Research Article

Mean Convergence Rate of Derivatives by Lagrange Interpolation on Chebyshev Grids

Wang Xiulian and Ning Jingrui

Department of Mathematics, Tianjin Normal University, Tianjin 300387, China

Correspondence should be addressed to Wang Xiulian, wangxiulian0205@gmail.com

Received 23 May 2011; Revised 30 August 2011; Accepted 19 September 2011

Academic Editor: Carlo Piccardi

Copyright © 2011 W. Xiulian and N. Jingrui. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We consider the rate of mean convergence of derivatives by Lagrange interpolation operators based on the Chebyshev nodes. Some estimates of error of the derivatives approximation in terms of the error of best approximation by polynomials are derived. Our results are sharp.

1. Introduction and Main Results

Mean convergence of Lagrange interpolation based on the zeros of orthogonal polynomials (and possibly some additional points) has been studied for at least 70 years. There is a vast literature on this topic. The authors of [1–3] considered the simultaneous approximation by the Hermite interpolation operators, and we will consider the simultaneous approximation by Lagrange interpolation operators based on the zeros of Chebyshev polynomials. The relevant results can be found in [4–6]. We introduce these results below.

Let

\[ \omega(x) = \prod_{k=1}^{N} |x - y_k|^{\Gamma_k} \quad (|x| \leq 1; -1 = y_1 < y_2 < \cdots < y_N = 1; \Gamma_k > -1; k = 1, \ldots, N) \] (1.1)

be a so-called generalized Jacobi weight \((\omega \in GJ)\), and let

\[ -1 \leq x_1 < x_2 < \cdots < x_n \leq 1 \] (1.2)

be the zeros of the \(n\)th orthogonal polynomial \(p_n(\omega)\) associated with the weight-function \(\omega \in GJ\). Let \(L_n(\omega, f)\) denote the Lagrange interpolating polynomial which interpolates \(f\) at...
the zeros of \( p_n(\omega) \). By using Markov-Bernstein type inequalities in \( L_p \) metric, J. Szabados and A. K. Varma [5] reduced the weighted mean convergence of derivatives \( L^{(r)}_n(\omega, f, x) \) to the weighted mean convergence of \( L_n(\omega, f, x) \) and obtained the following. If \( L^p \) means functional space equipped with \( L_p \) norm and

\[
\omega(x) \in G_j, \quad \frac{\omega(x)^{1/p-1/2}}{(1-x^2)^{1/4}} \in L^p, \tag{*}
\]

then, for \( f^{(r)}(x) \in C[-1,1] \), we have

\[
\int_{-1}^{1} \left| f^{(r)}(x) - L^{(r)}_n(f, x) \right|^p \left(1-x^2\right)^{rp/2} \omega(x) dx \leq C_r E^n_{n-r-1}(f^{(r)}) \quad (n \geq r + 1). \tag{1.3}
\]

Here and in the following, the constant \( C_r \) (may be different in the same expression) is independent of \( n \) and \( f \) but depends on \( r \), and \( E_n(\cdot) \) denotes the error of the best polynomial approximation of degree \( n \) of the corresponding function in the \( L_\infty \) metric.

Mastroianni and Nevai [4] get sharper estimates in terms of modulus of continuity instead of the best approximation. It improves some old results. But its proof also needs weighted Markov-Bernstein type inequality in \( L^p \) metric and the idea of additional points. For the weight functions not satisfying (\(*\)), it is not possible to discuss by their method. To deal with these case, Du and Xu [7] consider the most important special case \( \omega(x) = 1/\sqrt{1-x^2} \).

Let

\[
t_k = t_{kn} = \cos \frac{2k-1}{2n} \pi, \quad k = 1, \ldots, n, \tag{1.4}
\]

be the zeros of \( T_n(x) = \cos n\theta, \ x = \cos \theta \), the \( n \)th degree Chebyshev polynomial of the first kind. If \( f \in C[-1,1] \), then the well-known Lagrange interpolation polynomial of \( f \) based on \( \{t_k\}_{k=1}^n \) is given by (see [8])

\[
L_n(f, x) = \sum_{k=1}^{n} f(t_k) \ell_k(x), \tag{1.5}
\]

where

\[
\ell_k(x) = \frac{(-1)^{k+1} \sqrt{1-t_k^2} T_n(x)}{n(x-t_k)}, \quad k = 1, \ldots, n. \tag{1.6}
\]
Du and Xu [7] obtained the following.

**Theorem A.** Let $L_n(f, x)$ be as defined as above. Then, for $f \in C_{[-1,1]}^1$, we have

$$
\left( \int_{-1}^{1} |f'(x) - L_n'(f, x)|^p \left(1 - x^2 \right)^{\alpha} \, dx \right)^{1/p} \leq \begin{cases} 
CE_{n-2}(f'), & \alpha > \frac{p}{2} - 1, \\
\mathcal{C}(\ln n)^{1/p} E_{n-2}(f'), & \alpha = \frac{p}{2} - 1, \\
C n^{1-(2\alpha+2)/p} E_{n-2}(f'), & -1 < \alpha < \frac{p}{2} - 1,
\end{cases}
$$

and the estimation for $-1 < \alpha \leq (p/2) - 1$ is sharp.

We notice that although the sharp estimate is obtained, the upper bound is not $E_{n-2}(f')$ for $-1 \leq \alpha \leq (p/2) - 1$. Now we will give a Lagrange interpolation to improve their results. Let

$$x_k = x_{kn} = \cos \frac{k\pi}{n+1}, \quad k = 1, \ldots, n,$$

be the zeros of $U_n(x) = \sin(n+1)\theta / \sin \theta$, $x = \cos \theta$, the $n$th degree Chebyshev polynomial of the second kind. If $f \in C[-1,1]$, then the well-known Lagrange interpolation polynomial of $f$ based on $\{x_k\}_{k=1}^n \cup \{x_0 = 1, x_{n+1} = -1\}$ is given by (see [9])

$$Q_{n+2}(f, x) = \sum_{k=0}^{n+1} f(x_k) \varphi_k(x),$$

where

$$
\varphi_0(x) = \frac{(1 + x)U_n(x)}{2(n+1)}, \quad \varphi_{n+1}(x) = \frac{(-1)^n(x-1)U_n(x)}{2(n+1)}, \\
\varphi_k(x) = \frac{(-1)^{k+1}(1-x^2)U_n(x)}{(n+1)(x-x_k)}, \quad k = 1, \ldots, n.
$$

Firstly, we obtain the following.

**Theorem 1.1.** Let $Q_n(f, x)$ be as defined as above, $0 < p < +\infty$, $\alpha > -1$. Then, for $f \in C_{[-1,1]}^1$, we have

$$
\left( \int_{-1}^{1} |f'(x) - Q_n'(f, x)|^p \left(1 - x^2 \right)^{\alpha} \, dx \right)^{1/p} \leq CE_{n-2}(f').
$$

By Theorem A and Theorem 1.1, we know that $Q_{n+2}(f, x)$ have better convergence rate than $L_n(f, x)$ in the case $-1 \leq \alpha \leq (p/2) - 1$. But for continuous function approximation, we
noticed that \( Q_n \) have the same approximation order with \( L_n \), that is, if \( 0 < p < +\infty, \alpha > -1 \), then, for \( f \in C_{[-1,1]} \), from Hölder inequality [8, 9], it follows that

\[
\left( \int_{-1}^{1} |f(x) - L_n(f,x)|^p (1-x^2)^\alpha \, dx \right)^{1/p} \leq CE_{n-1}(f),
\]

\[
\left( \int_{-1}^{1} |f(x) - Q_n(f,x)|^p (1-x^2)^\alpha \, dx \right)^{1/p} \leq CE_{n-1}(f).
\]  

(1.12)

For high derivatives approximation, how the cases are? Secondly, we will consider second derivative approximation by \( L_n \) and \( Q_n \) and obtain the following.

**Theorem 1.2.** Let \( Q_n(f,x) \) and \( L_n(f,x) \) be as defined as above. Then, for \( f \in C^2_{[-1,1]} \), we have

\[
\left( \int_{-1}^{1} |f''(x) - Q_n''(f,x)|^p (1-x^2)^\alpha \, dx \right)^{1/p} \leq \begin{cases} 
CE_{n-3}(f''), & \alpha > \frac{p}{2} - 1, \\
C(\ln n)^{1/p}E_{n-3}(f''), & \alpha = \frac{p}{2} - 1, \\
Cn^{-(2\alpha + 2)/p}E_{n-3}(f''), & -1 < \alpha < \frac{p}{2} - 1,
\end{cases}
\]

\[
\left( \int_{-1}^{1} |f''(x) - L_n''(f,x)|^p (1-x^2)^\alpha \, dx \right)^{1/p} \leq \begin{cases} 
CE_{n-3}(f''), & \alpha > p - 1, \\
C(\ln n)^{1/p}E_{n-3}(f''), & \alpha = p - 1, \\
Cn^{-(2\alpha + 2)/p}E_{n-3}(f''), & -1 < \alpha < p - 1,
\end{cases}
\]  

(1.13)

and the estimation for \(-1 < \alpha \leq (p/2) - 1\) or \((-1 < \alpha \leq p - 1)\) is sharp.

From Theorem 1.2, we know that for the second derivative approximation, \( Q_n \) have better approximation orders than \( L_n \) in the case \(-1 < \alpha \leq p - 1\).

Using the same way as in the proof of Theorem 1.2, we can consider the \( r \) order derivatives approximation for \( r \geq 3 \), but the computation is more complicated, and we omit the detail.

2. Some Lemmas

We introduce some lemmas which are the main tools in our proof.

**Lemma 2.1** (see [10, p. 519]). If \( f \in C^r_{[-1,1]} \), then there exists an algebraic polynomial \( p_n(x) \) of degree at most \( n \) such that

\[
|f^{(j)}(x) - p_n^{(j)}(x)| \leq C \left[ \frac{\sqrt{1 - x^2}}{n} \right]^{r-j} E_{n-r}(f^{(j)}), \quad j = 0, 1, \ldots, r.
\]  

(2.1)
In the past, the error estimate depended on the Markov-Bernstein type inequalities in $L_p$ metric. In this paper, we will use the inequality in $L_\infty$ metric.

**Lemma 2.2** (see [7, p. 50]). Let $\varphi_k(x)$ be as defined by (1.10), $\alpha > -1$. Then, for any fixed $p > 0$,

$$
\left( \int_{-1}^{1} \left| \sum_{k=1}^{n} A_k \varphi_k(x) \right|^p \left( 1 - x^2 \right)^{\alpha} \, dx \right)^{1/p} \leq C \max_{1 \leq k \leq n} |A_k|.
$$

(2.2)

To prove our results, we need to build another polynomial integral inequality in $L_\infty$ metric. For its proof, we introduce two lemmas.

**Lemma 2.3** (see [8, p. 914]). Let $v_1, v_2, \ldots, v_{2N}$ be distinct integers between 1 and $n$. Then, we have

$$
\int_{-1}^{1} \ell_{v_1}(x) \ell_{v_2}(x) \cdots \ell_{v_{2N}}(x) \, dx \sqrt{1-x^2} = 0,
$$

(2.3)

and it is well known that

$$
\sum_{k=1}^{n} \ell_k^2(x) \leq 2.
$$

(2.4)

Let $x_1, \ldots, x_n$ be independent variables, $s$ are positive integers, and

$$
V_s = \left( \sum_{k=1}^{n} x_k^s \right)^{1/s}.
$$

(2.5)

By the mathematical induction we can obtain the following.

**Lemma 2.4.** If $N$ is a positive integer, $n > 2N$, then, the homogeneous symmetrical polynomial of degree $2N$:

$$
B_{2N} = \left( \sum_{i=1}^{n} x_i \right)^{2N} - (2N)! \sum_{k_1 < k_2 < \cdots < k_{2N}} x_{k_1} \cdots x_{k_{2N}},
$$

(2.6)

can be represented as a homogeneous polynomial of degree $2N$ about $V_1, \ldots, V_{2N}$:

$$
B_{2N} = \sum_{t_1 \leq 2N - 2t_2 \geq 0} B_{t_1 - t_2} V_1^{t_1} \cdots V_{2N}^{t_{2N}}.
$$

(2.7)

Now we give the inequality in $L_\infty$ metric which plays a key role in our paper.

**Lemma 2.5.** Let $\ell_k(x)$ be as defined by (2.1), $\alpha > -1$. Then, for any fixed $p > 0$,

$$
\left( \int_{-1}^{1} \left| \sum_{k=1}^{n} A_k \ell_k(x) \right|^p \left( 1 - x^2 \right)^{\alpha} \, dx \right)^{1/p} \leq C \max_{1 \leq k \leq n+1} |A_k|.
$$

(2.8)
Proof. Firstly, we will consider the special case $p = 2N, a = -1/2$ by induction on $N$. For $N = 1$, by (2.3) and (2.4), we obtain

$$\int_{-1}^{1} \left| \sum_{k=1}^{n} A_k \ell_k(x) \right|^2 \frac{dx}{\sqrt{1-x^2}} = \sum_{k=1}^{n} A_k^2 \int_{-1}^{1} \ell_k^2(x) \frac{dx}{\sqrt{1-x^2}} + 2 \sum_{k=1}^{n-1} \sum_{j=1}^{n} A_k A_j \int_{-1}^{1} \ell_k(x) \ell_j(x) \frac{dx}{\sqrt{1-x^2}}$$

$$\leq \max_{1 \leq k \leq n} |A_k|^2 \int_{-1}^{1} \sum_{k=1}^{n} \ell_k^2(x) \frac{dx}{\sqrt{1-x^2}} \leq 2 \pi \max_{1 \leq k \leq n} |A_k|^2.$$  

(2.9)

Suppose that for $0 < p \leq 2(N-1)$, we have

$$\int_{-1}^{1} \left| \sum_{k=1}^{n} A_k \ell_k(x) \right|^p \frac{dx}{\sqrt{1-x^2}} \leq C_p \max_{1 \leq k \leq n} |A_k|^p.$$  

(2.10)

For $p = 2N$, if $n \leq 2N$, then (2.4) gives

$$\int_{-1}^{1} \left| \sum_{k=1}^{n} A_k \ell_k(x) \right|^{2N} \frac{dx}{\sqrt{1-x^2}} \leq \pi (4N)^{2N} \max_{1 \leq k \leq n} |A_k|^{2N}.$$  

(2.11)

If $n > 2N$, then by Lemma 2.4, we know

$$\left| \sum_{k=1}^{n} A_k \ell_k(x) \right|^{2N} = (2N)! \sum_{k_1 < k_2 < \cdots < k_{2N}} A_{k_1} \cdots A_{k_{2N}} \ell_{k_1}(x) \cdots \ell_{k_{2N}}(x)$$

$$+ \sum_{t_1 \leq 2N-2, t_N \geq 0} B_{t_1 \cdots t_N} V^{t_1 \cdots t_N} (x)$$

$$= I_1(x) + I_2(x),$$  

(2.12)

where

$$V_s(x) = \left( \sum_{k=1}^{n} A_k^s \ell_k^s(x) \right)^{1/s}.$$  

(2.13)

From (2.3), it follows that

$$\int_{-1}^{1} I_1(x) \frac{dx}{\sqrt{1-x^2}} = 0.$$  

(2.14)

From (2.4), we know that, for $s \geq 2$,

$$|V_s(x)| \leq \max_{1 \leq k \leq n} |A_k| \left( \sum_{k=1}^{n} |\ell_k(x)|^p \right)^{1/s} \leq \sqrt{2} \max_{1 \leq k \leq n} |A_k|.$$  

(2.15)
By virtue of (2.12) and (2.15), we have

\[
\left| \int_{-1}^{1} I_2(x) \frac{dx}{\sqrt{1-x^2}} \right| \leq \sum_{t_i \leq N-2t_i \geq 0} |B_{t_i-t_iN}| \int_{-1}^{1} \left| V_1^{t_i}(x) \cdots V_{2N}^{t_i}(x) \right| \frac{dx}{\sqrt{1-x^2}} \\
\leq \sum_{t_i \leq N, t_i \geq 0} 2^N |B_{t_i-t_iN}| \max_{1 \leq k \leq N} |A_k|^{2N-t_i} \int_{-1}^{1} \left| V_1^{t_i}(x) \right| \frac{dx}{\sqrt{1-x^2}} \\
\leq \sum_{t_i \leq N-2t_i \leq 0} 2^N |B_{t_i-t_iN}| \left( \pi + \sum_{i \neq 1} C_i \right) \max_{1 \leq k \leq N} |A_k|^{2N}. \tag{2.16}
\]

From (2.11), (2.12), (2.14), and (2.16), it follows that

\[
\int_{-1}^{1} \left| \sum_{k=1}^{n} A_k e_k(x) \right|^{2N} \frac{dx}{\sqrt{1-x^2}} \leq C_{2N} \max_{1 \leq k \leq N} |A_k|^{2N}. \tag{2.17}
\]

Now we consider the general case. For arbitrary \( p > 0 \) and \( \alpha > -1 \), it is easy to see that we can choose a positive integer \( N \) satisfying \( p/4N < 1 \) and \( (\alpha + (p/4N))/(1 - (p/2N)) > -1 \). By Hlder inequality and (2.17), we can obtain

\[
\int_{-1}^{1} \left| \sum_{k=1}^{n} A_k e_k(x) \right|^p \left( 1 - x^2 \right)^{\alpha} dx \\
\leq \left( \int_{-1}^{1} \left| \sum_{k=1}^{n} A_k e_k(x) \right|^{2N} \frac{dx}{\sqrt{1-x^2}} \right)^{p/2N} \left( \int_{-1}^{1} \left( 1 - x^2 \right)^{(\alpha + (p/4N))/(1 - (p/2N))} dx \right)^{(1 - (p/2N))} \\
\leq C_p \max_{1 \leq k \leq N} |A_k|^p. \tag{2.18}
\]

Remark 2.6. P. Erdős and E. Feldheim [8] give a proof for \( p = 2, 4 \) and \( \alpha = -1/2 \). We give a mathematical induction proof for completion.

### 3. Proof of Theorem 1.1

We will consider \( Q_{n+2}(f, x) \) instead of \( Q_n(f, x) \) for simplicity. For \( f \in C_{[-1,1]} \), let \( p_{n+1}(x) \) be the polynomial of degree at most \( n + 1 \) satisfying (2.1). It is easily checked that for \(-1 \leq x \leq 1\),

\[
f(x) - Q_{n+2}(f, x) = f(x) - p_{n+1}(x) + Q_{n+2}(p_{n+1} - f, x). \tag{3.1}
\]

From (3.1), we can conclude that

\[
f'(x) - Q'_{n+2}(f, x) = f'(x) - p'_{n+1}(x) + Q'_{n+2}(p_{n+1} - f, x) = I_1(x) + I_2(x). \tag{3.2}
\]
From (2.1), we can derive

\[ \int_{-1}^{1} |I_{1}(x)|^p \left(1 - x^2 \right)^a \, dx \leq CE_n^p \int_{-1}^{1} \left(1 - x^2 \right)^a \, dx \leq CE_n^p(f'). \tag{3.3} \]

It is easy to see that \( I_2(x) \) is a polynomial of degree at most \( n \). Hence,

\[ I_2(x) = \sum_{k=1}^{n} \left( p_{n+1}(x_k) - f(x_k) \right) \varphi'_k(x) = L_{n+1}(I_2, x) \]
\[ = \sum_{s=1}^{n+1} \left[ \sum_{k=1}^{n} (p_{n+1}(x_k) - f(x_k)) \varphi'_k(t_s) \right] \ell_s(x). \tag{3.4} \]

By a direct computation, we know

\[ \varphi'_k(t_s) = \frac{(-1)^{k+s+1} \sqrt{1 - t_s^2}}{(n+1)(t_s - x_k)^2} + \frac{(-1)^{k+s+1} t_s}{(n+1) \sqrt{1 - t_s^2(t_s - x_k)}}. \tag{3.5} \]

Combining (3.4) and (3.5), we derive

\[ I_2(x) = \sum_{s=1}^{n+1} \left[ \sum_{k=1}^{n} (p_{n+1}(x_k) - f(x_k)) \frac{(-1)^{k+s+1} \sqrt{1 - t_s^2}}{(n+1)(t_s - x_k)^2} \right] \ell_s(x) \]
\[ + \sum_{s=1}^{n+1} \left[ \sum_{k=1}^{n} (p_{n+1}(x_k) - f(x_k)) \frac{(-1)^{k+s+1} t_s}{(n+1) \sqrt{1 - t_s^2(t_s - x_k)}} \right] \ell_s(x) \]
\[ = J_1(x) + J_2(x). \tag{3.6} \]

We consider \( J_1(x) \) first. For an arbitrary \( 1 \leq s \leq n+1 \),

\[ \left| \sum_{k=1}^{n} (p_{n+1}(x_k) - f(x_k)) \frac{(-1)^{k+s+1} \sqrt{1 - t_s^2}}{(n+1)(t_s - x_k)^2} \right| \leq CE_n(f') \left( \sum_{k=1}^{n} \frac{1 - t_s^2}{(t_s - x_k)^2} \right) \left( \sum_{k=1}^{n} \frac{1 - x_k^2}{(t_s - x_k)^2} \right) \]
\[ \leq CE_n(f') \frac{n}{n^2} \left( \sum_{k=1}^{n} \frac{1 - t_s^2}{(t_s - x_k)^2} + \sum_{k=1}^{n} \frac{1 - x_k^2}{(t_s - x_k)^2} \right). \tag{3.7} \]
Similar to [9, p. 71], we have

\[
\sum_{k=1}^{n} \frac{(1 - x^2)U_n^2(x)}{(x - x_k)^2} = \left(1 - x^2\right) \left[\left(U_n'(x)\right)^2 - U_n(x)U_n''(x)\right],
\]

\[
\sum_{k=1}^{n} \frac{(1 - x^2)U_n^2(x)}{(x - x_k)^2} = \sum_{k=1}^{n} \frac{(1 - x^2)U_n^2(x)}{(x - x_k)^2} + 2xU_n(x)U_n'(x) - nU_n^2(x).
\] (3.8)

By [9, p. 71], we know

\[
\left(1 - x^2\right)U_n'(x) = xU_n(x) - (n + 1)T_{n+1}(x),
\] (3.9)

\[
\left(1 - x^2\right)U_n''(x) = 3xU_n'(x) - n(n + 2)U_n(x).
\] (3.10)

Let \(x = t_s\), then by (3.8), (3.9), and (3.10), we obtain

\[
\sum_{k=1}^{n} \frac{1 - t_s^2}{(t_s - x_k)^2} = \frac{n(n + 2)}{1 - t_s^2} \leq n(n + 2),
\] (3.11)

\[
\sum_{k=1}^{n} \frac{1 - x_k^2}{(t_s - x_k)^2} = n^2 + n.
\] (3.12)

From (3.7), (3.11), and (3.12), we obtain that for an arbitrary \(1 \leq s \leq n + 1\),

\[
\left|\sum_{k=1}^{n} \left(p_{n+1}(x_k) - f(x_k)\right) \frac{1 - t_s^2}{(n + 1)(t_s - x_k)^2}\right| \leq CE_n(f').
\] (3.13)

From (2.8) and (3.13), we can obtain

\[
\left(\int_{-1}^{1} |f_1(x)|^p \left(1 - x^2\right)^a \right)^{1/p} \leq CE_n(f').
\] (3.14)

Now we consider \(J_2(x)\). Exchanging the summation order, we have

\[
J_2(x) = \sum_{k=1}^{n} \frac{(-1)^k \left(p_{n+1}(x_k) - f(x_k)\right)}{n + 1} \left[\sum_{s=1}^{n+1} \frac{(-1)^{s+1}t_s}{1 - t_s^2(t_s - x_k)} \ell_s(x)\right]
\]

\[
= \sum_{k=1}^{n} \frac{(-1)^k \left(p_{n+1}(x_k) - f(x_k)\right)}{(n + 1)^2} \left[\sum_{s=1}^{n+1} \frac{t_sT_{n+1}(x)}{(t_s - x_k)(x - t_s)}\right].
\] (3.15)
It is easy to know
\[
\sum_{s=1}^{n+1} T_{n+1}'(x) = T_{n+1}'(x) = (n + 1) U_n(x).
\]  \tag{3.16}

Let \( x = x_k \), then, we have
\[
\sum_{s=1}^{n+1} \frac{1}{x_k - t_s} = 0. \tag{3.17}
\]

By (3.16), (3.17), and the identity
\[
\frac{t_s T_{n+1}(x)}{(t_s - x_k)(x - t_s)} = \frac{x T_{n+1}(x)}{(x - x_k)(x - t_s)} - \frac{x_k T_{n+1}(x)}{(x - x_k)(x_k - t_s)}, \tag{3.18}
\]
we conclude that
\[
\sum_{s=1}^{n+1} \frac{t_s T_{n+1}(x)}{(t_s - x_k)(x - t_s)} = \frac{(n + 1) x U_n(x)}{x - x_k}. \tag{3.19}
\]

From (3.15) and (3.19), it follows that
\[
J_2(x) = \sum_{k=1}^{n} \frac{(-1)^k (p_{n+1}(x_k) - f(x_k))}{n + 1} \cdot \frac{x_k U'_n(x_k)}{x - x_k}. \tag{3.20}
\]

For an arbitrary \( 1 \leq k \leq n \), by (3.20), (2.1), \( |U'_n(x_k)| = (n + 1)/(1 - x_k^2) \), \( k = 1, 2, \ldots, n \), and a simple computation, we can obtain
\[
|J_2(x_k)| = \left| \frac{(-1)^k (p_{n+1}(x_k) - f(x_k))}{n + 1} \cdot x_k U'_n(x_k) \right| \leq CE_n(f'). \tag{3.21}
\]

For \( k = 0 \), by (2.1), \( U_n(1) = n + 1 \) and a simple computation we obtain
\[
|J_2(1)| = \left| \frac{\sum_{k=1}^{n} (-1)^k (p_{n+1}(x_k) - f(x_k))}{1 - x_k} \right| \leq CE_n(f') \frac{n}{n} \sum_{k=1}^{n} \frac{1}{\sqrt{1 - x_k}}. \tag{3.22}
\]

From \( 2x/\pi \leq \sin x \leq x \), for all \( x \in [0, \pi/2] \), we derive
\[
\sum_{k=1}^{n} \frac{1}{\sqrt{1 - x_k}} = \sum_{k=1}^{n} \frac{1}{\sqrt{2} \sin k \pi / (n + 1)} \leq \sum_{k=1}^{n} \frac{n + 1}{k} \leq Cn \ln n. \tag{3.23}
\]
Hence,

$$|J_2(1)| \leq C \ln nE_n(f').$$

(3.24)

Similarly,

$$|J_2(-1)| \leq C \ln nE_n(f').$$

(3.25)

The fact that $J_2(x)$ is an algebraic polynomial of degree at most $n$ implies

$$J_2(x) = Q_{n+2}(J_2, x) = J_2(1)\varphi_0(x) + J_2(-1)\varphi_{n+1}(x) + \sum_{k=1}^{n} J_2(x_k)\varphi_k(x).$$

(3.26)

Let $x = \cos \theta$. By (3.24) and a simple computation similar to [11, p. 204], we obtain that, for $p > 0$ and $\alpha > -1$,

$$\int_{-1}^{1} |J_2(1)\varphi_0(x)|^p \left(1 - x^2\right)^{\alpha} dx \leq \frac{\text{Ch}^n nE_n^p(f')}{(n + 1)^p} \int_{0}^{\pi} \frac{\sin n\theta |p|}{\sin^{p-2\alpha-1}\theta} d\theta \leq CE_n^p(f').$$

(3.27)

Similarly,

$$\int_{-1}^{1} |J_2(-1)\varphi_{n+1}(x)|^p \left(1 - x^2\right)^{\alpha} dx \leq CE_n^p(f').$$

(3.28)

By virtue of (2.2) and (3.21), we have

$$\int_{-1}^{1} \left| \sum_{k=1}^{n} J_2(x_k)\varphi_k(x) \right|^p \left(1 - x^2\right)^{\alpha} dx \leq CE_n^p(f').$$

(3.29)

From (3.26), (3.27), (3.28), and (3.29), it follows that

$$\left( \int_{-1}^{1} |J_2(x)|^p \left(1 - x^2\right)^{\alpha} dx \right)^{1/p} \leq CE_n(f').$$

(3.30)

By (3.2), (3.3), (3.6), (3.14), and (3.30), we obtain the upper estimate.

### 4. Proof of Theorem 1.2

We consider $Q_n(f, x)$ first. We will consider $Q_{n+2}(f, x)$ instead of $Q_n(f, x)$ for simplicity. For $f \in C_{[-1,1]}^1$ let $p_{n+1}(x)$ be the polynomial of degree at most $n + 1$ satisfying (2.1). From (3.1), it follows that

$$f''(x) - Q''_{n+2}(f, x) = f''(x) - p''_{n+1}(x) + Q''_{n+2}(p_{n+1} - f, x) = M_1(x) + M_2(x).$$

(4.1)
From (2.1), we can derive

\[ \int_{-1}^{1} |M_1(x)|^p \left(1 - x^2\right)^a \, dx \leq C \ddot{E}_n^p(f''). \]  \hspace{1cm} (4.2)

Similar to (3.4),

\[ M_2(x) = \sum_{s=1}^{n+1} \left\{ \sum_{k=1}^{n} \left[ (p_{n+1}(x_k) - f(x_k)) \varphi''_s(t_s) \right] \right\} \ell_s(x). \]  \hspace{1cm} (4.3)

By a direct computation, we get

\[ \varphi''_s(t_s) = \frac{2(-1)^{k+s} \sqrt{1 - t_s^2}}{(n+1)(t_s - x_k)^3} + \frac{2(-1)^{k+s} t_s}{(n+1) \sqrt{1 - t_s^2(t_s - x_k)^2}} + \frac{(n+1)(-1)^{k+s+1}}{\sqrt{1 - t_s^2(t_s - x_k)}} + \frac{(-1)^{k+s+1}}{(n+1)(1 - t_s^2)^{3/2}(t_s - x_k)}. \]  \hspace{1cm} (4.4)

Equations (4.3) and (4.4) yield

\[ M_2(x) = \sum_{s=1}^{n+1} \left\{ \sum_{k=1}^{n} \left[ (p_{n+1}(x_k) - f(x_k)) \frac{2(-1)^{k+s} \sqrt{1 - t_s^2}}{(n+1)(t_s - x_k)^3} \right] \right\} \ell_s(x) \\
+ \sum_{s=1}^{n+1} \left\{ \sum_{k=1}^{n} \left[ (p_{n+1}(x_k) - f(x_k)) \frac{2(-1)^{k+s} t_s}{(n+1) \sqrt{1 - t_s^2(t_s - x_k)^2}} \right] \right\} \ell_s(x) \\
+ \sum_{s=1}^{n+1} \left\{ \sum_{k=1}^{n} \left[ (p_{n+1}(x_k) - f(x_k)) \frac{(n+1)(-1)^{k+s+1}}{\sqrt{1 - t_s^2(t_s - x_k)}} \right] \right\} \ell_s(x) \\
+ \sum_{s=1}^{n+1} \left\{ \sum_{k=1}^{n} \left[ (p_{n+1}(x_k) - f(x_k)) \frac{(-1)^{k+s+1}}{(n+1)(1 - t_s^2)^{3/2}(t_s - x_k)} \right] \right\} \ell_s(x) \\
= N_1(x) + N_2(x) + N_3(x) + N_4(x). \]
We consider $N_1(x)$ now. For an arbitrary $1 \leq s \leq n+1$, from (2.1), (3.12), and $\sum_{k=1}^{n} |q_k(x)|^2 \leq 2$ (see [9]), it follows that

$$
\left| \sum_{k=1}^{n} (p_{n+1}(x_k) - f(x_k)) \right| \frac{2(-1)^{k+s} \sqrt{1 - t_s^2}}{(n+1)(t_s - x_k)^3} \leq \frac{CE_{n-1}(f'')}{(n+1)^3} \sum_{k=1}^{n} \frac{(1 - x_k^2) \sqrt{1 - t_s^2}}{|t_s - x_k|^3}
$$

$$= \frac{CE_{n-1}(f'')}{(n+1)^3} \sum_{k=1}^{n} \frac{(1 - x_k^2)|q_k(t_s)|}{|t_s - x_k|^3} \leq \frac{CE_{n-1}(f'')}{(n+1)^3} \sum_{k=1}^{n} \frac{(1 - x_k^2)}{|t_s - x_k|^2} \leq CE_{n-1}(f'').$$

(4.6)

From (2.8) and (4.6), we can obtain

$$\left( \int_{-1}^{1} [N_1(x)]^p \left( 1 - x^2 \right)^a dx \right)^{1/p} \leq CE_{n-1}(f''). \quad (4.7)$$

Now we consider $N_2(x)$. From $2x/\pi \leq \sin x \leq x$, for all $x \in [0, \pi/2]$, it follows that $\sqrt{1 - t_s^2} \geq \sin(\pi/2(n+1)) \geq 1/(n+1)$. By (2.1) and (3.12), we have that, for an arbitrary $1 \leq s \leq n+1$,

$$\left| \sum_{k=1}^{n} (p_{n+1}(x_k) - f(x_k)) \right| \frac{2(-1)^{k+s} t_s}{(n+1) \sqrt{1 - t_s^2}(t_s - x_k)^2} \leq \frac{CE_{n-1}(f'')}{(n+1)^3} \sum_{k=1}^{n} \frac{1 - x_k^2}{\sqrt{1 - t_s^2}(t_s - x_k)^2} \leq CE_{n-1}(f'').$$

(4.8)

From (2.8) and (4.8), we can obtain

$$\left( \int_{-1}^{1} [N_2(x)]^p \left( 1 - x^2 \right)^a dx \right)^{1/p} \leq CE_{n-1}(f''). \quad (4.9)$$

For the $N_3(x)$, similar to (3.15), we have

$$N_3(x) = \sum_{k=1}^{n} (-1)^k (p_{n+1}(x_k) - f(x_k)) \left[ \sum_{k=1}^{n+1} \frac{T_{n+1}(x_k) (x - t_s)}{(t_s - x_k)(x - t_s)} \right]$$

$$= (n+1) \sum_{k=1}^{n} (-1)^k (p_{n+1}(x_k) - f(x_k)) \frac{U_n(x)}{x - x_k}. \quad (4.10)$$

For an arbitrary $1 \leq k \leq n$, by (2.1) and a simple computation, we can obtain

$$|N_3(x_k)| = (n+1) \left| (p_{n+1}(x_k) - f(x_k)) \cdot U_n(x_k) \right| \leq CE_{n-1}(f''). \quad (4.11)$$
For $k = 0$, (2.1) leads to

$$|N_3(1)| = (n + 1)^2 \left| \sum_{k=1}^{n} \frac{(-1)^k (p_{n+1}(x_k) - f(x_k))}{1 - x_k} \right| \leq CnE_{n-1}(f''). \quad (4.12)$$

Similarly,

$$N_3(-1) \leq CnE_{n-1}(f''). \quad (4.13)$$

Similar to (3.30), from (4.10), (4.11), (4.12), and (4.13), it follows that

$$\left( \int_{-1}^{1} |N_3(x)|^p (1 - x^2)^{\alpha} \, dx \right)^{1/p} \leq \begin{cases} CE_{n-1}(f''), & \alpha > \frac{p}{2} - 1, \\ C(\ln n)^{1/p}E_{n-1}(f''), & \alpha = \frac{p}{2} - 1, \\ Cn^{1-(2\alpha+2)/p}E_{n-1}(f''), & -1 < \alpha < \frac{p}{2} - 1. \end{cases} \quad (4.14)$$

For the $N_4(x)$, similar to (3.15), we have

$$N_4(x) = \sum_{k=1}^{n} (-1)^k \frac{(p_{n+1}(x_k) - f(x_k))}{(n + 1)^2} \left[ \sum_{s=1}^{n+1} \frac{T_{n+1}(x)}{(1 - t_s^2)(t_s - x_k)(x - t_s)} \right]. \quad (4.15)$$

It is easy to verify

$$\frac{T_{n+1}(x)}{(1 - t_s^2)(t_s - x_k)(x - t_s)} = \frac{1}{x - x_k} \left[ \frac{T_{n+1}(x)}{(1 - t_s^2)(x - t_s)} + \frac{T_{n+1}(x)}{(1 - t_s^2)(t_s - x_k)} \right]. \quad (4.16)$$

For $a \neq \pm 1$, it is easy to verify that

$$\frac{1}{(1 - x^2)(x - a)} = -\frac{1}{2(1 + a)(1 + x)} + \frac{1}{2(1 - a)(1 - x)} + \frac{1}{(1 - a^2)(x - a)}. \quad (4.17)$$

From (4.17), (3.16) and

$$\sum_{s=1}^{n+1} \frac{1}{1 + t_s} = \sum_{s=1}^{n+1} \frac{1}{1 - t_s} = (n + 1)^2, \quad (4.18)$$
we obtain

\[
\sum_{s=1}^{n+1} \frac{T_{n+1}(x)}{(1-t_s^2)(x-t_s)} = \frac{T_{n+1}(x)}{2(1+x)} \sum_{s=1}^{n+1} \frac{1}{1+t_s} \frac{T_{n+1}(x)}{2(1-x)} \sum_{s=1}^{n+1} \frac{1}{1-t_s} + \frac{1}{1-x^2} \sum_{s=1}^{n+1} T_{n+1}(x)\]

\[
= -\frac{(n+1)^2 x T_{n+1}(x)}{1-x^2} + \frac{(n+1)U_n(x)}{1-x^2},
\]

\[
\sum_{s=1}^{n+1} \frac{T_{n+1}(x)}{(1-t_s^2)(t_s-x_k)} = \frac{(n+1)^2 x_k T_{n+1}(x)}{1-x_k^2}.
\]

From (4.16), (4.19), (61), (3.9), and a direct computation, we get

\[
\sum_{s=1}^{n+1} \frac{T_{n+1}(x)}{(1-t_s^2)(t_s-x_k)} = -\frac{(n+1)^2 (1+x x_k) T_{n+1}(x)}{(1-x^2)(1-x_k^2)} + \frac{(n+1)U_n(x)}{(1-x)(x-x_k)}
\]

\[
= \frac{(n+1)(1+x x_k) U_n(x)}{1-x_k^2} + \frac{(n+1)x_k U_n(x)}{1-x_k^2} + \frac{(n+1) U_n(x)}{(1-x_k^2)(x-x_k)}.
\]

(4.20)

From (4.15) and (4.20), we obtain

\[
N_n(x) = \sum_{k=1}^{n} \frac{(-1)^k (p_{n+1}(x_k) - f(x_k))}{(n+1)} \left[ \frac{(1+x x_k) U_n(x)}{1-x_k^2} + \frac{x_k U_n(x)}{1-x_k^2} \right]
\]

\[
+ \sum_{k=1}^{n} \frac{(-1)^k (p_{n+1}(x_k) - f(x_k))}{(n+1)} \frac{U_n(x)}{(1-x_k^2)(x-x_k)} = N_{n1}(x) + N_{n2}(x).
\]

(4.21)

For \(N_{n1}(x)\), from (2.1), we can obtain

\[
|N_{n1}(x)| \leq \frac{|U_n(x)| + |U_n(x)|}{(n+1)^2} E_{n-1}(f^n).
\]

(4.22)

By (3.9), Markov inequality, and \(\|U_n(x)\|_\infty = n + 1\), we obtain

\[
|U_n(x)| \leq \frac{2(n+1)}{1-x^2}, \quad |U_n(x)| \leq n^2(n+1).
\]

(4.23)

So for an arbitrary \(0 \leq A \leq 1\),

\[
|U_n(x)| \leq \frac{2^{1-A}(n+1)^{1+2A}}{(1-x^2)^{1-A}} \leq \frac{8n^{1+2A}}{(1-x^2)^{1-A}}.
\]

(4.24)
Let $x = \cos \theta$. From $2\alpha + 1 > -1$, we can choose $A$ such that $0 < A < 1$ and $2\alpha + 1 - 2p + 2pA > -1$. Then by (4.24) and $2x/\pi \leq \sin x \leq x$, for all $x \in [0, \pi/2]$, we can obtain

\[
\int_{-1}^{1} |U'_n(x)|^p (1 - x^2)^\alpha \, dx = 2 \int_{-1}^{0} |U'_n(x)|^p (1 - x^2)^\alpha \, dx \\
\leq 2^{3p+1} \left( \int_{0}^{\pi/2(n+1)} n^p(1+2A) \sin^{2\alpha+1-2p+2pA} \theta \, d\theta \\
+ \int_{\pi/2(n+1)}^{\pi/2} n^p \sin^{1+2\alpha-2p} \theta \, d\theta \right) \tag{4.25}
\]

From $\|U_n(x)\|_\infty = n + 1$, it follows that

\[
\int_{-1}^{1} |U_n(x)|^p (1 - x^2)^\alpha \, dx \leq Cn^p. \tag{4.26}
\]

From (4.22), (4.25), and (4.26), it follows that

\[
\left( \int_{-1}^{1} |N_{41}(x)|^p (1 - x^2)^\alpha \, dx \right)^{1/p} \leq \begin{cases} CE_{n-1}(f''), \\ Cn^{1-(2\alpha+2)/p} E_{n-1}(f''), \end{cases} \quad \alpha \geq \frac{p}{2} - 1, \tag{4.27}
\]

For $N_{42}(x)$, from (2.1) and a simple computation, we can obtain that, for $1 \leq k \leq n$,

\[
|N_{42}(x_k)| = \frac{1}{(n+1)(1-x_k^2)} \left| (p_{n+1}(x_k) - f(x_k)) \cdot U'_n(x_k) \right| \leq CE_{n-1}(f''). \tag{4.28}
\]

Let $x = 1$. Then, from (3.10) and

\[
\sum_{s=1}^{n-1} \frac{U_n(x)}{x - x_s} = U'_n(x), \tag{4.29}
\]

we obtain

\[
\sum_{k=1}^{n} \frac{1}{1-x_k} = \frac{n(n+2)}{3}. \tag{4.30}
\]
From (2.1), it follows that

\[
\left| N_{42}^{(1)} \right| = \left| \sum_{k=1}^{n} \frac{(-1)^k (p_{n+1}(x_k) - f(x_k))}{(1 - x_k)(1 - x_k^2)} \right| \leq \frac{C E_{n-1}(f'')}{(n+1)^2} \sum_{k=1}^{n} \frac{1}{1 - x_k} \leq C E_{n-1}(f''). \tag{4.31}
\]

Similarly,

\[
\left| N_{42}^{(-1)} \right| \leq C E_{n-1}(f''). \tag{4.32}
\]

Similar to (3.30), from (4.28), (4.31), and (4.32), we can obtain

\[
\left( \int_{-1}^{1} |N_{42}^{(1)}(x)|^p \left( 1 - x^2 \right)^a \, dx \right)^{1/p} \leq C E_{n-1}(f''). \tag{4.33}
\]

From (4.1), (4.2), (4.5), (4.7), (4.9), (4.14), (4.21), (4.27), and (4.33), we obtain the upper estimate.

On the other hand, for \( p \geq 2a + 2 \), let \( f(x) = (1 - x^2)U_n(x) \). Then,

\[
f''(x) = -2(n + 2)(n + 1)T_n(x) + q_{n-1}(x), \tag{4.34}
\]

here, \( q_{n-1}(x) \) is a polynomial of degree at most \( n - 1 \). Hence,

\[
E_{n-1}(f'') = 2(n + 2)(n + 1). \tag{4.35}
\]

It is easy to verify that

\[
Q''_{n+2}(f, x) = 0,
\]

\[
f''(x) = -(n + 1)^2 U_n(x) + \frac{-U_n(x) + (n + 1)x T_{n+1}(x)}{1 - x^2}. \tag{4.36}
\]
Let \( x = \cos \theta \), then, \((2k\pi + \pi)/2(n + 1) \leq \theta \leq (2k\pi + 2\pi)/2(n + 1)\) implies that \(T_{n+1}(x)U_n(x) \leq 0\). Therefore,

\[
\int_{-1}^{1} |f''(x) - Q''_n(f, x)|^p (1 - x^2)^a \, dx \geq \int_{0}^{1} |f''(x) - Q''_n(f, x)|^p (1 - x^2)^a \, dx \\
\geq n^{2p} \sum_{k=0}^{[(n+1)/2]} \int_{2k\pi/2(n+1)}^{(2(k+1)\pi/2(n+1))} \left|\sin(n+1)\theta\right|^p \sin^{p-2a-1} \theta \, d\theta
\]

\[
\geq \begin{cases} 
C n^{2p} \ln n, & a = \frac{p}{2} - 1, \\
C n^{3p-2a-2}, & -1 < a < \frac{p}{2} - 1,
\end{cases}
(4.37)
\]

We consider \(L_n\) in the following. For \( f \in C_{[-1,1]}^2 \), let \( p_{n-1}(x) \) be the polynomial of degree at most \( n-1 \) satisfying (2.1). Then,

\[
f''(x) - L''_n(f, x) = f''(x) - p''_{n-1}(x) + L''_n(p_{n-1} - f, x) = K_1(x) + K_2(x).
(4.38)
\]

From (2.1), we can derive

\[
\int_{-1}^{1} |K_1(x)|^p (1 - x^2)^a \, dx \leq C E_{n-3}^p (f'').
(4.39)
\]

If \( f \in C_{[-1,1]} \), then the well-known Lagrange interpolation polynomial of \( f \) based on \( \{x_k\}_{k=1}^n \) is given by

\[
R_n(f, x) = \sum_{k=1}^{n} f(x_k) \phi_k(x),
(4.40)
\]

where

\[
\phi_k(x) = \frac{(-1)^{k+1}(1 - x_k^2)U_n(x)}{(n + 1)(x - x_k)}, \quad k = 1, \ldots, n.
(4.41)
\]

Similar to (3.4), we have

\[
K_2(x) = R_{n-1}(K_2, x) = \sum_{s=1}^{n-1} \left[ \sum_{k=1}^{n} (p_{n-1}(t_k) - f(t_k)) \phi''_k(x) \right] \phi_s(x).
(4.42)
\]
By a direct computation, we obtain

\[
\varphi_k^{(s)}(x_s) = \frac{n(-1)^{k+s} \sqrt{1-t_k^2}}{(1-x_x^2)(x_s-t_k)} + \frac{2(-1)^{k+s+1} \sqrt{1-t_k^2}}{n(x_s-t_k)^3}, \quad s = 1, \ldots, n-1. \tag{4.43}
\]

From (4.42) and (4.43), it follows that

\[
K_2(x) = \sum_{s=1}^{n-1} \left[ \sum_{k=1}^{n} \left( p_{n-1}(t_k) - f(t_k) \right) \frac{n(-1)^{k+s} \sqrt{1-t_k^2}}{(1-x_x^2)(x_s-t_k)} \right] \phi_s(x)
+ \sum_{s=1}^{n-1} \left[ \sum_{k=1}^{n} \left( p_{n-1}(t_k) - f(t_k) \right) \frac{2(-1)^{k+s+1} \sqrt{1-t_k^2}}{n(x_s-t_k)^3} \right] \phi_s(x)
= A_1(x) + A_2(x).
\tag{4.44}
\]

Exchanging the summation order, we have

\[
A_1(x) = \sum_{k=1}^{n} (-1)^{k+1} \left( p_{n-1}(t_k) - f(t_k) \right) \sqrt{1-t_k^2} \left[ \sum_{s=1}^{n-1} \frac{U_{n-1}(x)}{(x_s-t_k)(x-x_s)} \right]. \tag{4.45}
\]

For an arbitrary \( 1 \leq s \leq n-1 \),

\[
\frac{U_{n-1}(x)}{(x_s-t_k)(x-x_s)} = \frac{1}{x-t_k} \left( \frac{U_{n-1}(x)}{x_s-t_k} + \frac{U_{n-1}(x)}{x-x_s} \right). \tag{4.46}
\]

It is well known that

\[
\sum_{s=1}^{n-1} \frac{U_{n-1}(x)}{x-x_s} = U'_{n-1}(x). \tag{4.47}
\]

Let \( x = t_k \). Then, from (4.47) and (3.9), it follows that

\[
\frac{n-1}{\sum_{s=1}^{n-1} \frac{1}{t_k-x_s}} = \frac{U'_{n-1}(t_k)}{U_{n-1}(t_k)} = \frac{t_k}{1-t_k^2}. \tag{4.48}
\]
Combining (4.47), (4.48), and (3.9), we obtain

\[
\sum_{s=1}^{n-1} \frac{1}{x-t_k} \left( \frac{U_{n-1}(x)}{x_{s-t_k}} + \frac{U_{n-1}(x)}{x-x_s} \right) = \frac{1}{x-t_k} \frac{-t_k U_{n-1}(x) + (1 - t_k^2) U'_{n-1}(x)}{1-t_k^2} + \frac{1}{x-t_k} \frac{-x U_{n-1}(x) + (1 - x^2) U'_{n-1}(x)}{1-t_k^2} \]

\[
= \frac{U_{n-1}(x) + (x + t_k) U'_{n-1}(x)}{1-t_k^2} + \frac{1}{x-t_k} \frac{-x U_{n-1}(x) + (1 - x^2) U'_{n-1}(x)}{1-t_k^2} \]

\[
= \frac{U_{n-1}(x) + (x + t_k) U'_{n-1}(x)}{1-t_k^2} - \frac{nT_n(x)}{(x-t_k)(1-t_k^2)}. \tag{4.49}
\]

From (4.45) and (4.49), it follows that

\[
A_1(x) = \sum_{k=1}^{n} (-1)^{k+1} (p_{n-1}(t_k) - f(t_k)) \frac{U_{n-1}(x) + (x + t_k) U'_{n-1}(x)}{\sqrt{1-t_k^2}} \]

\[
+ \sum_{k=1}^{n} (-1)^k (p_{n-1}(t_k) - f(t_k)) \frac{nT_n(x)}{(x-t_k)\sqrt{1-t_k^2}} = A_{11}(x) + A_{12}(x). \tag{4.50}
\]

By (2.1), we have

\[
|A_{11}(x)| \leq \frac{CE_{n-3}(f'')(\max |U'_{n-1}(x)| + |U_n(x)|)}{n}. \tag{4.51}
\]

From (4.51), (4.25), and (4.26), it follows that

\[
\left( \int_{-1}^{1} |A_{11}(x)|^p (1-x^2)^{\alpha} \, dx \right)^{1/p} \leq \begin{cases} CE_{n-3}(f''), & \alpha > p - 1, \\ C(\ln n)^{1/p} E_{n-3}(f''), & \alpha = p - 1, \\ Cn^{2-(2\alpha+2)/p} E_{n-3}(f''), & -1 < \alpha < p - 1. \end{cases} \tag{4.52}
\]

From (2.1) and \( |T_n'(t_k)| = n/\sqrt{1-t_k^2} \), it follows that, for \( 1 \leq k \leq n \),

\[
|A_{12}(t_k)| = \left| (-1)^{k+1} (p_{n-1}(t_k) - f(t_k)) \frac{nT_n'(t_k)}{\sqrt{1-t_k^2}} \right| \leq CE_{n-3}(f''). \tag{4.53}
\]

From (4.53), (2.8), and

\[
A_{12}(x) = \sum_{k=1}^{n} A_{12}(t_k) \zeta_k(x), \tag{4.54}
\]
we know

\[
\left( \int_{-1}^{1} |A_{12}(x)|^p \left( 1 - x^2 \right)^{\alpha} \, dx \right)^{1/p} \leq C E_{n-3}(f'').
\]...

4.38

On the other hand, let \( f(x) = T_n(x) \). Then, it is easy to see

\[
L_n(f, x) = 0,
\]

\[
f''(x) = 4n(n - 1)T_{n-2}(x) + q_{n-3}(x),
\]

here, \( q_{n-3}(x) \) is a polynomial of degree at most \( n - 3 \). Consequently, due to (4.61), we get

\[
E_{n-3}(f'' \prime) = 4n(n - 1).
\]
Let \( x = \cos \theta \). From (4.60), (4.62), (3.9), \( T_n'(x) = nU_{n-1}'(x) \), and the odevity of \( U_{n-1}'(x) \), it follows that

\[
\int_{-1}^{1} \left| f''(x) - L_n''(f, x) \right|^p (1 - x^2)^{\alpha} \, dx \geq \frac{E_{n-3}^p (f'')}{4^p (n - 1)^p} \int_{0}^{1} \left| U_{n-1}'(x) \right|^p (1 - x^2)^{\alpha} \, dx
\]

\[
\geq \frac{E_{n-3}^p (f'')}{8^p} \sum_{k=1}^{n/2} \frac{1}{(k\pi - x/4)^\alpha} |\sin \theta|^{2p-2\alpha-1} d\theta
\]

(4.63)

\[
\begin{cases}
CE_{n-3}^p (f''), & \alpha > p - 1, \\
C \ln n E_{n-3}^p (f''), & \alpha = p - 1, \\
C n^{2p-2\alpha-2} E_{n-3}^p (f''), & -1 < \alpha < p - 1.
\end{cases}
\]

References

Submit your manuscripts at http://www.hindawi.com