Research Article
Falling $d$-Ideals in $d$-Algebras

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Based on the theory of a falling shadow which was first formulated by Wang (1985), a theoretical approach of the ideal structure in $d$-algebras is established. The notions of a falling $d$-subalgebra, a falling $d$-ideal, a falling $BCK$-ideal, and a falling $d^\#$-ideal of a $d$-algebra are introduced. Some fundamental properties are investigated. Relations among a falling $d$-subalgebra, a falling $d$-ideal, a falling $BCK$-ideal, and a falling $d^\#$-ideal are stated. Characterizations of falling $d$-ideals and falling $d^\#$-ideals are discussed. A relation between a fuzzy $d$-subalgebra and a falling $d$-subalgebra is provided.

1. Introduction

Išeki and Tanaka introduced two classes of abstract algebras $BCK$-algebras and $BCI$-algebras [1, 2]. It is known that the class of $BCK$-algebras is a proper subclass of the class of $BCI$-algebras. $BCK$-algebras have several connections with other areas of investigation, such as: lattice ordered groups, $MV$-algebras, Wajsberg algebras, and implicitive commutative semigroups. Font et al. [3] have discussed Wajsberg algebras which are term-equivalent to $MV$-algebras. Mundici [4] proved that $MV$-algebras are categorically equivalent to bounded commutative $BCK$-algebras. Meng [5] proved that implicitive commutative semigroups are equivalent to a class of $BCK$-algebras. Neggers and Kim [6] introduced the notion of $d$-algebras which is another useful generalization of $BCK$-algebras. They investigated several relations between $d$-algebras and $BCK$-algebras as well as several other relations between $d$-algebras and oriented digraphs. After that, some further aspects were studied in [7, 8]. Neggers et al. [9] introduced the concept of $d$-fuzzy function which generalizes the concept of fuzzy subalgebra to a much larger class of functions in a natural way. In addition, they discussed a method of fuzzification of a wide class of algebraic systems onto $[0, 1]$ along with some consequences.

In this paper, we establish a theoretical approach to define a falling $d$-subalgebra, a falling $d$-ideal, a falling $BCK$-ideal, and a falling $d^{\#}$-ideal in $d$-algebras based on the theory of falling shadows which was first formulated by Wang [12]. We provide relations among a falling $d$-subalgebra, a falling $d$-ideal, a falling $BCK$-ideal, and a falling $d^{\#}$-ideal. We consider characterizations of falling $d$-ideals and falling $d^{\#}$-ideals and discuss a relation between a fuzzy $d$-subalgebra and a falling $d$-subalgebra.

2. Preliminaries

A $d$-algebra is a nonempty set $X$ with a constant $0$ and a binary operation “$*$” satisfying the following axioms:

(i) $x * x = 0$,
(ii) $0 * x = 0$,
(iii) $x * y = 0$ and $y * x = 0$ imply $x = y$,

for all $x, y \in X$.

A $BCK$-algebra is a $d$-algebra $(X, *, 0)$ satisfying the following additional axioms:

(iv) $((x * y) * (x * z)) * (z * y) = 0$,
(v) $(x * (x * y)) * y = 0$,

for all $x, y, z \in X$.

Any $BCK$-algebra $(X, *, 0)$ satisfies the following conditions:

(a1) (for all $x, y \in X$) $((x * y) * x = 0)$,
(a2) (for all $x, y, z \in X$) $(((x * z) * (y * z)) * (x * y) = 0)$. 
A subset $I$ of a BCK-algebra $X$ is called a BCK-ideal of $X$ if it satisfies

(b1) $0 \in I$,
(b2) (for all $x \in X$) (for all $y \in I$) $(x \ast y \in I \Rightarrow x \in I)$.

We now display the basic theory on falling shadows. We refer the reader to the papers [10–14] for further information regarding the theory of falling shadows.

Given a universe of discourse $U$, let $\mathcal{P}(U)$ denote the power set of $U$. For each $u \in U$, let

$$\hat{u} := \{E \mid u \in E \text{ and } E \subseteq U\},$$

and for each $E \in \mathcal{P}(U)$, let

$$\check{E} := \{\hat{u} \mid u \in E\}.$$ (2.1)

An ordered pair $(\mathcal{P}(U), \mathcal{B})$ is said to be a hypermeasurable structure on $U$ if $\mathcal{B}$ is a $\sigma$-field in $\mathcal{P}(U)$ and $U \subseteq \mathcal{B}$. Given a probability space $(\Omega, \mathcal{A}, P)$ and a hypermeasurable structure $(\mathcal{P}(U), \mathcal{B})$ on $U$, a random set on $U$ is defined to be a mapping $\xi : \Omega \to \mathcal{P}(U)$ which is $\mathcal{A}$-$\mathcal{B}$ measurable, that is,

$$\forall C \in \mathcal{B} \quad \{\xi^{-1}(C) = \{\omega \mid \omega \in \Omega \text{ and } \xi(\omega) \in C\} \in \mathcal{A}\}. \quad (2.2)$$

Suppose that $\xi$ is a random set on $U$. Let

$$\bar{H}(u) := P(\omega \mid u \in \xi(\omega)) \text{ for each } u \in U.$$ (2.4)

Then $\bar{H}$ is a kind of fuzzy set in $U$. We call $\bar{H}$ a falling shadow of the random set $\xi$, and $\xi$ is called a cloud of $\bar{H}$.

For example, $(\Omega, \mathcal{A}, P) = ([0,1], \mathcal{A}, m)$, where $\mathcal{A}$ is a Borel field on $[0, 1]$ and $m$ is the usual Lebesgue measure. Let $\bar{H}$ be a fuzzy set in $U$ and let $\bar{H}_t := \{u \in U \mid \bar{H}(u) \geq t\}$ be a $t$-cut of $\bar{H}$. Then

$$\xi : [0,1] \to \mathcal{P}(U), \quad t \mapsto \bar{H}_t$$

is a random set and $\xi$ is a cloud of $\bar{H}$. We will call $\xi$ defined above as the cut-cloud of $\bar{H}$ (see [10]).

### 3. Falling $d$-Subalgebras/Ideals

In what follows let $X$ denote a $d$-algebra unless otherwise specified.

A nonempty subset $S$ of $X$ is called a $d$-subalgebra of $X$ (see [8]) if $x \ast y \in S$ whenever $x \in S$ and $y \in S$. 

A subset $I$ of $X$ is called a BCK-ideal of $X$ (see [8]) if it satisfies conditions (b1) and (b2).

A subset $I$ of $X$ is called a $d$-ideal of $X$ (see [8]) if it satisfies conditions (b2) and (b3) (for all $x, y \in X$ $(x \in I \Rightarrow x * y \in I)$).

**Definition 3.1.** Let $(\Omega, \mathcal{A}, P)$ be a probability space, and let

$$\xi : \Omega \rightarrow \mathcal{P}(X)$$

be a random set. If $\xi(\omega)$ is a $d$-subalgebra (resp., BCK-ideal and $d$-ideal) of $X$ for any $\omega \in \Omega$ with $\xi(\omega) \neq \emptyset$, then the falling shadow $\widetilde{H}$ of the random set $\xi$, that is,

$$\widetilde{H}(x) = P(\omega \mid x \in \xi(\omega))$$

is called a falling $d$-subalgebra (resp., falling BCK-ideal and falling $d$-ideal) of $X$.

**Example 3.2.** Let $(\Omega, \mathcal{A}, P)$ be a probability space and let

$$F(X) := \{f \mid f : \Omega \rightarrow X \text{ is a mapping}\}.$$  

Define an operation $\otimes$ on $F(X)$ by

$$(\forall \omega \in \Omega) \quad ((f \otimes g)(\omega) = f(\omega) * g(\omega))$$

for all $f, g \in F(X)$. Let $\theta \in F(X)$ be defined by $\theta(\omega) = 0$ for all $\omega \in \Omega$. It is routine to check that $(F(X); \otimes, \theta)$ is a $d$-algebra. For any $d$-subalgebra (resp., BCK-ideal and $d$-ideal) $A$ of $X$ and $f \in F(X)$, let

$$A_f := \{\omega \in \Omega \mid f(\omega) \in A\},$$

$$\xi : \Omega \rightarrow \mathcal{P}(F(X)), \quad \omega \mapsto \{f \in F(X) \mid f(\omega) \in A\}.$$  

Then $A_f \in \mathcal{A}$ and $\xi(\omega) = \{f \in F(X) \mid f(\omega) \in A\}$ is a $d$-subalgebra (resp., BCK-ideal and $d$-ideal) of $F(X)$. Since

$$\xi^{-1}(f) = \{\omega \in \Omega \mid f \in \xi(\omega)\} = \{\omega \in \Omega \mid f(\omega) \in A\} = A_f \in \mathcal{A},$$

$\xi$ is a random set of $F(X)$. Hence the falling shadow $\widetilde{H}(f) = P(\omega \mid f(\omega) \in A)$ on $F(X)$ is a falling $d$-subalgebra (resp., falling BCK-ideal and falling $d$-ideal) of $F(X)$. 


Example 3.3. Let \( X := \{0, a, b, c\} \) be a \( d \)-algebra which is not a \( BCK \)-algebra with the following Cayley table:

\[
\begin{array}{c|cccc}
* & 0 & a & b & c \\
\hline
0 & 0 & 0 & 0 & 0 \\
a & a & 0 & 0 & a \\
b & b & b & 0 & 0 \\
c & c & c & a & 0 \\
\end{array}
\]

(3.7)

Let \((\Omega, \mathcal{A}, P) = ([0,1], \mathcal{A}, m)\) and define a random set \( \xi : [0,1] \to \mathcal{P}(X) \) as follows:

\[
\xi(t) := \begin{cases} 
\emptyset, & \text{if } t \in [0,0.2), \\
\{0, a, c\}, & \text{if } t \in [0.2,0.6), \\
X, & \text{if } t \in [0.6,1]. 
\end{cases}
\]

(3.8)

Then the falling shadow \( \widetilde{H} \) of \( \xi \) is a falling \( d \)-subalgebra of \( X \).

Example 3.4. Let \( X := \{0, a, b, c\} \) be a \( d \)-algebra which is not a \( BCK \)-algebra with the Cayley table as follows:

\[
\begin{array}{c|cccc}
* & 0 & a & b & c \\
\hline
0 & 0 & 0 & 0 & 0 \\
a & a & 0 & 0 & b \\
b & b & c & 0 & 0 \\
c & c & c & c & 0 \\
\end{array}
\]

(3.9)

Let \((\Omega, \mathcal{A}, P) = ([0,1], \mathcal{A}, m)\) and define a random set \( \xi : [0,1] \to \mathcal{P}(X) \) as follows:

\[
\xi(t) := \begin{cases} 
\{0, a, b\}, & \text{if } t \in [0,0.9), \\
X, & \text{if } t \in [0.9,1]. 
\end{cases}
\]

(3.10)

Then the falling shadow \( \widetilde{H} \) of \( \xi \) is a falling \( BCK \)-ideal of \( X \).
Example 3.5. Let $X := \{0, a, b, c, d\}$ be a $d$-algebra which is not a $BCK$-algebra with the Cayley table as follows:

$$
\begin{array}{c|ccccc}
\ast & 0 & a & b & c & d \\
\hline
0 & 0 & 0 & 0 & 0 & 0 \\
a & a & 0 & a & 0 & a \\
b & b & b & 0 & c & 0 \\
c & c & c & b & 0 & c \\
d & c & c & a & a & 0 \\
\end{array}
$$

Let $(\Omega, \mathcal{A}, P) = ([0, 1], \mathcal{A}, m)$ and define a random set $\xi : [0, 1] \rightarrow \mathcal{P}(X)$ as follows:

$$
\xi(t) := \begin{cases} 
\{0, a\}, & \text{if } t \in [0, 0.3), \\
X, & \text{if } t \in [0.3, 0.8), \\
\emptyset, & \text{if } t \in [0.8, 1].
\end{cases}
$$

Then the falling shadow $\overline{H}$ of $\xi$ is a falling $d$-ideal of $X$.

Note that the falling shadow $\overline{H}$ of $\xi$ in Example 3.4 is not a falling $d$-subalgebra of $X$ because if we take $t \in [0, 0.9)$, then $\xi(t) = \{0, a, b\}$ is not a $d$-subalgebra of $X$. This shows that, in a $d$-algebra, a falling $BCK$-ideal need not be a falling $d$-subalgebra.

The following example shows that a falling $d$-subalgebra need not be a falling $BCK$-ideal in $d$-algebras.

Example 3.6. Consider the $d$-algebra $X$ which is given in Example 3.4. Let $(\Omega, \mathcal{A}, P) = ([0, 1], \mathcal{A}, m)$ and define a random set

$$
\xi : [0, 1] \rightarrow \mathcal{P}(X), \quad t \mapsto \begin{cases} 
\{0, c\}, & \text{if } t \in [0, 0.4), \\
X, & \text{if } t \in [0.4, 1].
\end{cases}
$$

Then the falling shadow $\overline{H}$ of $\xi$ is a falling $d$-subalgebra of $X$, but it is not a falling $BCK$-ideal of $X$ since $\xi(t) = \{0, c\}$ is not a $BCK$-ideal of $X$ for $t \in [0, 0.4)$.

Theorem 3.7. Every falling $d$-ideal is a falling $d$-subalgebra.

Proof. It is clear, and we omit the proof.
The following example shows that the converse of Theorem 3.7 is not true.

**Example 3.8.** Let $X := \{0, a, b, c\}$ be a $d$-algebra which is not a BCK-algebra with the Cayley table as follows:

\[
\begin{array}{c|cccc}
* & 0 & a & b & c \\
\hline
0 & 0 & 0 & 0 & 0 \\
a & a & 0 & 0 & b \\
b & b & b & 0 & 0 \\
c & c & c & c & 0 \\
\end{array}
\]  

(3.14)

Let $(\Omega, \mathcal{A}, P) = ([0, 1], \mathcal{A}, m)$ and define a random set

\[
\xi : [0, 1] \to \mathcal{P}(X), \quad t \mapsto \begin{cases} 
\emptyset, & \text{if } t \in [0, 0.2), \\
\{0, a\}, & \text{if } t \in [0.2, 0.5), \\
X, & \text{if } t \in [0.5, 1]. 
\end{cases}
\]  

(3.15)

Then the falling shadow $\tilde{H}$ of $\xi$ is a falling $d$-subalgebra of $X$, but not a falling $d$-ideal of $X$, since $\xi(t) = \{0, a\}$ is not a $d$-ideal of $X$ for $t \in [0.2, 0.5)$.

Let $(\Omega, \mathcal{A}, P)$ be a probability space and $\tilde{H}$ a falling shadow of a random set $\xi : \Omega \to \mathcal{P}(X)$. For any $x \in X$, let

\[
\Omega(x; \xi) := \{\omega \in \Omega \mid x \in \xi(\omega)\}.
\]  

(3.16)

Then $\Omega(x; \xi) \in \mathcal{A}$.

**Lemma 3.9.** If $\tilde{H}$ is a falling $d$-subalgebra of $X$, then

\[
(\forall x \in X) \quad (\Omega(x; \xi) \subseteq \Omega(0; \xi)).
\]  

(3.17)

*Proof.* If $\Omega(x; \xi) = \emptyset$, then it is clear. Assume that $\Omega(x; \xi) \neq \emptyset$ and let $\omega \in \Omega$ be such that $\omega \in \Omega(x; \xi)$. Then $x \in \xi(\omega)$, and so $0 = x * x \in \xi(\omega)$ since $\xi(\omega)$ is a $d$-subalgebra of $X$. Hence $\omega \in \Omega(0; \xi)$, and therefore $\Omega(x; \xi) \subseteq \Omega(0; \xi)$ for all $x \in X$.

Combining Theorem 3.7 and Lemma 3.9, we have the following corollary.

**Corollary 3.10.** If $\tilde{H}$ is a falling $d$-ideal of $X$, then (3.17) is valid.

We provide a characterization of a falling $d$-ideal.
Theorem 3.11. Let $\tilde{H}$ be a falling shadow of a random set $\xi$ on $X$. Then $\tilde{H}$ is a falling $d$-ideal of $X$ if and only if the following conditions are valid:

(a) (for all $x, y \in X$) $(\Omega(x \ast y; \xi) \cap \Omega(y; \xi) \subseteq \Omega(x; \xi))$,

(b) (for all $x, y \in X$) $(\Omega(x; \xi) \subseteq \Omega(x \ast y; \xi))$.

Proof. Assume that $\tilde{H}$ is a falling $d$-ideal of $X$. For any $x, y \in X$, if

$$\omega \in \Omega(x \ast y; \xi) \cap \Omega(y; \xi),$$

then $x \ast y \in \xi(\omega)$ and $y \in \xi(\omega)$. Since $\xi(\omega)$ is a $d$-ideal of $X$, it follows from (b2) that $x \in \xi(\omega)$ so that $\omega \in \Omega(x; \xi)$. Hence $\Omega(x \ast y; \xi) \cap \Omega(y; \xi) \subseteq \Omega(x; \xi)$ for all $x, y \in X$. Now let $x, y \in X$ and $\omega \in \Omega$ be such that $\omega \in \Omega(x; \xi)$. Then $x \in \xi(\omega)$ and so $x \ast y \in \xi(\omega)$ by (b3). Thus $\omega \in \Omega(x \ast y; \xi)$, and therefore $\Omega(x; \xi) \subseteq \Omega(x \ast y; \xi)$ for all $x, y \in X$.

Conversely, suppose that two conditions (a) and (b) are valid. Let $x, y \in X$ and $\omega \in \Omega$ be such that $x \ast y \in \xi(\omega)$ and $y \in \xi(\omega)$. Then $\omega \in \Omega(x \ast y; \xi)$ and $\omega \in \Omega(y; \xi)$. It follows from (a) that $\omega \in \Omega(x \ast y; \xi) \cap \Omega(y; \xi) \subseteq \Omega(x; \xi)$ so that $x \in \xi(\omega)$. Now, assume that $x \in \xi(\omega)$ for every $x \in X$ and $\omega \in \Omega$. Then $\omega \in \Omega(x; \xi) \subseteq \Omega(x \ast y; \xi)$ for all $y \in X$, and so $x \ast y \in \xi(\omega)$. Therefore $\xi(\omega)$ is a $d$-ideal of $X$ for all $\omega \in \Omega$. Hence $\tilde{H}$ is a falling $d$-ideal of $X$.

Proposition 3.12. For a falling shadow $\tilde{H}$ of a random set $\xi$ on $X$, if $\tilde{H}$ is a falling BCK-ideal of $X$, then

(a) (for all $x, y \in X$) $(x \ast y = 0 \Rightarrow \Omega(y; \xi) \subseteq \Omega(x; \xi))$,

(b) (for all $x, y \in X$) $(\Omega(x \ast y; \xi) \cap \Omega(y; \xi) \subseteq \Omega(x; \xi))$,

(c) (for all $x \in X$) $(\Omega(x; \xi) \subseteq \Omega(0; \xi))$.

Proof. (a) Let $x, y \in X$ and $\omega \in \Omega$ be such that $x \ast y = 0$ and $\omega \in \Omega(y; \xi)$. Then $y \in \xi(\omega)$ and $x \ast y = 0 \in \xi(\omega)$ by (b1). It follows from (b2) that $x \in \xi(\omega)$ so that $\omega \in \Omega(x; \xi)$. Hence $\Omega(y; \xi) \subseteq \Omega(x; \xi)$ for all $x, y \in X$ with $x \ast y = 0$.

(b) Let $x, y \in X$ and $\omega \in \Omega$ be such that $\omega \in \Omega(x \ast y; \xi) \cap \Omega(y; \xi)$. Then $x \ast y \in \xi(\omega)$ and $y \in \xi(\omega)$. Since $\xi(\omega)$ is a BCK-ideal of $X$, it follows from (b2) that $x \in \xi(\omega)$ so that $\omega \in \Omega(x; \xi)$. Hence $\Omega(x \ast y; \xi) \cap \Omega(y; \xi) \subseteq \Omega(x; \xi)$ for all $x, y \in X$.

(c) It follows from (ii) and (a).

We give conditions for a falling shadow to be a falling BCK-ideal.

Theorem 3.13. For a falling shadow $\tilde{H}$ of a random set $\xi$ on $X$, assume that the following conditions are satisfied:

(a) $\Omega = \Omega(0; \xi)$,

(b) (for all $x, y \in X$) $(\Omega(x \ast y; \xi) \cap \Omega(y; \xi) \subseteq \Omega(x; \xi))$.

Then $\tilde{H}$ is a falling BCK-ideal of $X$.

Proof. Using (a), we have $0 \in \xi(\omega)$ for all $\omega \in \Omega$. Let $x, y \in X$ and $\omega \in \Omega$ be such that $x \ast y \in \xi(\omega)$ and $y \in \xi(\omega)$. Then $\omega \in \Omega(x \ast y; \xi) \cap \Omega(y; \xi) \subseteq \Omega(x; \xi)$ by (b), and so $x \in \xi(\omega)$. Therefore $\xi(\omega)$ is a BCK-ideal of $X$ for all $\omega \in \Omega$. Hence $\tilde{H}$ is a falling BCK-ideal of $X$. 


Proposition 3.14. If \( \widetilde{H} \) is a falling \( d \)-ideal of \( X \), then

\[
(\forall x, y \in X) \quad (y \ast x = 0 \implies \Omega(x; \xi) \subseteq \Omega(y; \xi)).
\] (3.19)

Proof. Let \( x, y \in X \) be such that \( y \ast x = 0 \). Let \( \omega \in \Omega(x; \xi) \). Then \( x \in \xi(\omega) \) and \( \omega \in \Omega(0; \xi) \) by Corollary 3.10. Hence \( y \ast x = 0 \in \xi(\omega) \). Since \( \xi(\omega) \) is a \( d \)-ideal of \( X \), it follows from (b2) that 
\( y \in \xi(\omega) \). Therefore (3.19) holds. \( \square \)

A \( d \)-ideal \( I \) of \( X \) is called a \( \xi \)-ideal of \( X \) (see [8]) if, for arbitrary \( x, y, z \in X \), (b4)
\( x \ast z \in I \) whenever \( x \ast y \in I \) and \( y \ast z \in I \).

Definition 3.15. Let \((\Omega, \mathcal{A}, P)\) be a probability space, and let

\[
\xi : \Omega \rightarrow \mathcal{P}(X)
\] (3.20)

be a random set. If \( \xi(\omega) \) is a \( \xi \)-ideal of \( X \) for any \( \omega \in \Omega \) with \( \xi(\omega) \neq \emptyset \), then the falling shadow \( \widetilde{H} \) of the random set \( \xi \) is called a falling \( \xi \)-ideal of \( X \).

Example 3.16. Let \( X \) be a \( d \)-algebra as in Example 3.8. Let \((\Omega, \mathcal{A}, P) = ([0,1], \mathcal{A}, m)\) and define a random set

\[
\xi : \Omega \rightarrow \mathcal{P}(X), \quad \omega \mapsto \begin{cases} 
\{0, a, b\}, & \text{if } \omega \in [0,0.3), \\
X, & \text{if } \omega \in [0.3,0.8), \\
\emptyset, & \text{if } \omega \in [0.8,1]. 
\end{cases}
\] (3.21)

Then the falling shadow \( \widetilde{H} \) of \( \xi \) is a falling \( \xi \)-ideal of \( X \), and it is represented as follows:

\[
\widetilde{H}(x) = \begin{cases} 
0.8, & \text{if } x \in \{0, a, b\}, \\
0.5, & \text{if } x = c. 
\end{cases}
\] (3.22)

Theorem 3.17. Every falling \( \xi \)-ideal is a falling \( d \)-ideal.

Proof. Straightforward. \( \square \)

We provide an example to show that the converse of Theorem 3.17 is not true.

Example 3.18. Consider the falling \( d \)-ideal \( \widetilde{H} \) of \( X \) which is given in Example 3.5. For \( t \in [0,0.3) \), \( \xi(t) = \{0, a\} \) is not a \( \xi \)-ideal of \( X \) since \( b \ast d = 0 \in \xi(t) \), \( d \ast c = a \in \xi(t) \), but \( b \ast c = c \notin \xi(t) \). Hence \( \widetilde{H} \) is not a falling \( \xi \)-ideal of \( X \).
In the above discussion, we can see the following relations:

\[
\begin{array}{c}
\text{Falling } d^f\text{-ideal} \\
\downarrow \\
\text{Falling } d\text{-ideal} \\
\downarrow \\
\text{Falling } d\text{-subalgebra} & \text{Falling } BCK\text{-ideal}
\end{array}
\] (3.23)

In this diagram, the reverse implications are not true, and we need additional conditions for considering the reverse implications.

A \(d\)-algebra \(X\) is called a \(d^*\)-algebra (see [8]) if it satisfies the identity \((x \ast y) \ast x = 0\) for all \(x, y \in X\).

**Theorem 3.19.** In a \(d^*\)-algebra, every falling BCK-ideal is a falling \(d\)-ideal.

**Proof.** Let \(\tilde{H}\) be a falling BCK-ideal of a \(d^*\)-algebra \(X\). Then \(\Omega(x \ast y; \xi) \cap \Omega(y; \xi) \subseteq \Omega(x; \xi)\) for all \(x, y \in X\) by Proposition 3.12. Let \(x, y \in X\) and \(\omega \in \Omega(x; \xi)\). Since \(X\) is a \(d^*\)-algebra, we have \((x \ast y) \ast x = 0 \in \xi(\omega)\) and so \(x \ast y \in \xi(\omega)\) by (b2). Hence \(\omega \in \Omega(x \ast y; \xi)\), which shows that \(\Omega(x; \xi) \subseteq \Omega(x \ast y; \xi)\) for all \(x, y \in X\). Using Theorem 3.11, we conclude that \(\tilde{H}\) is a falling \(d\)-ideal of \(X\). \(\square\)

**Corollary 3.20.** In a \(d^*\)-algebra, every falling BCK-ideal is a falling \(d\)-subalgebra.

**Proof.** It follows from Theorems 3.7 and 3.19. \(\square\)

The following example shows that, in a \(d^*\)-algebra, any falling \(d\)-subalgebra is neither a falling BCK-ideal nor a falling \(d\)-ideal.

**Example 3.21.** Let \(X := \{0, a, b, c\}\) be a \(d^*\)-algebra which is not a BCK-algebra with the following Cayley table:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>b</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>c</td>
<td>c</td>
<td>c</td>
<td>a</td>
<td>0</td>
</tr>
</tbody>
</table>

(3.24)

Let \((\Omega, \mathcal{A}, P) = ([0, 1], \mathcal{A}, m)\) and define a random set \(\xi : [0, 1] \rightarrow \mathcal{P}(X)\) as follows:

\[
\xi(t) := \begin{cases} 
\emptyset, & \text{if } t \in [0, 0.3), \\
\{0, a, c\}, & \text{if } t \in [0.3, 0.7), \\
X, & \text{if } t \in [0.7, 1]. 
\end{cases}
\] (3.25)
Then the falling shadow $\widetilde{H}$ of $\xi$ is a falling $d$-subalgebra of $X$, but it is neither falling $BCK$-ideal nor a falling $d$-ideal of $X$ since $\xi(t) = \{0, a, c\}$ is neither a $BCK$-ideal nor a $d$-ideal of $X$ for $t \in [0.3, 0.7]$.

Hence, in a $d^*$-algebra, we have the following relations among falling $d$-ideals, falling $d$-subalgebras, and falling $BCK$-ideals:

\[
\begin{align*}
\text{Falling } d\text{-ideal} & \quad \leftrightarrow \quad \text{Falling } BCK\text{-ideal} \\
\text{Falling } d\text{-subalgebra} & \quad \leftrightarrow \quad \text{Falling } BCK\text{-ideal}
\end{align*}
\]  

(3.26)

We now establish a characterization of a falling $d^\#$-ideal.

**Theorem 3.22.** For a falling shadow $\widetilde{H}$ of a random set $\xi$ on $X$, the followings are equivalent.

(a) $\widetilde{H}$ is a falling $d^\#$-ideal of $X$.

(b) $\widetilde{H}$ is a falling $d$-ideal of $X$ that satisfies the following inclusion:

\[
(\forall x, y, z \in X) \quad (\Omega(x \ast y; \xi) \cap \Omega(y \ast z; \xi) \subseteq \Omega(x \ast z; \xi)).
\]  

(3.27)

**Proof.** Assume that $\widetilde{H}$ is a falling $d^\#$-ideal of $X$. Let $x, y, z \in X$ and $\omega \in \Omega$ be such that $\omega \in \Omega(x \ast y; \xi) \cap \Omega(y \ast z; \xi)$. Then $x \ast y \in \xi(\omega)$ and $y \ast z \in \xi(\omega)$, and so $x \ast z \in \xi(\omega)$ since $\xi(\omega)$ is a $d^\#$-ideal of $X$. Hence $\omega \in \Omega(x \ast z; \xi)$, and therefore $\Omega(x \ast y; \xi) \cap \Omega(y \ast z; \xi) \subseteq \Omega(x \ast z; \xi)$ for all $x, y, z \in X$.

Conversely, let $\widetilde{H}$ be a falling $d$-ideal of $X$ satisfying the condition (3.27). Then $\xi(\omega)$ is a $d$-ideal of $X$. Let $x, y, z \in X$ and $\omega \in \Omega$ be such that $x \ast y \in \xi(\omega)$ and $y \ast z \in \xi(\omega)$. Then $\omega \in \Omega(x \ast y; \xi) \cap \Omega(y \ast z; \xi) \subseteq \Omega(x \ast z; \xi)$ by (3.27), and thus $x \ast z \in \xi(\omega)$. Hence $\widetilde{H}$ is a falling $d^\#$-ideal of $X$. \(\square\)

We now discuss relations between a falling $d$-subalgebra and a fuzzy $d$-subalgebra. As a result, we can make a statement that the notion of a falling $d$-subalgebra is a generalization of the notion of a fuzzy $d$-subalgebra.

A fuzzy set $\mu$ on $X$ is called a fuzzy $d$-subalgebra of $X$ (see [7]) if $\mu(x \ast y) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in X$.

**Lemma 3.23** (see [7]). A fuzzy set $\mu$ of $X$ is a fuzzy $d$-subalgebra of $X$ if and only if, for every $\lambda \in [0, 1]$, $\mu_\lambda := \{x \in X \mid \mu(x) \geq \lambda\}$ is a $d$-subalgebra of $X$ when it is nonempty.

**Theorem 3.24.** If one takes the probability space $(\Omega, \mathcal{A}, P) = ([0, 1], \mathcal{A}, m)$, where $\mathcal{A}$ is a Borel field on $[0, 1]$ and $m$ is the usual Lebesgue measure, then every fuzzy $d$-subalgebra of $X$ is a falling $d$-subalgebra of $X$. 

Proof. Let $\mu$ be a fuzzy $d$-subalgebra of $X$. Then $\mu_\lambda$ is a $d$-subalgebra of $X$ for all $\lambda \in [0, 1]$ by Lemma 3.23. Let

$$\xi : [0, 1] \rightarrow \mathcal{P}(X)$$

be a random set and $\xi(\lambda) = \mu_\lambda$ for every $\lambda \in [0, 1]$. Then $\mu$ is a falling $d$-subalgebra of $X$. \qed

We provide an example to show that the converse of Theorem 3.24 is not true.

Example 3.25. Let $X$ be a $d$-algebra as in Example 3.4. Let $(\Omega, \mathcal{A}, P) = ([0, 1], \mathcal{A}, m)$ and define a random set

$$\xi : \Omega \rightarrow \mathcal{P}(X), \quad \omega \mapsto \begin{cases} 
\{0, c\}, & \text{if } t \in [0, 0.2), \\
\emptyset, & \text{if } t \in [0.2, 0.3), \\
\{0, b\}, & \text{if } t \in [0.3, 0.6), \\
\{0, a\}, & \text{if } t \in [0.6, 0.85), \\
X, & \text{if } t \in [0.85, 1].
\end{cases} \quad (3.29)$$

Then the falling shadow $\tilde{H}$ of $\xi$ is a falling $d$-subalgebra of $X$, and it is represented as follows:

$$\tilde{H}(x) = \begin{cases} 
0.9, & \text{if } x = 0, \\
0.4, & \text{if } x = a, \\
0.45, & \text{if } x = b, \\
0.35, & \text{if } x = c.
\end{cases} \quad (3.30)$$

We know that $\tilde{H}$ is not a fuzzy $d$-subalgebra of $X$ since

$$\tilde{H}(b \ast a) = \tilde{H}(c) = 0.35 \ngeq 0.4 = \min\{\tilde{H}(b), \tilde{H}(a)\}. \quad (3.31)$$

Theorem 3.26. Every falling $d$-subalgebra of $X$ is a $T_m$-fuzzy $d$-subalgebra of $X$; that is, if $\tilde{H}$ is a falling $d$-subalgebra of $X$, then

$$(\forall x, y \in X) \quad (\tilde{H}(x \ast y) \geq T_m(\tilde{H}(x), \tilde{H}(y))), \quad (3.32)$$

where $T_m(s, t) = \max\{s + t - 1, 0\}$ for any $s, t \in [0, 1]$. 
Proof. By Definition 3.1, $\xi(\omega)$ is a $d$-subalgebra of $X$ for any $\omega \in \Omega$ with $\xi(\omega) \neq \emptyset$. Hence
\[
\{ \omega \in \Omega \mid x \in \xi(\omega) \} \cap \{ \omega \in \Omega \mid y \in \xi(\omega) \} \subseteq \{ \omega \in \Omega \mid x \ast y \in \xi(\omega) \},
\] (3.33)
which implies that
\[
\widetilde{H}(x \ast y) = P(\omega \mid x \ast y \in \xi(\omega)) \geq P(\{ \omega \mid x \in \xi(\omega) \} \cap \{ \omega \mid y \in \xi(\omega) \}) \geq P(\omega \mid x \in \xi(\omega)) + P(\omega \mid y \in \xi(\omega)) - P(\omega \mid x \in \xi(\omega) \text{ or } \omega \mid y \in \xi(\omega)) \geq \widetilde{H}(x) + \widetilde{H}(y) - 1.
\]
Hence
\[
\widetilde{H}(x \ast y) \geq \max\left\{ \widetilde{H}(x) + \widetilde{H}(y) - 1, 0 \right\} = T_m\left(\widetilde{H}(x), \widetilde{H}(y)\right).
\] (3.35)
This completes the proof. \hfill \blacksquare

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References


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