n,q}^{(\alpha)}(0, k \mid x) &= \frac{[2]_q^k}{[\alpha]_q^n (1-q)^n} \sum_{l=0}^n \binom{n}{l} (-1)^l q^{alx} \frac{1}{(-q^{al-k+1}; q)_k}, \\ \sum_{l=0}^n \binom{n}{l} [\alpha]_q^l (q-1)^l \tilde{E}_{l,q}^{(\alpha)}(0, k \mid x) &= \frac{q^{anx} [2]_q^k}{(-q^{an-k+1}; q)_k}, \end{aligned} \quad (3.4)$$

where $(a : q)_0 = 1$ and $(a : q)_k = (1-a)(1-aq) \cdots (1-aq^{k-1})$.

Let $d \in \mathbb{N}$ with $d \equiv 1 \pmod{2}$. Then we have

$$\begin{aligned} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left[x + \sum_{j=1}^k x_j \right]_{q^a}^n q^{-\sum_{j=1}^k jx_j} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k) \\ = \frac{[d]_{q^a}^n}{[d]_{-q}^k} \sum_{a_1, \dots, a_k=0}^{d-1} q^{-\sum_{j=2}^k (j-1)a_j} (-1)^{\sum_{j=1}^k a_j} \\ \times \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left[\frac{x + \sum_{j=1}^k a_j}{d} + \sum_{j=1}^k x_j \right]_{q^{ad}}^n q^{-d \sum_{j=1}^k jx_j} d\mu_{-q^d}(x_1) \cdots d\mu_{-q^d}(x_k). \end{aligned} \quad (3.5)$$

Thus, by (3.5), we obtain the following theorem.

Theorem 3.2. For $d \in \mathbb{N}$ with $d \equiv 1 \pmod{2}$, one has

$$\tilde{E}_{n,q}^{(\alpha)}(0, k \mid x) = \frac{[d]_{q^a}^n}{[d]_{-q}^k} \sum_{a_1, \dots, a_k=0}^{d-1} q^{-\sum_{j=2}^k (j-1)a_j} (-1)^{\sum_{j=1}^k a_j} \tilde{E}_{n,q^d}^{(\alpha)}\left(0, k \mid \frac{x + a_1 + \cdots + a_k}{d}\right). \quad (3.6)$$

Moreover,

$$\tilde{E}_{n,q}^{(\alpha)}(0, k \mid dx) = \frac{[d]_{q^\alpha}^n}{[d]_{-q}^k} \sum_{a_1, \dots, a_k=0}^{d-1} q^{-\sum_{j=2}^k (j-1)a_j} (-1)^{\sum_{j=1}^k a_j} \tilde{E}_{n,q^d}^{(\alpha)}\left(0, k \mid x + \frac{a_1 + \dots + a_k}{d}\right). \quad (3.7)$$

By (3.1), we get

$$\tilde{E}_{n,q}^{(\alpha)}(0, k \mid x) = \sum_{l=0}^n \binom{n}{l} [x]_{q^\alpha}^{n-l} q^{alx} \tilde{E}_{l,q}^{(\alpha)}(0, k), \quad (3.8)$$

where $\tilde{E}_{n,q}^{(\alpha)}(0, k \mid 0) = \tilde{E}_{n,q}^{(\alpha)}(0, k)$.

Thus, we note that

$$\tilde{E}_{n,q}^{(\alpha)}(0, k \mid x + y) = \sum_{l=0}^n \binom{n}{l} [y]_{q^\alpha}^{n-l} q^{aly} \tilde{E}_{l,q}^{(\alpha)}(0, k \mid x). \quad (3.9)$$

4. Polynomials $\tilde{E}_{n,q}^{(\alpha)}(h, 1 \mid x)$

Let us define polynomials $\tilde{E}_{n,q}^{(\alpha)}(h, 1 \mid x)$ as follows:

$$\tilde{E}_{n,q}^{(\alpha)}(h, 1 \mid x) = \int_{\mathbb{Z}_p} [x + x_1]_{q^\alpha}^n q^{x_1(h-1)} d\mu_{-q}(x_1). \quad (4.1)$$

From (4.1), we have

$$\tilde{E}_{n,q}^{(\alpha)}(h, 1 \mid x) = \frac{[2]_q}{(1 - q^\alpha)^n} \sum_{l=0}^n \binom{n}{l} (-1)^l q^{alx} \frac{1}{(1 + q^{al+h})}. \quad (4.2)$$

By the calculation of the fermionic p -adic q -integral on $\mathbb{Z}_p$, we see that

$$\begin{aligned} & q^{\alpha x} \int_{\mathbb{Z}_p} [x + x_1]_{q^\alpha}^n q^{x_1(h-1)} d\mu_{-q}(x_1) \\ &= (q^\alpha - 1) \int_{\mathbb{Z}_p} [x + x_1]_{q^\alpha}^{n+1} q^{x_1(h-\alpha-1)} d\mu_{-q}(x_1) + \int_{\mathbb{Z}_p} [x + x_1]_{q^\alpha}^n q^{x_1(h-\alpha-1)} d\mu_{-q}(x_1). \end{aligned} \quad (4.3)$$

Thus, by (4.3), we obtain the following theorem.

Theorem 4.1. For $\alpha \in \mathbb{N}$ and $h \in \mathbb{Z}$, one has

$$q^{\alpha x} \tilde{E}_{n,q}^{(\alpha)}(h, 1 \mid x) = (q^\alpha - 1) \tilde{E}_{n+1,q}^{(\alpha)}(h - \alpha - 1, 1 \mid x) + \tilde{E}_{n,q}^{(\alpha)}(h - \alpha - 1, 1 \mid x). \quad (4.4)$$

It is easy to show that

$$\begin{aligned}
 \tilde{E}_{n,q}^{(\alpha)}(h, 1 \mid x) &= \int_{\mathbb{Z}_p} [x + x_1]_{q^\alpha}^n q^{x_1(h-1)} d\mu_{-q}(x_1) \\
 &= \sum_{l=0}^n \binom{n}{l} [x]_{q^\alpha}^{n-l} q^{\alpha l x} \int_{\mathbb{Z}_p} [x_1]_{q^\alpha}^l q^{x_1(h-1)} d\mu_{-q}(x_1) \\
 &= \sum_{l=0}^n \binom{n}{l} [x]_{q^\alpha}^{n-l} q^{\alpha l x} \tilde{E}_{l,q}^{(\alpha)}(h, 1) \\
 &= \left(q^{\alpha x} \tilde{E}_q^{(\alpha)}(h, 1) + [x]_{q^\alpha} \right)^n, \quad \text{for } n \geq 1,
 \end{aligned} \tag{4.5}$$

with the usual convention about replacing $(\tilde{E}_q^{(\alpha)}(h, 1))^n$ by $\tilde{E}_{n,q}^{(\alpha)}(h, 1)$.

From $qI_{-q}(f_1) + I_{-q}(f) = [2]_q f(0)$, we have

$$q^h \int_{\mathbb{Z}_p} [x + x_1 + 1]_{q^\alpha}^n q^{x_1(h-1)} d\mu_{-q}(x_1) + \int_{\mathbb{Z}_p} [x + x_1]_{q^\alpha}^n q^{x_1(h-1)} d\mu_{-q}(x_1) = [2]_q [x]_{q^\alpha}^n. \tag{4.6}$$

By (4.3) and (4.6), we get

$$q^h \tilde{E}_{n,q}^{(\alpha)}(h, 1 \mid x + 1) + \tilde{E}_{n,q}^{(\alpha)}(h, 1 \mid x) = [2]_q [x]_{q^\alpha}^n. \tag{4.7}$$

For $x = 0$ in (4.7), we have

$$q^h \tilde{E}_{n,q}^{(\alpha)}(h, 1 \mid 1) + \tilde{E}_{n,q}^{(\alpha)}(h, 1) = \begin{cases} [2]_q, & \text{if } n = 0, \\ 0, & \text{if } n > 0. \end{cases} \tag{4.8}$$

Therefore, by (4.8), we obtain the following theorem.

Theorem 4.2. For $h \in \mathbb{Z}$ and $n \in \mathbb{Z}_+$, one has

$$q^h \left(q^\alpha \tilde{E}_q^{(\alpha)}(h, 1) + 1 \right)^n + \tilde{E}_{n,q}^{(\alpha)}(h, 1) = \begin{cases} [2]_q, & \text{if } n = 0, \\ 0, & \text{if } n > 0, \end{cases} \tag{4.9}$$

with the usual convention about replacing $(\tilde{E}_q^{(\alpha)}(h, 1))^n$ by $\tilde{E}_{n,q}^{(\alpha)}(h, 1)$.

From the fermionic p -adic q -integral on $\mathbb{Z}_p$, we easily get

$$\tilde{E}_{0,q}^{(\alpha)}(h, 1) = \int_{\mathbb{Z}_p} q^{x_1(h-1)} d\mu_{-q}(x_1) = \frac{[2]_q}{[2]_{q^h}}. \tag{4.10}$$

By (4.1), we see that

$$\begin{aligned}
 \tilde{E}_{n,q^{-1}}^{(\alpha)}(h, 1 \mid 1-x) &= \int_{\mathbb{Z}_p} [1-x+x_1]_{q^{-\alpha}}^n q^{-x_1(h-1)} d\mu_{-q^{-1}}(x_1) \\
 &= (-1)^n q^{\alpha n+h-1} \frac{[2]_q}{(1-q^\alpha)^n} \sum_{l=0}^n \binom{n}{l} (-1)^l q^{\alpha l x} \frac{1}{1+q^{\alpha l+h}} \\
 &= (-1)^n q^{\alpha n+h-1} \tilde{E}_{n,q^{-1}}^{(\alpha)}(h, 1 \mid x).
 \end{aligned} \tag{4.11}$$

Therefore, by (4.11), we obtain the following theorem.

Theorem 4.3. For $\alpha \in \mathbb{N}$, $h \in \mathbb{Z}$, and $n \in \mathbb{Z}_+$, one has

$$\tilde{E}_{n,q^{-1}}^{(\alpha)}(h, 1 \mid 1-x) = (-1)^n q^{\alpha n+h-1} \tilde{E}_{n,q}^{(\alpha)}(h, 1 \mid x). \tag{4.12}$$

In particular, for $x = 1$, one gets

$$\begin{aligned}
 \tilde{E}_{n,q}^{(\alpha)}(h, 1) &= (-1)^n q^{\alpha n+h-1} \tilde{E}_{n,q}^{(\alpha)}(h, 1 \mid 1) \\
 &= (-1)^{n-1} q^{\alpha n-1} \tilde{E}_{n,q}^{(\alpha)}(h, 1) \quad \text{if } n \geq 1.
 \end{aligned} \tag{4.13}$$

Let $d \in \mathbb{N}$ with $d \equiv 1 \pmod{2}$. Then one has

$$\begin{aligned}
 &\int_{\mathbb{Z}_p} q^{x_1(h-1)} [x+x_1]_{q^\alpha}^n d\mu_{-q}(x_1) \\
 &= \frac{[d]_{q^\alpha}^n}{[d]_{-q}} \sum_{a=0}^{d-1} q^{ha} (-1)^a \int_{\mathbb{Z}_p} \left[\frac{x+a}{d} + x_1 \right]_{q^{\alpha d}}^n q^{x_1(h-1)d} d\mu_{-q^d}(x_1).
 \end{aligned} \tag{4.14}$$

Therefore, by (4.14), we obtain the following theorem.

Theorem 4.4 (Multiplication formula). For $d \in \mathbb{N}$ with $d \equiv 1 \pmod{2}$, we have

$$\tilde{E}_{n,q}^{(\alpha)}(h, 1 \mid x) = \frac{[d]_{q^\alpha}^n}{[d]_{-q}} \sum_{a=0}^{d-1} q^{ha} (-1)^a \tilde{E}_{n,q^d}^{(\alpha)}\left(h, 1 \mid \frac{x+a}{d}\right). \tag{4.15}$$

5. Polynomials $\tilde{E}_{n,q}^{(\alpha)}(h, k \mid x)$ and $k = h$

In (2.1), we know that

$$\begin{aligned} & \tilde{E}_{n,q}^{(\alpha)}(h, k \mid x) \\ &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x_1 + \cdots + x_k + x]_{q^\alpha}^n q^{(h-1)x_1 + \cdots + (h-k)x_k} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k). \end{aligned} \quad (5.1)$$

Thus, we get

$$\begin{aligned} (q^\alpha - 1)^n \tilde{E}_{n,q}^{(\alpha)}(h, k \mid x) &= [2]_q^k \sum_{l=0}^n \binom{n}{l} (-1)^{n-l} \frac{q^{\alpha l x}}{(1 + q^{\alpha l + h}) \cdots (1 + q^{\alpha l + h - k + 1})}, \\ q^h \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x + 1 + x_1 + \cdots + x_k]_{q^\alpha}^n q^{(h-1)x_1 + \cdots + (h-k)x_k} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k) \\ &= - \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x + x_1 + \cdots + x_k]_{q^\alpha}^n q^{(h-1)x_1 + \cdots + (h-k)x_k} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k) \\ &\quad + [2]_q \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x + x_2 + \cdots + x_k]_{q^\alpha}^n q^{(h-2)x_2 + \cdots + (h-k)x_k} d\mu_{-q}(x_2) \cdots d\mu_{-q}(x_k). \end{aligned} \quad (5.2)$$

Therefore, by (2.1) and (5.2), we obtain the following theorem.

Theorem 5.1. For $h \in \mathbb{Z}$, $\alpha \in \mathbb{N}$, and $n \in \mathbb{Z}_+$, one has

$$q^h \tilde{E}_{n,q}^{(\alpha)}(h, k \mid x + 1) + \tilde{E}_{n,q}^{(\alpha)}(h, k \mid x) = [2]_q \tilde{E}_{n,q}^{(\alpha)}(h - 1, k - 1 \mid x). \quad (5.3)$$

Note that

$$\begin{aligned} & q^{\alpha x} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x + x_1 + \cdots + x_k]_{q^\alpha}^n q^{hx_1 + (h-1)x_2 + \cdots + (h+1-k)x_k} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k) \\ &= (q^\alpha - 1) \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x + x_1 + \cdots + x_k]_{q^\alpha}^{n+1} q^{(h-\alpha)x_1 + \cdots + (h+1-\alpha-k)x_k} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k) \\ &\quad + \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x + x_1 + \cdots + x_k]_{q^\alpha}^n q^{(h-\alpha)x_1 + \cdots + (h+1-\alpha-k)x_k} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k) \\ &= (q^\alpha - 1) \tilde{E}_{n+1,q}^{(\alpha)}(h + 1 - \alpha, k \mid x) + \tilde{E}_{n,q}^{(\alpha)}(h + 1 - \alpha, k \mid x). \end{aligned} \quad (5.4)$$

Therefore, by (5.4), we obtain the following theorem.

Theorem 5.2. For $n \in \mathbb{Z}_+$, one has

$$q^{\alpha x} \tilde{E}_{n,q}^{(\alpha)}(h+1, k | x) = (q^\alpha - 1) \tilde{E}_{n+1,q}^{(\alpha)}(h+1 - \alpha, k | x) + \tilde{E}_{n,q}^{(\alpha)}(h+1 - \alpha, k | x). \quad (5.5)$$

Let $d \in \mathbb{N}$ with $d \equiv 1 \pmod{2}$. Then we get

$$\begin{aligned} & \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left[x + \sum_{j=1}^k x_j \right]_{q^\alpha}^n q^{\sum_{j=1}^k (h-j)x_j} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k) \\ &= \frac{[d]_{q^\alpha}^n}{[d]_{-q}^k} \sum_{a_1, \dots, a_k=0}^{d-1} q^{h \sum_{j=1}^k a_j - \sum_{j=2}^k (j-1)a_j} (-1)^{\sum_{j=1}^k a_j} \\ & \quad \times \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left[\frac{x + \sum_{j=1}^k a_j}{d} + \sum_{j=1}^k x_j \right]_{q^{ad}}^n q^{d \sum_{j=1}^k (h-j)x_j} d\mu_{-q^d}(x_1) \cdots d\mu_{-q^d}(x_k). \end{aligned} \quad (5.6)$$

Therefore, by (5.6), we obtain the following theorem.

Theorem 5.3. For $d \in \mathbb{N}$ with $d \equiv 1 \pmod{2}$, one has

$$\begin{aligned} & \tilde{E}_{n,q}^{(\alpha)}(h, k | dx) \\ &= \frac{[d]_{q^\alpha}^n}{[d]_{-q}^k} \sum_{a_1, \dots, a_k=0}^{d-1} q^{h \sum_{j=1}^k a_j - \sum_{j=2}^k (j-1)a_j} (-1)^{\sum_{j=1}^k a_j} \tilde{E}_{n,q^d}^{(\alpha)}\left(h, k | x + \frac{a_1 + \cdots + a_k}{d}\right). \end{aligned} \quad (5.7)$$

Let $\tilde{E}_{n,q}^{(\alpha)}(k, k | x) = \tilde{E}_{n,q}^{(\alpha)}(k | x)$. Then we get

$$\begin{aligned} & (q^\alpha - 1)^n \tilde{E}_{n,q}^{(\alpha)}(k | x), \\ &= \sum_{l=0}^n \binom{n}{l} (-1)^{n-l} q^{\alpha l x} \frac{[2]_q^k}{(1 + q^{\alpha l + k}) \cdots (1 + q^{\alpha l + 1})} \\ & \quad \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [k - x + x_1 + \cdots + x_k]_{q^{-\alpha}}^n q^{-(k-1)x_1 - \cdots - (k-k)x_k} d\mu_{-q^{-1}}(x_1) \cdots d\mu_{-q^{-1}}(x_k) \\ &= \frac{q^{\alpha \binom{k+1}{2} - k}}{(1 - q^{-\alpha})^n} [2]_q^k \sum_{l=0}^n \binom{n}{l} (-1)^l q^{\alpha l x} \frac{1}{(1 + q^{\alpha l + 1}) \cdots (1 + q^{\alpha l + k})} \\ &= (-1)^n q^{n\alpha} q^{\alpha \binom{k+1}{2} - k} \frac{[2]_q^k}{(1 - q^\alpha)^n} \sum_{l=0}^n \frac{\binom{n}{l} (-1)^l q^{\alpha l x}}{(1 + q^{\alpha l + 1}) \cdots (1 + q^{\alpha l + k})} \\ &= (-1)^n q^{\alpha \left(n + \binom{k+1}{2}\right) - k} \tilde{E}_{n,q}^{(\alpha)}(k | x). \end{aligned} \quad (5.8)$$

Therefore, by (5.8), we obtain the following theorem.

Theorem 5.4. *For $n \in \mathbb{Z}_+$, one has*

$$\tilde{E}_{n,q^{-1}}^{(\alpha)}(k \mid k-x) = (-1)^n q^{\alpha(n+\binom{k+1}{2})-k} \tilde{E}_{n,q}^{(\alpha)}(k \mid x). \quad (5.9)$$

Let $x = k$ in Theorem 5.4. Then we see that

$$\tilde{E}_{n,q^{-1}}^{(\alpha)}(k \mid 0) = (-1)^n q^{\alpha(n+\binom{k+1}{2})-k} \tilde{E}_{n,q}^{(\alpha)}(k \mid k). \quad (5.10)$$

From (4.6) and Theorem 5.1, we note that

$$q^k \tilde{E}_{n,q}^{(\alpha)}(k \mid x+1) + \tilde{E}_{n,q}^{(\alpha)}(k \mid x) = [2]_q \tilde{E}_{n,q}^{(\alpha)}(k-1 \mid x). \quad (5.11)$$

It is easy to show that

$$(q^\alpha - 1)^n \tilde{E}_{n,q}^{(\alpha)}(k \mid 0) = \sum_{l=0}^n \binom{n}{l} (-1)^{l+n} \frac{[2]_q^k}{(1+q^{\alpha l+1}) \cdots (1+q^{\alpha l+k})}. \quad (5.12)$$

By simple calculation, we get

$$\begin{aligned} \sum_{l=0}^n \binom{n}{l} (q^\alpha - 1)^l \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x_1 + \cdots + x_k]_{q^\alpha}^l q^{\sum_{l=1}^k (k-l)x_l} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k) \\ = \frac{[2]_q^k}{(1+q^{\alpha n+k})(1+q^{\alpha n+k-1}) \cdots (1+q^{\alpha n+1})}. \end{aligned} \quad (5.13)$$

From (5.13), we note that

$$\begin{aligned} \sum_{l=0}^n \binom{n}{l} (q^\alpha - 1)^l \tilde{E}_{l,q}^{(\alpha)}(k \mid 0) &= \frac{[2]_q^k}{(1+q^{\alpha n+k})(1+q^{\alpha n+k-1}) \cdots (1+q^{\alpha n+1})}, \\ \tilde{E}_{n,q}^{(\alpha)}(k \mid x) &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x + x_1 + \cdots + x_k]_{q^\alpha}^n q^{(k-1)x_1 + \cdots + (k-k)x_k} d\mu_{-q}(x_1) \\ &\quad \cdots d\mu_{-q}(x_k) \\ &= \sum_{l=0}^n \binom{n}{l} q^{\alpha l x} \tilde{E}_{l,q}^{(\alpha)}(k \mid 0) [x]_{q^\alpha}^{n-l} \\ &= \left(q^{x\alpha} \tilde{E}_q^{(\alpha)}(k \mid 0) + [x]_{q^\alpha} \right)^n \quad \text{for } n \in \mathbb{Z}_+, \end{aligned} \quad (5.14)$$

with the usual convention about replacing $(\tilde{E}_q^{(\alpha)}(k \mid 0))^n$ by $\tilde{E}_{n,q}^{(\alpha)}(k \mid 0)$.

Put $x = 0$ in (5.11); we get

$$q^k \tilde{E}_{n,q}^{(\alpha)}(k | 1) + \tilde{E}_{n,q}^{(\alpha)}(k | 0) = [2]_q \tilde{E}_{n,q}^{(\alpha)}(k - 1 | 0). \quad (5.15)$$

Thus, we have

$$q^k \left(q^\alpha \tilde{E}_q^{(\alpha)}(k | 0) + 1 \right)^n + \tilde{E}_{n,q}^{(\alpha)}(k | 0) = [2]_q \tilde{E}_{n,q}^{(\alpha)}(k - 1 | 0), \quad (5.16)$$

with the usual convention about replacing $(\tilde{E}_q^{(\alpha)}(k | 0))^n$ by $\tilde{E}_{n,q}^{(\alpha)}(k | 0)$.

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