New Construction Weighted
\((h,q)\)-Genocchi Numbers and Polynomials
Related to Zeta Type Functions

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The fundamental aim of this paper is to construct \((h,q)\)-Genocchi numbers and polynomials with weight \(\alpha\). We shall obtain some interesting relations by using \(p\)-adic \(q\)-integral on \(\mathbb{Z}_p\) in the sense of fermionic. Also, we shall derive the \((h,q)\)-extensions of zeta type functions with weight \(\alpha\) from the Mellin transformation of this generating function which interpolates the \((h,q)\)-Genocchi numbers and polynomials with weight \(\alpha\) at negative integers.

1. Introduction, Definitions, and Notations

Let \(p\) be a fixed odd prime number. Throughout this paper we use the following notations. \(\mathbb{Z}_p\) denotes the ring of \(p\)-adic rational integers, \(\mathbb{Q}\) denotes the field of rational numbers, \(\mathbb{Q}_p\) denotes the field of \(p\)-adic rational numbers, and \(\mathbb{C}_p\) denotes the completion of algebraic closure of \(\mathbb{Q}_p\). Let \(\mathbb{N}\) be the set of natural numbers and \(\mathbb{N}^* = \mathbb{N} \cup \{0\}\). The \(p\)-adic absolute value is defined by \(|p|_p = 1/p\). In this paper, we assume \(|q-1|_p < 1\) as an indeterminate. In [1–3], Kim defined the fermionic \(p\)-adic \(q\)-integral on \(\mathbb{Z}_p\) as follows:

\[
I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) \, d\mu_{-q}(x) = \lim_{N \to \infty} \frac{1}{[pN]_{-q}} \sum_{x=0}^{pN-1} f(x) (-q)^x.
\] (1.1)
\([x]_q\) is a \(q\)-extension of \(x\) which is defined by

\[
[x]_q = \frac{1 - q^x}{1 - q},
\]

(1.2)

see [1–15].

Note that \(\lim_{q\to1}[x]_q = x\).

Let \(f_n(x) = f(x + n)\). By the definition (1.1) we easily get

\[
-qI_q(f_1) = \lim_{N \to \infty} \frac{1}{[pN]_{-q}} \sum_{x=0}^{pN-1} f(x + 1)(-q)^x
\]

\[
= \lim_{N \to \infty} \frac{1}{[pN]_{-q}} \sum_{x=0}^{pN-1} f(x)(-q)^x - (1 + q) \lim_{N \to \infty} \frac{f(pN)q^{pN} + f(0)}{1 + q^{pN}}
\]

\[
= I_q(f) - [2]_qf(0).
\]

Continuing this process, we obtain easily the relation

\[
q^nI_q(f_n) + (-1)^{n-1}I_q(f) = [2]_q\sum_{l=0}^{n-1}(-1)^{n-1}q^lf(l),
\]

(1.4)

\((h, q)\)-Genocchi numbers are defined as follows:

\[
G_{0,q}^{(h)} = 0, \quad q^{-2h}(qG_{q}^{(h)} + 1)^n + G_{n,q}^{(h)} = \begin{cases} 
[2]_{q^h} & \text{if } n = 1, \\
0 & \text{if } n > 1,
\end{cases}
\]

(1.5)

with the usual convention about replacing \((G_{q}^{(h)})^n\) by \(G_{n,q}^{(h)}\) (see [6]).

In this paper, we constructed \((h, q)\)-Genocchi numbers and polynomials with weight \(\alpha\). By using fermionic \(p\)-adic \(q\)-integral equations on \(\mathbb{Z}_p\), we investigated some interesting identities and relations on the \((h, q)\)-Genocchi numbers and polynomials with weight \(\alpha\). Furthermore, we derive the \(q\)-extensions of zeta type functions with weight \(\alpha\) from the Mellin transformation of this generating function which interpolates the \((h, q)\)-Genocchi polynomials with weight \(\alpha\).

2. On the Weighted \((h, q)\)-Genocchi Numbers and Polynomials

In this section, by using fermionic \(p\)-adic \(q\)-integral equations on \(\mathbb{Z}_p\), some interesting identities and relation on the \((h, q)\)-Genocchi numbers and polynomials with weight \(\alpha\) are shown.

Definition 2.1. Let \(\alpha, n \in \mathbb{N}^*\) and \(h \in \mathbb{N}\). Then the \((h, q)\)-Genocchi numbers with weight \(\alpha\) defined by as follows:

\[
\tilde{G}_{n+1,q}^{\alpha+1, h} = [2]_q\sum_{m=0}^{\infty}(-1)^m q^{mh} [m]_q^n.
\]

(2.1)
If we take \( h = 1 \) to (2.1), then we have, \( \widetilde{G}_{n+1,q}^{(\alpha,1)} = \widetilde{G}_{n+1,q}^{(\alpha)} \) (see [5]). From (2.1), we obtain

\[
\frac{\widetilde{G}_{n+1,q}^{(\alpha,h)}}{n+1} = \frac{[2]_q}{(1-q^n)^n} \sum_{m=0}^{\infty} (-1)^m q^{mh} (1 - q^{ma})^n
\]

\[
= \frac{[2]_q}{(1-q^n)^n} \left[ \sum_{m=0}^{\infty} (-1)^m q^{mh} \sum_{l=0}^{n} \binom{n}{l} (-1)^l (q^{ma})^l \right]
\]

(2.2)

\[
= \frac{[2]_q}{(1-q^n)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{1}{1 + q^{al+h}}
\]

Therefore, we obtain the following theorem.

**Theorem 2.2.** For \( \alpha, n \in \mathbb{N}^* \) and \( h \in \mathbb{N} \). Then

\[
\frac{\widetilde{G}_{n+1,q}^{(\alpha,h)}}{n+1} = \frac{[2]_q}{(1-q^n)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{1}{1 + q^{al+h}}. \tag{2.3}
\]

In (1.1), one takes \( f(x) = q^{(h-1)x}[x]_{q,p}^n \),

\[
\int_{\mathbb{Z}_p} q^{(h-1)x}[x]_{q,p}^n d\mu_q(x) = \frac{1}{(1-q^n)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \int_{\mathbb{Z}_p} q^{x(al+h-1)} d\mu_q(x)
\]

\[
= \frac{1}{(1-q^n)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \lim_{N \to \infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} (-q^{al+h})^x
\]

\[
= \frac{1}{(1-q^n)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{(1+q)}{1+q^{al+h}} \lim_{N \to \infty} \frac{1 + (q^{al+h})^{p^N}}{1 + q^{p^N}} \tag{2.4}
\]

\[
= \frac{[2]_q}{(1-q^n)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{1}{1 + q^{al+h}}
\]

\[
= \frac{\widetilde{G}_{n+1,q}^{(\alpha,h)}}{n+1}
\]

From [12], we obtain \((h,q)\)-Genocchi numbers with weight \( \alpha \) witt’s type formula as follows.
Theorem 2.3. For \( \alpha, n \in \mathbb{N}^* \) and \( h \in \mathbb{N} \). Then

\[
\frac{\tilde{G}_{n+1,q}^{(\alpha,h)}}{n+1} = \int_{\mathbb{Z}_p} q^{(h-1)x}[x]^n_{q^\alpha} d\mu_{-q}(x). \tag{2.5}
\]

From (2.1), one easily gets

\[
\int_{\mathbb{Z}_p} q^{(h-1)x}e^{[x]_q}_{\alpha,h} d\mu_{-q}(x) = t[2]_q \sum_{m=0}^{\infty} (-1)^m q^{mh}[m]_{q^\alpha}. \tag{2.6}
\]

By (2.6), one has

\[
\sum_{n=0}^{\infty} \tilde{G}_{n,q}^{(\alpha,h)} \frac{t^n}{n!} = t[2]_q \sum_{m=0}^{\infty} (-1)^m q^{mh}[m]_{q^\alpha}. \tag{2.7}
\]

Therefore, we obtain the following corollary.

Corollary 2.4. If \( \tilde{G}_{0,q}^{(\alpha,h)} = 0 \). Let \( D_q^{(\alpha,h)}(t) = \sum_{n=0}^{\infty} \tilde{G}_{n,q}^{(\alpha,h)} (t^n/n!) \). Then

\[
D_q^{(\alpha,h)}(t) = t[2]_q \sum_{m=0}^{\infty} (-1)^m q^{mh}[m]_{q^\alpha}. \tag{2.8}
\]

Now, one considers the \((h,q)\)-Genocchi polynomials with weight \( \alpha \) as follows:

\[
\frac{\tilde{G}_{n+1,q}^{(\alpha,h)}(x)}{n+1} = \int_{\mathbb{Z}_p} q^{(h-1)y}[x+y]^n_{q^\alpha} d\mu_{-q}(y), \quad n \in \mathbb{N}, \ \alpha \in \mathbb{N}^*. \tag{2.9}
\]

From (2.9), one sees that

\[
\frac{\tilde{G}_{n+1,q}^{(\alpha,h)}(x)}{n+1} = [2]_q \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{alx} \frac{1}{1+q^{al+h}} = [2]_q \sum_{m=0}^{\infty} (-1)^m q^{mh}[m+x]_{q^\alpha}. \tag{2.10}
\]

Let \( D_q^{(\alpha,h)}(t,x) = \sum_{n=0}^{\infty} \tilde{G}_{n,q}^{(\alpha,h)}(x) (t^n/n!) \). Then, one has

\[
D_q^{(\alpha,h)}(t,x) = t[2]_q \sum_{m=0}^{\infty} (-1)^m q^{mh}[m+x]_{q^\alpha} = \sum_{n=0}^{\infty} \tilde{G}_{n,q}^{(\alpha,h)}(x) \frac{t^n}{n!}. \tag{2.11}
\]

By (1.4), one sees that

\[
q^{hm} \frac{\tilde{G}_{m+1,q}^{(\alpha,h)}(n)}{m+1} + (-1)^{n-1} \frac{\tilde{G}_{m+1,q}^{(\alpha,h)}}{m+1} = [2]_q \sum_{l=0}^{n-1} (-1)^{n-l-1} q^{hl}[l]_{q^\alpha}. \tag{2.12}
\]
Therefore, we obtain the following theorem.

**Theorem 2.5.** For $m, h \in \mathbb{N}$, and $\alpha, n \in \mathbb{N}^*$, one has

$$q^{hm}\sum_{m=1}^{q(m^h)}(n)\frac{n}{m+1} + (-1)^{n-m}q^{m+1}\sum_{m=0}^{n-1}(-1)^{n-m}q^{m+1}\left[\mathcal{L}_q^m\right]_{\alpha,h}^{\mathbb{N}}.$$  

(2.13)

In (1.3), it is known that

$$q\mathcal{L}_q(f_1) + I_{-q}(f) = [2]_qf(0).$$

(2.14)

If we take $f(x) = q^{(h-1)x}e^{[x]_q}a$, then one has

$$[2]_q = q\int_{\mathbb{Z}_p}q^{(h-1)(x+1)}e^{[x]_q}d\mu_{-q}(x) + \int_{\mathbb{Z}_p}q^{(h-1)x}e^{[x]_q}d\mu_{-q}(x)$$

$$= \sum_{m=0}^{\infty}q\frac{\tilde{G}^{(a,h)}_{m+1,q}(1)}{m+1} + \sum_{m=0}^{\infty}q\frac{\tilde{G}^{(a,h)}_{m+1,q}(1)}{m+1} = \frac{[2]_q}{m}.$$  

(2.15)

Therefore, by (2.15), we obtain the following theorem.

**Theorem 2.6.** For $\alpha \in \mathbb{N}^*$ and $m, h \in \mathbb{N}$, one has

$$\mathcal{G}^{(a,h)}_0 = 0, \quad q^h\mathcal{G}^{(a,h)}_{m+1,q}(1) = \left\{ \begin{array}{ll} [2]_q, & \text{if } m = 0, \\ 0, & \text{if } m \neq 0. \end{array} \right.$$  

(2.16)

From (2.9), one can easily derive the following:

$$\int_{\mathbb{Z}_p}q^{(h-1)y}[x+y]_q^n d\mu_{-q}(y) = \left[\frac{d^n}{[d]_{-q}a^0}\right] \sum_{a=0}^{d-1}(-1)^a q^{ha} \int_{\mathbb{Z}_p}q^{d[y - 1]_q}a \left[\frac{x+a}{d} + y\right]^n_\mathcal{L}_q \mu_{-q}^d(y)$$

$$= \left[\frac{d^n}{[d]_{-q}a^0}\right] \sum_{a=0}^{d-1}(-1)^a q^{ha} \mathcal{G}^{(a,h)}_{n+1,q}((x+a)/d, n+1).$$

(2.17)

Therefore, by (2.17), we obtain the following theorem.

**Theorem 2.7.** For $d \equiv 1 \pmod{2}$, $n \in \mathbb{N}^*$ and $\alpha, h \in \mathbb{N}$

$$\mathcal{G}^{(a,h)}_{n+1,q}(x) = \left[\frac{d^n}{[d]_{-q}a^0}\right] \sum_{a=0}^{d-1}(-1)^a q^{ha} \mathcal{G}^{(a,h)}_{n+1,q}((x+a)/d).$$

(2.18)
3. Interpolation Function of the Polynomials $\tilde{G}_{n,q}(x)$

In this section, we give interpolation function of the generating functions of $(h,q)$-Genocchi polynomials with weight $\alpha$. For $s \in \mathbb{C}$ and $h \in \mathbb{N}$, by applying the Mellin transformation to (2.11), we obtain

$$I_q^{(\alpha,h)}(s,x) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-2} \left(-D_q^{(\alpha,h)}(-t,x)\right) dt = [2q] \sum_{m=0}^\infty (-1)^m q^{mh} \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-t[m+x]} dt, \quad (3.1)$$

so we have

$$I_q^{(\alpha,h)}(s,x) = [2q] \sum_{m=0}^\infty (-1)^m q^{mh} \frac{1}{[m+x]_q^s}. \quad (3.2)$$

We define $q$-extension zeta type function as follows.

**Theorem 3.1.** For $s \in \mathbb{C}$, $h \in \mathbb{N}$, and $\alpha \in \mathbb{N}^*$. One has

$$I_q^{(\alpha,h)}(s,x) = [2q] \sum_{m=0}^\infty (-1)^m q^{mh} \frac{1}{[m+x]_q^s}. \quad (3.3)$$

$I_q^{(\alpha,h)}(s,x)$ can be continued analytically to an entire function.

By substituting $s = -n$ into (3.3) one easily gets

$$I_q^{(\alpha,h)}(-n,x) = [2q] \sum_{m=0}^\infty (-1)^m q^{mh} \frac{1}{[m+x]_q^s} \quad (3.4)$$

We obtain the following theorem.

**Theorem 3.2.** For $h \in \mathbb{N}$ and $q, s \in \mathbb{C}$, $|q| < 1$. Then one defines

$$I_q^{(\alpha,h)}(-n,x) = \frac{\tilde{G}_{n+1,q}^{(\alpha,h)}(x)}{n+1}. \quad (3.5)$$

**References**


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