Research Article
On the Values of the Weighted
$q$-Zeta and $L$-Functions

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Received 17 August 2011; Accepted 3 October 2011

Academic Editor: Binggen Zhang

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Recently, the modified $q$-Bernoulli numbers and polynomials are introduced in (D. V. Dolgy et al., in press). These numbers are valuable to study the weighted $q$-zeta and $L$-functions. In this paper, we study the weighted $q$-zeta functions and weighted $L$-functions from the modified $q$-Bernoulli numbers and polynomials with weight $\alpha$.

1. Introduction

Let $q \in \mathbb{C}$ with $|q| < 1$. The modified $q$-Bernoulli numbers and polynomials with weight $\alpha$ are defined by

$$
\tilde{B}^{(\alpha)}_{b,q} = \frac{q^{1-1}}{\log q}, \quad \left( q^{n} \tilde{B}^{(\alpha)}_{q} + 1 \right)^{n} - \tilde{B}^{(\alpha)}_{n,q} = \begin{cases} 
\frac{\alpha}{[\alpha]_{q}} & \text{if } n = 1, \\
0 & \text{if } n > 1,
\end{cases}
$$

(1.1)

with the usual convention about replacing $(\tilde{B}^{(\alpha)}_{q})^{n}$ by $\tilde{B}^{(\alpha)}_{n,q}$ (see [1, 2]).

Throughout this paper, we use the notation of $q$-number as

$$[x]_{q} = \frac{1 - q^{x}}{1 - q},
$$

(1.2)

(see [1–14]).
From (1.1), we note that

\[ \tilde{B}_{n,q}^{(a)} = \frac{1}{(1 - q^a)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{al}{[al]_q}, \]

\[ = \frac{1}{(1 - q)^n [a]_q^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{al}{[al]_q}. \]  

(1.3)

Let \( \tilde{F}_q^{(a)}(t) = \sum_{n=0}^{\infty} \tilde{B}_{n,q}^{(a)} t^n / n! \), then, by (1.3), we get

\[ \tilde{F}_q^{(a)}(t) = \alpha \frac{q-1}{\log q} e^{(1/(1-q^a))t} - \alpha t \frac{q^a}{[\alpha]_q} \sum_{m=0}^{\infty} q^{am} e^{[m]_q t}. \]  

(1.4)

Let us define the modified \( q \)-Bernoulli polynomials with weight \( \alpha \) as follows:

\[ \tilde{B}_{n,q}^{(a)}(x) = \sum_{l=0}^{n} \binom{n}{l} [x]_{q^a}^{n-l} q^{alx} \tilde{B}_{l,q}^{(a)} = ([x]_{q^a} + q^{xa} \tilde{B}_{l,q}^{(a)})^n, \]  

(1.5)

with the usual convention about replacing \( (\tilde{B}_q^{(a)})^n \) by \( \tilde{B}_{n,q}^{(a)} \) (see [1–13]).

From (1.5), we can derive the following equation:

\[ \tilde{B}_{n,q}^{(a)}(x) = \frac{1}{(1 - q^a)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{alx} \frac{al}{[al]_q}, \]

\[ = \frac{1}{(1 - q)^n [a]_q^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{alx} \frac{al}{[al]_q}. \]  

(1.6)

(see [2]).

Let \( \tilde{F}_q^{(a)}(t, x) = \sum_{n=0}^{\infty} \tilde{B}_{n,q}^{(a)}(x) t^n / n! \), then, by (1.6), we get

\[ \tilde{F}_q^{(a)}(t, x) = \alpha \frac{q-1}{\log q} e^{(1/(1-q^a))t} - t \frac{q^a}{[\alpha]_q} \sum_{m=0}^{\infty} q^{amx} e^{[m]_q t}. \]  

(1.7)

In this paper, we consider the generalized \( q \)-Bernoulli numbers with weight \( \alpha \), and we study the weighted \( q \)-zeta function and \( q \)-analogue of \( L \)-function with weight \( \alpha \) from the modified \( q \)-Bernoulli numbers and polynomials with weight \( \alpha \).

### 2. Weighted \( q \)-Zeta Function and Weighted \( q \)-\( L \)-Function

From (1.7), we note that

\[ \tilde{B}_{n,q}^{(a)}(x) = \frac{\alpha}{(1-q)^n [a]_q^n} \left( \frac{q-1}{\log q} \right) - \frac{na}{[\alpha]_q} \sum_{m=0}^{\infty} q^{amx} [m+x]_{q^a}^{n-1}. \]  

(2.1)
For \( n \in \mathbb{N} \), we have
\[
\frac{-\tilde{B}_{n,q}^{(a)}(x)}{n} = \left( \frac{\alpha}{[\alpha]_q} \right) \left( \frac{1}{1 - q^x} \right)^{n-1} \left( \frac{1}{\log q} \right) + \frac{\alpha}{[\alpha]_q} \sum_{m=0}^{\infty} q^{\alpha(m+x)} \left( m + x \right)^{-n}. \tag{2.2}
\]

Let \( \Gamma(s) \) be the gamma function, then we consider the following complex integral. For \( s \in \mathbb{C} \),
\[
\frac{1}{\Gamma(s)} \int_0^\infty F_q^{(a)}(-t, x)t^{s-2}dt = \frac{\alpha}{s-1} \left( \frac{q-1}{\log q} \right)^{s-1} + \frac{\alpha}{[\alpha]_q} \sum_{m=0}^{\infty} q^{\alpha(m+x)} \left( m + x \right)^{s}, \tag{2.3}
\]
where \( x \neq 0, -1, -2, -3, \ldots \).

Now, we define the twisted Hurwitz's type \( q \)-zeta function as follows.
For \( s \in \mathbb{C} \), define
\[
\tilde{\zeta}_q^{(a)}(s, x) = \frac{\alpha}{[\alpha]_q} \frac{1}{1 - s} \left( \frac{1 - q^x}{\log q} \right)^{s} + \frac{\alpha}{[\alpha]_q} \sum_{m=0}^{\infty} q^{\alpha(m+x)} \left( m + x \right)^{s}, \tag{2.4}
\]
where \( x \neq 0, -1, -2, -3, \ldots \).

Note that \( \tilde{\zeta}_q^{(a)}(s, x) \) is meromorphic function whole in complex \( s \)-plane except for \( s = 1 \).

From (2.3) and (2.4), we can derive the following equation:
\[
\tilde{\zeta}_q^{(a)}(s, x) = \frac{1}{\Gamma(s)} \int_0^\infty F_q^{(a)}(-t, x)t^{s-2}dt. \tag{2.5}
\]
By (1.7), (2.3), (2.4), (2.5), and Laurent series, we get
\[
\tilde{\zeta}_q^{(a)}(1 - k, x) = -\frac{\tilde{B}_{k,q}^{(a)}(x)}{k}, \tag{2.6}
\]
where \( k \in \mathbb{N} \).

Therefore, by (2.6), we obtain the following theorem.

**Theorem 2.1.** For \( k \in \mathbb{N} \), one has
\[
\tilde{\zeta}_q^{(a)}(1 - k, x) = -\frac{\tilde{B}_{k,q}^{(a)}(x)}{k}. \tag{2.7}
\]

From (2.4), one notes that
\[
\tilde{\zeta}_q^{(a)}(s, 1) = \frac{\alpha}{[\alpha]_q} \frac{1}{1 - s} \left( \frac{1 - q^1}{\log q} \right)^{s} + \frac{\alpha}{[\alpha]_q} \sum_{m=0}^{\infty} q^{\alpha(m+1)} \left( m + 1 \right)^{s}, \tag{2.8}
\]
\[
= \frac{\alpha}{[\alpha]_q} \frac{1}{1 - s} \left( \frac{1 - q^1}{\log q} \right)^{s} + \frac{\alpha}{[\alpha]_q} \sum_{m=0}^{\infty} q^{am} \left( m \right)^{s}.
\]
Now, by (2.8), one defines the weighted $q$-zeta function as follows:

$$
\tilde{\zeta}^{(a)}_{b_q}(s) = \frac{\alpha}{[\alpha]_q} \frac{1}{1 - s} \log q + \frac{\alpha}{[\alpha]_q} \sum_{m=1}^{\infty} \frac{q^{am}}{[m]_q^s}.
$$

(2.9)

For $k \in \mathbb{N}$, by (1.1) and (1.5), one gets

$$
\tilde{\zeta}^{(a)}_{b_q}(1 - k) = \tilde{\zeta}^{(a)}_{b_q}(1 - k, 1) = -\frac{\bar{B}^{(a)}_{k,q}(1)}{k}.
$$

(2.10)

Therefore, by (2.10), one obtains the following corollary.

**Corollary 2.2.** For $k \in \mathbb{N}$, one has

$$
\tilde{\zeta}^{(a)}_{b_q}(1 - k) = \begin{cases} 
-\left(\frac{\alpha}{[\alpha]_q} + \bar{B}^{(a)}_{1,q}\right) & \text{if } k = 1, \\
-\bar{B}^{(a)}_{k,q} & \text{if } k > 1.
\end{cases}
$$

(2.11)

Let $\chi$ be the Dirichlet’s character with conductor $d \in \mathbb{N}$. Let us consider the generalized $q$-Bernoulli polynomials with weight $\alpha$ as follows:

$$
\tilde{F}^{(a)}_{q,\chi}(t, x) = \frac{\alpha}{[\alpha]_q} t \sum_{m=0}^{\infty} \chi(m) q^{a(m + x)} e^{[m + x]_q \frac{t}{q}}.
$$

(2.12)

The sequence $\tilde{B}^{(a)}_{n,\chi,q}(x)$ will be called the $n$th generalized $q$-Bernoulli polynomials with weight $\alpha$ attached to $\chi$.

In the special case, $x = 0$, $\tilde{B}^{(a)}_{n,\chi,q}(0) = \tilde{B}^{(a)}_{n,\chi,q}$ are called the $n$th generalized $q$-Bernoulli numbers with weight $\alpha$ attached to $\chi$.

From (1.7) and (2.12), one notes that

$$
\tilde{F}^{(a)}_{q,\chi}(t, x) = \frac{1}{[d]_q} \sum_{a=0}^{d-1} \chi(a) \tilde{F}^{(a)}_{q}(\frac{[d]_q t}{q}, \frac{x + a}{d}).
$$

(2.13)
Thus, by (2.13), one gets

\[
\bar{B}^{(a)}_{n,x,q}(x) = \frac{[d]_q^n}{[d]_q} \sum_{a=0}^{d-1} \chi(a) \bar{B}^{(a)}_{n,q^a} \left( \frac{x+a}{d} \right). \tag{2.14}
\]

Therefore, by (2.14), one obtains the following theorem.

**Theorem 2.3.** For \( n \in \mathbb{Z}_+ \), one has

\[
\bar{B}^{(a)}_{n,x,q}(x) = \frac{[d]_q^n}{[d]_q} \sum_{a=0}^{d-1} \chi(a) \bar{B}^{(a)}_{n,q^a} \left( \frac{x+a}{d} \right). \tag{2.15}
\]

In the special case, \( x = 0 \), one obtains the following corollary.

**Corollary 2.4.** For \( n \in \mathbb{Z}_+ \), one has

\[
\bar{B}^{(a)}_{n,x,q} = \frac{[d]_q^n}{[d]_q} \sum_{a=0}^{d-1} \chi(a) \bar{B}^{(a)}_{n,q^a} \left( \frac{a}{d} \right). \tag{2.16}
\]

Let

\[
\bar{F}^{(a)}_{q,x}(t) = \frac{\alpha}{[\alpha]_q} \sum_{m=0}^{\infty} \chi(m) q^m e^{[m]_q t} \\
\quad = \sum_{n=0}^{\infty} \bar{B}^{(a)}_{n,x,q} \frac{t^n}{n!}, \tag{2.17}
\]

then, by (2.12) and (2.17), one easily gets

\[
\frac{\bar{B}^{(a)}_{n,x,q}(d) - \bar{B}^{(a)}_{n,x,q}}{n} = \frac{\alpha}{[\alpha]_q} \sum_{l=0}^{d-1} \chi(l) q^l [l]_q^{n-1}. \tag{2.18}
\]

For \( s \in \mathbb{C} \), consider

\[
\frac{1}{\Gamma(s)} \int_0^{\infty} \bar{F}^{(a)}_{q,x}(-t, x) t^{s-2} dt = \frac{\alpha}{[\alpha]_q} \frac{1}{\Gamma(s)} \int_0^{\infty} \sum_{m=0}^{\infty} \chi(m) q^m e^{-[m+1]_q^s t} t^{s-1} dt \\
\quad = \frac{\alpha}{[\alpha]_q} \sum_{m=0}^{\infty} \chi(m) q^m [m+x]_q^s \frac{1}{\Gamma(s)} \int_0^{\infty} e^{-y} y^{s-1} dy \\
\quad = \frac{\alpha}{[\alpha]_q} \sum_{m=0}^{\infty} \chi(m) q^m [m+x]_q^s, \tag{2.19}
\]

where \( x \neq 0, -1, -2, -3, \ldots \).
Now, one defines Hurwitz’s type $q$-L-function with weight $\alpha$ as follows. For $s \in \mathbb{C}$,

$$
\tilde{L}_{q}^{(\alpha)} (s, \chi \mid x) (-t, x) = \frac{\alpha}{\left[ \alpha \right]_{q}^{*}} \sum_{n=0}^{\infty} \frac{\chi(n)q^{(n+x)\alpha}}{[n + x]_{q}^{\alpha}},
$$

(2.20)

where $x \neq 0, -1, -2, -3, \ldots$.

From (2.19) and (2.20), one notes that

$$
\tilde{L}_{q}^{(\alpha)} (s, \chi \mid x) = \frac{1}{\Gamma(s)} \int_{0}^{\infty} \tilde{F}_{q}(s, t) t^{s-1} dt.
$$

(2.21)

By (1.7) and (2.21) and Laurent series, one obtains the following theorem.

**Theorem 2.5.** For $k \in \mathbb{N}$, one has

$$
\tilde{L}_{q}^{(\alpha)} (1-k, \chi \mid x) = -\frac{\tilde{B}_{k, q}^{(\alpha)} (x)}{k}.
$$

(2.22)

In the special case, $x = 0$, $\tilde{L}_{q}^{(\alpha)} (1-k, \chi \mid 0) = \tilde{L}_{q}^{(\alpha)} (1-k, \chi)$ are called the $q$-L-function with weight $\alpha$.

Let

$$
\tilde{F}_{q}^{(s)} (s, a \mid F) = \frac{\alpha}{[F]_{q}^{\alpha}} \left( \sum_{n=0}^{\infty} \frac{q^{am}}{[a]_{q}^{\alpha}} + \frac{(1-q^{a})^{s}}{F(1-s) \log q} \right)
$$

(2.23)

$$
\tilde{F}_{q}^{(s)} (s, a \mid F)
$$

where $a$ and $F$ are positive integers with $0 < a < F$.

Then, by (2.23), one gets

$$
\tilde{H}_{q}^{(a)} (1-n, a \mid F) = \frac{[F]_{q}^{n} \tilde{B}_{q, \chi, q}^{(a)} (a/F)}{[F]_{q}^{n}}, \quad n \geq 1,
$$

(2.24)

and $\tilde{H}_{q}^{(a)} (s, a \mid F)$ has as simple pole as $s = 1$ with residue $(a/[F]_{q})((q - 1)/\log q^{F})$.

Let $\chi$ be the Dirichlet character with conductor $F$, then one easily sees that

$$
\tilde{L}_{q}^{(\alpha)} (s, \chi) = \sum_{a=1}^{F} \chi(a) \tilde{H}_{q}^{(a)} (s, a \mid F).
$$

(2.25)
References


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