Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2011, Article ID 561015, 17 pages
doi:10.1155/2011/561015

Research Article

An Operator-Difference Method for Telegraph Equations Arising in Transmission Lines

Mehmet Emir Koksal

Department of Elementary Mathematics Education, Mevlana University, 42003, Konya, Turkey

Correspondence should be addressed to Mehmet Emir Koksal, mekoksal@mevlana.edu.tr

Received 12 June 2011; Accepted 3 August 2011

Academic Editor: Hassan A. El-Morshedy

A second-order linear hyperbolic equation with time-derivative term subject to appropriate initial and Dirichlet boundary conditions is considered. Second-order unconditionally absolutely stable difference scheme in (Ashyralyev et al. 2011) generated by integer powers of space operator is modified for the equation. This difference scheme is unconditionally absolutely stable. Stability estimates for the solution of the difference scheme are presented. Various numerical examples are tested for showing the usefulness of the difference scheme. Numerical solutions of the examples are provided using modified unconditionally absolutely stable second-order operator-difference scheme. Finally, the obtained results are discussed by comparing with other existing numerical solutions. The modified difference scheme is applied to analyze a real engineering problem related with a lossy power transmission line.

1. Introduction

Second-order linear hyperbolic partial differential equations with both constant and variable coefficients arise in many branches of science and engineering, for example, electromagnetic, electrodynamics, thermodynamics, hydrodynamics, elasticity, fluid dynamics, wave propagation, and materials science, see [1–13]. For example, they are used frequently for modelling power transmission lines [7–10, 13]. In numerical methods for solving these equations, the problem of stability has received a great deal of importance and attention (see [14–20]). Specially, a suitable model for analyzing the stability is provided by a proper unconditionally absolutely stable difference scheme with an unbounded operator.

In the literature, the work on new finite difference schemes has drawn important attention for numerical solutions of linear hyperbolic partial differential equations, (see [21–26] and the references given therein). However, these difference schemes are conditionally stable since the stability estimates of them are based on some restrictions on choice of the grid step sizes τ and h with respect to the time and space variables, respectively.

Many scientists have investigated unconditionally stable difference schemes for linear hyperbolic differential equations. Such a difference scheme for approximately solving linear hyperbolic differential equations was studied for the first time in [14]. The first-order difference scheme was constructed for approximately solving the abstract Cauchy problem

$$\frac{d^2u(t)}{dt^2} + A(t)u(t) = f(t), \quad 0 \le t \le T,$$
$$u(0) = \varphi, \qquad u'(0) = \psi, \tag{1.1}$$

where $A(t)$ is an unbounded selfadjoint positive linear operator with domain $D(A(t))$ in an arbitrary Hilbert space H. It is known (see [27, 28]) that various initial-boundary value problems for hyperbolic equations can be reduced to the initial value problem (1.1). Note that (1.1) is the well-known wave equation in the special case when $A(t)$ is equal to Laplace operator Δ. The stability estimates for the solution of this difference scheme and for the first- and second-order difference derivatives were established in Hilbert space.

Then, in the past decade, a huge variety of works on finite difference method for numerical solutions of linear hyperbolic partial differential equations were studied. For the problem (1.1) when $A(t) = A$, Ashyralyev and Sobolevskii [15] developed the first- and two different types of second-order difference schemes. The stability estimates for the solutions of these difference schemes and for the first- and second-order difference derivatives have been established. For the same problem, they developed also the high-order two-step difference methods generated by an exact difference scheme, and by the Taylor expansion on three points in [16]; here, the stability estimates for approximate solutions by these difference methods are also discussed. In [17, 18], two different types of second-order difference methods for the problem (1.1) were studied, and the stability estimates for the solutions of these difference schemes were established. But, the difference methods developed in these references are generated by square roots of $A(t)$. Further, an operator $A^{1/2}(t)$ of which construction is not easy is required to realize these difference methods practically. Therefore, in spite of derived theoretical results, it is not very practical to apply these methods for solving an initial-value problem numerically. Another two different types of second-order difference methods generated by integer powers of $A(t)$ for the problem (1.1) were developed and the stability estimates for the solutions of these difference methods were established in [19, 20]. We should mention that all difference methods in references [15–20] are unconditionally absolutely stable and applicable for multidimensional linear hyperbolic equations with space variable coefficient. Specially, the difference methods in [17–20] are also applicable for multidimensional linear hyperbolic equations with time and space variable coefficients.

For approximately solving the problem

$$u_{tt} + 2\alpha u_t + \beta^2 u = u_{xx} + f(t,x), \quad \alpha > \beta \ge 0,\ a < x < b,\ t > 0,$$
$$u(0,x) = \varphi(x), \qquad u_t(0,x) = \psi(x), \quad a \le x \le b, \tag{1.2}$$
$$u(t,a) = G_0(t), \qquad u(t,b) = G_1(t), \quad t \ge 0,$$

when $a = 0$ and $b = 1$, a three-level implicit difference scheme of $O(\tau^2 + h^2)$ was discussed in [29] wheres α and β are real numbers. For $\alpha > 0$, $\beta = 0$, and $\alpha > \beta > 0$, (1.2) is called as a damped wave equation and as a telegraph equation, respectively, see [22]. In [30], also a three-level implicit difference scheme, whose order is the same as the previous literature for approximately solving the equation

$$u_{tt} + 2\alpha(t,x)u_t + \beta^2(t,x)u = \gamma(t,x)u_{xx} + f(t,x), \tag{1.3}$$

defined in the region $\Omega \times [0 < t < T]$, where $\Omega = \{x \mid 0 < x < 1\}$ with the same initial and boundary conditions of (1.2) at different points of the space variable, was developed. Gao and Chi. [31] developed two explicit difference schemes to solve the problem (1.2) numerically. The accuracy orders of these difference schemes are $O(\tau^3 + h^2)$ and $O(\tau^5 + h^2)$, respectively. Recently, H.-W. Liu and L.-B. Liu. [32] developed a new implicit difference scheme based on quartic spline interpolation in space direction and finite discretization in time direction for solving the same problem. The accuracy order of this difference scheme is second order in time direction and fourth order in space direction. In [33], a difference scheme based on alternating direction implicit methods was presented for solving multidimensional telegraph equations with different coefficients numerically. The order of accuracy of the proposed scheme, in which two free parameters are introduced, is also $O(\tau^2 + h^4)$. All the difference schemes in references [29–33] are unconditionally stable.

Note that Dehghan and Shokri [34] proposed a new numerical scheme based on radial-based function method (Kansa's method) for solving the equation

$$u_{tt} + \alpha u_t + \beta u = u_{xx} + f(t,x), \tag{1.4}$$

with the same initial and Dirichlet boundary conditions in (1.2), where α and β are known constants.

Although there are other important unconditionally stable difference schemes in the literature [35, 36] to solve the equation in (1.2) numerically, it is not possible to consider all of them in detail in a single paper.

The only alternative is the application of stable numerical methods for solving the initial-boundary value problem (1.2) since the analytical solution cannot be determined for arbitrary $f(t,x)$. Being easy to implement and universally applicable, the finite difference method outstands among many of the available numerical methods. For solving a linear hyperbolic partial differential equation with time or space variable coefficient analytically, there is no specific method in the literature.

In the present paper, the second order in both time and space direction unconditionally absolutely stable operator-difference scheme in [20] generated by integer powers of $A(t)$ is modified for approximately solving the equation

$$\frac{d^2u(t)}{dt^2} + B(t)\frac{du(t)}{dt} + A(t)u(t) = f(t), \qquad 0 \le t \le T. \tag{1.5}$$

Here, $A(t)$ is an unbounded selfadjoint positive linear operator with domain $D(A(t))$ in an arbitrary Hilbert space H. The initial conditions of the above equation consists of

$$u(0) = \varphi, \qquad u'(0) = \psi. \tag{1.6}$$

Numerical solutions of the equations (1.2), (1.3), and (1.4) are computed with the modified difference scheme. The results are provided and compared with the difference schemes in [29–34]. The differences among all these difference schemes are illustrated with numerical examples by considering various initial-boundary value problems.

2. Operator Difference Schemes-Stability Estimates

Using the finite difference formulas

$$\begin{gathered}\frac{u(t_{k+1})-2u(t_k)+u(t_{k-1})}{\tau^2}-u''(t_k)=O\left(\tau^2\right),\\ \frac{u(t_{k+1})-u(t_{k-1})}{2\tau}-u'(t_k)=O\left(\tau^2\right),\end{gathered} \tag{2.1}$$

in the equation

$$u''(t_k)+B(t_k)u'(t_k)=-A(t_k)u(t_k)+f(t_k), \tag{2.2}$$

we obtain

$$\begin{aligned}&\frac{u(t_{k+1})-2u(t_k)+u(t_{k-1})}{\tau^2}+B(t_k)\frac{u(t_{k+1})-u(t_{k-1})}{2\tau}+A(t_k)\left(u(t_k)+\frac{\tau^2}{4}A(t_k)u(t_{k+1})\right)\\ &\quad=f(t_k)+O\left(\tau^2\right).\end{aligned} \tag{2.3}$$

Further, we have

$$\left(I+\tau^2A(0)\right)\frac{u(\tau)-u(0)}{\tau}=\frac{\tau}{2}(-A(0)u(0)-B(0)\psi+f(0))+\psi+O\left(\tau^2\right). \tag{2.4}$$

Neglecting small terms $O(\tau^2)$, we obtain the following second-order difference scheme:

$$\begin{gathered}\frac{u_{k+1}-2u_k+u_{k-1}}{\tau^2}+B_k\frac{u_{k+1}-u_{k-1}}{2\tau}+A_ku_k+\frac{\tau^2}{4}A_k^2u_{k+1}=f_k,\\ A_k=A(t_k),\quad B_k=B(t_k),\quad f_k=f(t_k),\quad t_k=k\tau,\quad 1\le k\le N-1,\ N\tau=T,\\ \left(I+\tau^2A_0\right)\tau^{-1}(u_1-u_0)=\frac{\tau}{2}(f_0-A_0u_0-B_0u_0')+\psi,\quad f_0=f(0),\ u_0=\varphi,\ u_0'=\psi\end{gathered} \tag{2.5}$$

for approximately solving (1.5).

Theorem 2.1. *For the solution of the difference scheme* (2.5), *the stability estimate*

$$\left\|\left\{\frac{u_k-u_{k-1}}{\tau}\right\}_1^{N-1}\right\|_{C_\tau}+\|u^\tau\|_{C_\tau}\le C_1\left[\left\|A^{1/2}(0)u_0\right\|_H+\|u_0'\|_H+\tau\sum_{s=0}^{N-1}\|f_s\|_H\right], \tag{2.6}$$

holds, where $u(0)\in D(A^{1/2}(0))$ *and* C_1 *does not depend on* u_0, u_0', $f_s(0\le s\le N-1)$ *and* τ.

Theorem 2.2. *For the solution of the difference scheme* (2.5)*, the stability estimate*

$$\begin{aligned}&\left\|\left\{\tau^{-2}(u_{k+1}-2u_k+u_{k-1})\right\}_1^{N-1}\right\|_{C_\tau}+\left\|\{A_k u_k\}_1^N\right\|_{C_\tau}\\&\quad\le C_2\left[\|A(0)u_0\|_H+\left\|A^{1/2}(0)u_0'\right\|_H+\|f_0\|_H+\sum_{s=0}^{N}\|f_{s+1}-f_s\|_H\right]\end{aligned}\tag{2.7}$$

holds, where $u(0)\in D(A(0))$, $u'(0)\in D(A^{1/2}(0))$ *and* C_2 *does not depend on* u_0, u_0', $f_s(0\le s\le N)$ *and* τ.

Proof. Theorems of the existence and uniqueness of the classical solutions of the initial value problem

$$\begin{aligned}&\frac{d^2u(t)}{dt^2}+A(t)u(t)=f\left(t,u,\frac{du(t)}{dt}\right),\quad 0\le t\le T,\\&u(0)=\varphi,\qquad u'(0)=\psi,\end{aligned}\tag{2.8}$$

for the hyperbolic equation in a Hilbert space H with the selfadjoint positive operator $A(t)$ with a region of definition that does not depend on t under some additional condition of differentiability on the nonlinear maps were established in [37].

The proofs of the above Theorems are based on the symmetry properties of the operators $A_h^x(t)$ defined by formula (2.5) and the following theorem on the coercivity inequality for the solution of the elliptic difference problem in L_{2h}. □

Theorem 2.3. *For the solutions of the elliptic difference problem*

$$A_h^x u^h(x)=w^h(x),\quad x\in\Omega_h,\quad u^h(x)=0,\quad x\in S_h,\tag{2.9}$$

the following coercivity inequality holds (Sobolevskii) [38]:

$$\sum_{r=1}^{n}\left\|u^h_{x_r\bar{x}_r,j_r}\right\|_{L_{2h}}\le C_2\left\|w^h\right\|_{L_{2h}}.\tag{2.10}$$

3. Numerical Examples and Discussion

We have not been able to obtain a sharp estimate for the constants figuring in the stability inequalities. So, in this section, the numerical results are presented by considering various initial-boundary value problems. The executions in all examples are carried out by MATLAB

7.01 and obtained by a PC Pentium (R) 2CPV, 2.00 6 Hz, 2.87 GB of RAM. The errors in the numerical solutions are computed by the root mean square (RMS) error [39]

$$\mathrm{RMS}_{\mathrm{error}} = \sqrt{\frac{\sum_{n=0}^{M} \left(u(T, x_n) - \hat{u}(T, x_n)\right)^2}{M+1}}, \tag{3.1}$$

and by the absolute error,

$$\text{Absolute error} = \max_{0 \le n \le M} \left| u(T, x_n) - \hat{u}(T, x_n) \right|, \tag{3.2}$$

where $u(T, x_n)$ and $\hat{u}(T, x_n)$ represent the exact and numerical solutions at final time, respectively.

Example 1. First, the following initial-boundary value problem [29]:

$$\begin{aligned}
&u_{tt}(t,x) + 2\alpha u_t(t,x) + \beta^2 u(t,x) \\
&\quad = u_{xx}(t,x) + \left(4 - 4\alpha + \beta^2 - 1\right) e^{-2t} \sinh x, \quad \alpha > \beta \ge 0,\ 0 < x < 1,\ t > 0, \\
&u(0,x) = \sinh x, \qquad u_t(0,x) = -2\sinh x, \qquad 0 \le x \le 1, \\
&u(t,0) = 0, \qquad u(t,1) = e^{-2t}\sinh 1, \qquad t \ge 0
\end{aligned} \tag{3.3}$$

with the exact solution $u(t,x) = e^{-2t}\sinh x$ is considered. According to the problem (3.3), the differential operator can be defined follows:

$$Au(t,x) = -u_{xx}(t,x) + \beta^2 u(t,x). \tag{3.4}$$

Concerning the above differential operator, discrete operators can be defined as

$$\begin{aligned}
Au(t_k, x_n) &= -u_{xx}(t_k, x_n) + \beta^2 u(t_k, x_n), \\
A^2 u(t_{k+1}, x_n) &= u_{xxxx}(t_{k+1}, x_n) - 2\beta^2 u_{xx}(t_{k+1}, x_n) + \beta^4 u(t_{k+1}, x_n).
\end{aligned} \tag{3.5}$$

For the approximate solution of the problem (3.3) and the ones appearing in the sequel, using the second order of approximation of second and fourth derivatives in the above

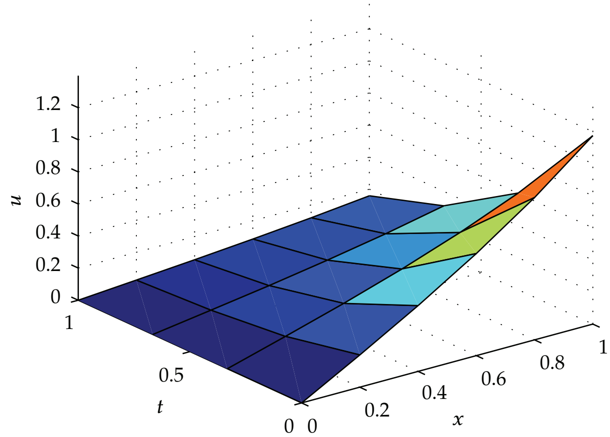

Figure 1: Exact solution.

discrete operators and applying the second-order difference scheme (2.5), the system of linear equations

$$\begin{aligned}
&\frac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2}+2\alpha\frac{u_n^{k+1}-u_n^{k-1}}{2\tau}-\frac{u_{n+1}^k-2u_n^k+u_{n-1}^k}{h^2}+\beta^2u_n^k\\
&\quad-\frac{\tau^2}{4}\left(\frac{u_{n-2}^{k+1}-4u_{n-1}^{k+1}+6u_n^{k+1}-4u_{n+1}^{k+1}+u_{n+2}^{k+1}}{h^4}-2\beta^2\frac{u_{n+1}^{k+1}-2u_n^{k+1}+u_{n-1}^{k+1}}{h^2}+\beta^4u_n^{k+1}\right)\\
&\quad=f(t_k,x_n),\\
&f(t_k,x_n)=\left(4-4\alpha+\beta^2-1\right)e^{-2t_k}\sinh x_n,\qquad x_n=nh,\ t_k=k\tau,\ 1\le k\le N-1,\ 2\le n\le M-2,\\
&u_n^0=\sinh x_n,\quad x_n=nh,\ 1\le n\le M-1,\\
&\frac{u_n^1-u_n^0}{\tau}=\frac{\tau}{2}\left(\frac{u_{n+1}^1-2u_n^1+u_{n-1}^1}{h^2}-\beta^2u_n^1+f(0,x_n)\right)+(1-\tau\alpha)\psi(x_n),\\
&\psi(x_n)=-2\sinh x_n,\quad x_n=nh,\ 1\le n\le M-1,\\
&u_0^k=0,\qquad u_M^k=e^{-2t_k}\sinh,\qquad t_k=k\tau,\quad 0\le k\le N
\end{aligned}\tag{3.6}$$

is obtained. Then, writing the system in the matrix form, a second-order difference equation with respect to k with matrix coefficients is arrived. To solve this resulting difference equation, iterative method is applied (see the third Chapter of [40]).

The exact and numerical solution obtained for $\alpha = 40$ and $\beta = 4$ by using the second-order difference scheme (2.5) with $h = 1/5$ and $\tau = 1/3(\lambda = 1.\overline{6})$ is shown in Figures 1 and 2, respectively, as an example. The exact and numerical solutions of the next examples are shown in odd and even figure number, respectively.

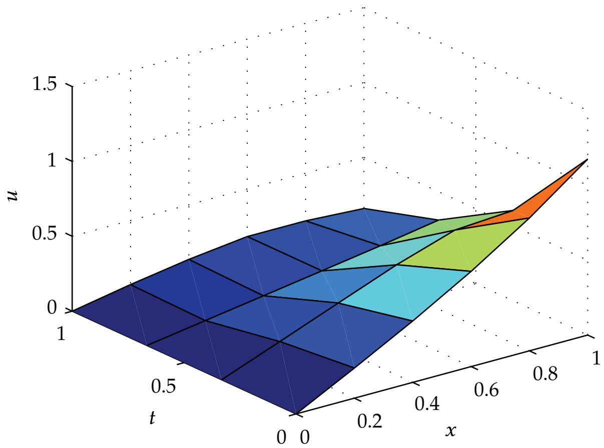

Figure 2: Numerical solution.

Table 1: RMS errors of scheme (2.5) and the scheme in [29] for $\alpha = 10$, $\beta = 0$ when $\lambda = 3.2$.

	$t = 1.0$		$t = 2.0$	
h	The scheme (2.5)	[29]	The scheme (2.5)	[29]
1/32	0.1865 (−02)	0.3404 (−02)	0.1369 (−02)	0.2523 (−02)
1/64	0.4910 (−03)	0.8702 (−03)	0.3599 (−03)	0.6434 (−03)

The difference between Figures 1 and 2 is not clearly obvious though the largest step-size values which correspond to minimum grid numbers, $N = 3$, $M = 5$, are used to graph them. The errors should be computed for the accurate comparison of the numerical and exact solutions as well as for the comparison of the two different difference schemes. RMS errors at $t = 1$, $t = 2$ for different values of α and $\beta(\alpha > \beta \geq 0)$ are tabulated in Tables 1 and 2, and the results are compared with the solutions obtained by [29].

Though both difference schemes are of $O(\tau^2 + h^2)$, considering the errors in the numerical results tabulated in Tables 1 and 2 for the different values of the parameters α, β and the step sizes h, $\tau = h\lambda$, it can be seen easily in Table 1 that the present scheme is approximately 2 times better in the sense of accuracy than the scheme in [29] for $\alpha = 10$, $\beta = 0$. For the values of parameters indicated in Table 2, the present scheme is a little bit better than the scheme in [29] for $h = 1/32$, but it yields approximately 1.5 times smaller error for $h = 1/64$.

Although both of the schemes compared in this example are of $O(\tau^2 + h^2)$, the scheme in [29] requires adequate choice of its free parameters. Otherwise, the results come out to be worse than the results of this paper.

Example 2. When the following equation [31, 32, 34]:

$$u_{tt}(t,x) + 4u_t(t,x) + 2u(t,x) = u_{xx}(t,x), \quad 0 < x < \pi, \quad t > 0 \tag{3.7}$$

with initial conditions

$$u(0,x) = \sin x, \qquad u_t(0,x) = -\sin x, \quad 0 \leq x \leq \pi \tag{3.8}$$

Table 2: RMS errors of scheme (2.5) and the scheme in [29] for $\alpha = 40$, $\beta = 4$ when $\lambda = 1.6$.

	$t = 1.0$		$t = 2.0$	
h	The scheme (2.5)	[29]	The scheme (2.5)	[29]
1/32	0.8123 (−03)	0.9134 (−03)	0.6010 (−03)	0.7485 (−03)
1/64	0.1655 (−03)	0.2588 (−03)	0.1240 (−03)	0.2018 (−03)

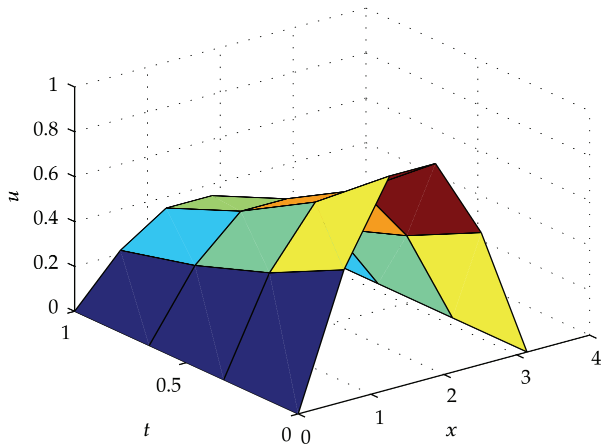

Figure 3: Exact solution.

and boundary conditions

$$u(t,0) = u(t,1) = 0, \quad t \geq 0 \tag{3.9}$$

is considered, then the analytical solution of this problem is $u(t,x) = e^{-t}\sin x$. We apply the same procedure (3.3) for the approximate solution of the above problem.

The exact solution and the numerical solution obtained for $\alpha = 2$ and $\beta = \sqrt{2}$ with the largest step sizes $h = \pi/5$ and $\tau = h\lambda = 1/3$ are shown in Figures 3 and 4.

The difference between the exact solution and numerical solutions for the given example is not obvious though the largest step-size values are used to graph them. For smaller values of h and τ, there is apparently no difference between the graphs of the exact and numerical solutions without depending on the values of the parameters α and β. So again, the errors should be computed for the accurate comparison. The absolute errors involved by the scheme (2.5) and by the scheme (16) in [31] at $t = 1$, $t = 2$ are listed in Tables 3, 4 and 5 for different values of h and τ.

All the results obtained by the scheme (2.5) are almost the same with those obtained by using the scheme in [31] except the results listed in the first rows of Tables 4 and 5. Although the order of accuracy $O(\tau^5 + h^2)$ of the scheme (16) in [31] is much better than $O(\tau^2 + h^2)$ of the present paper, this is not apparent in the results of Tables 3, 4 and 5. In fact, its effect shows up when larger enough τ values (e.g., $\tau = 0.1$) are used [31, 32].

The absolute errors given by the scheme (2.5) and by the scheme in [32] in numerical solutions of the same example are listed in Tables 6 and 7 for $h = \pi/30$ and $h = \pi/300$, respectively, with the same step size $\tau = 0.1$ for time variable.

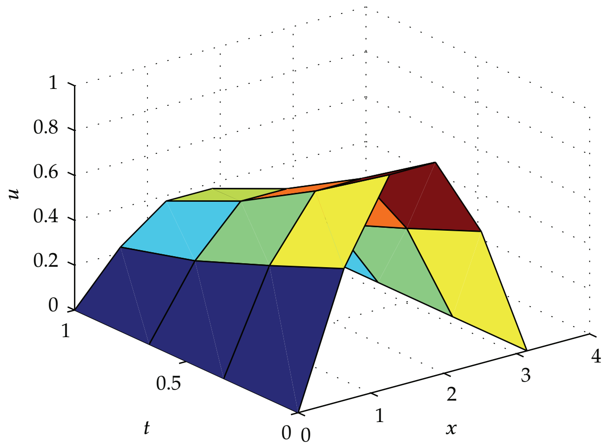

Figure 4: Numerical solution.

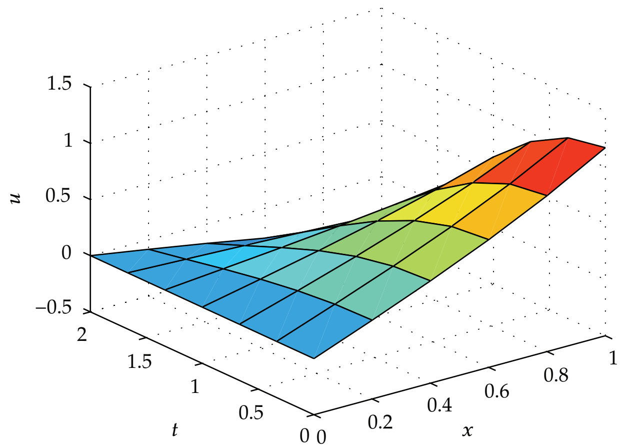

Figure 5: Exact solution.

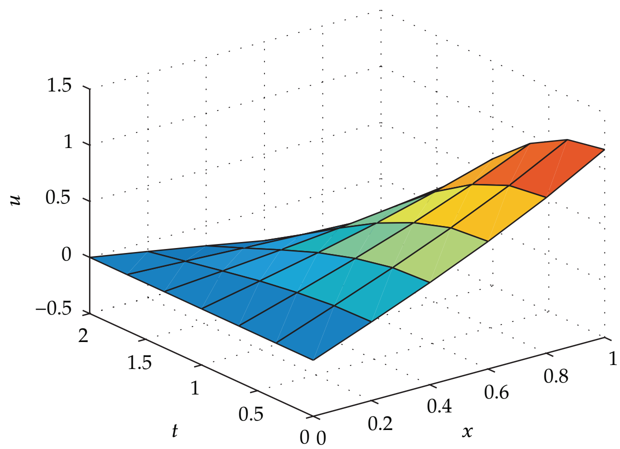

Figure 6: Numerical solution.

Table 3: Absolute errors of scheme (2.5) and scheme (16) in [31] ($h = \pi/30,\quad \tau = 0.001$).

	$t = 1.0$		$t = 2.0$	
x	The scheme (2.5)	[31]	The scheme (2.5)	[31]
$\pi/10$	0.27545 (−04)	0.29477 (−04)	0.27995 (−04)	0.28834 (−04)
$3\pi/10$	0.76583 (−04)	0.77174 (−04)	0.74829 (−04)	0.75489 (−04)
$5\pi/10$	0.94484 (−04)	0.95392 (−04)	0.92662 (−04)	0.93309 (−04)

Table 4: Absolute errors of the present scheme (2.5) and scheme (16) in [31] ($h = \pi/20,\quad s\tau = 0.001$).

	$t = 1.0$		$t = 2.0$	
x	The scheme (2.5)	[31]	The scheme (2.5)	[31]
$\pi/10$	0.05154 (−03)	0.09740 (−03)	0.05867 (−03)	0.09529 (−03)
$3\pi/10$	0.16851 (−03)	0.19116 (−03)	0.16509 (−03)	0.18702 (−03)
$5\pi/10$	0.21460 (−03)	0.21191 (−03)	0.20535 (−03)	0.20732 (−03)

Table 5: Absolute errors of the present scheme (2.5) and scheme (16) in [31] ($h = \pi/20,\quad \tau = 0.01$).

	$t = 1.0$		$t = 2.0$	
x	The scheme (2.5)	[31]	The scheme (2.5)	[31]
$\pi/10$	0.05457 (−03)	0.09740 (−03)	0.05699 (−03)	0.09529 (−03)
$3\pi/10$	0.17669 (−03)	0.19116 (−03)	0.16073 (−03)	0.18702 (−03)
$5\pi/10$	0.22444 (−03)	0.21191 (−03)	0.19990 (−03)	0.20732 (−03)

Table 6: Absolute errors of scheme (2.5) and the scheme in [32] ($h = \pi/30,\ \tau = 0.1$).

	$t = 1.0$		$t = 2.0$	
x	The scheme (2.5)	[32]	The scheme (2.5)	[32]
$\pi/30$	0.1723 (−03)	0.0321 (−03)	0.0117 (−03)	0.0532 (−03)
$15\pi/30$	1.6753 (−03)	0.2033 (−03)	0.1033 (−03)	0.3128 (−03)
$29\pi/30$	0.1723 (−03)	0.0412 (−03)	0.0117 (−03)	0.0331 (−03)

Table 7: Absolute errors of scheme (2.5) and the scheme in [32] ($h = \pi/300,\ \tau = 0.1$).

	$t = 1.0$		$t = 2.0$	
x	The scheme (2.5)	[32]	The scheme (2.5)	[32]
$\pi/30$	0.1650 (−03)	0.0379 (−03)	0.0205 (−03)	0.0389 (−03)
$15\pi/30$	1.5787 (−03)	0.4033 (−03)	0.1961 (−03)	0.4128 (−03)
$29\pi/30$	0.1650 (−03)	0.0463 (−03)	0.0205 (−03)	0.0475 (−03)

Table 8: RMS errors of scheme (2.5) and the scheme in [34].

	$\alpha = 4, \beta = 2$		$\alpha = 6, \beta = 2$	
t	The scheme (2.5)	[34]	The scheme (2.5)	[34]
0.5	1.3139 (−06)	6.3239 (−06)	0.1050 (−05)	1.1210 (−05)
1.0	0.2457 (−05)	1.1579 (−05)	0.1838 (−05)	1.9542 (−05)
1.5	0.2489 (−05)	1.2645 (−05)	0.2054 (−05)	2.1819 (−05)
2.0	0.2402 (−05)	1.1285 (−05)	0.1960 (−05)	2.0821 (−05)

Table 9: RMS errors of the scheme in [32].

h	$\alpha = 10, \beta = 2$	$\alpha = 100, \beta = 2$	$\alpha = 10, \beta = 5$	$\alpha = 100, \beta = 5$
1/8	0.4116 (−02)	0.6115 (−02)	0.1149 (−01)	0.1737 (−01)
1/16	0.7773 (−04)	0.8872 (−03)	0.2174 (−02)	0.2570 (−02)
1/32	0.2591 (−04)	0.5430 (−04)	0.7906 (−04)	0.1684 (−03)
1/64	0.9078 (−05)	0.1220 (−05)	0.3331 (−05)	0.8204 (−05)

Table 10: RMS errors of the scheme in [33].

h	$\alpha = 10, \beta = 2$	$\alpha = 100, \beta = 2$	$\alpha = 10, \beta = 5$	$\alpha = 100, \beta = 5$
1/8	0.5803 (−02)	0.2435 (−02)	0.2568 (−02)	0.1258 (−01)
1/16	0.1306 (−03)	0.5218 (−03)	0.2294 (−02)	0.3035 (−02)
1/32	0.5256 (−04)	0.4042 (−04)	0.6335 (−04)	0.2087 (−03)
1/64	0.1038 (−04)	0.8602 (−05)	0.4467 (−05)	0.2256 (−04)

For all values of h, the errors at $t = 1.0$ of the scheme in [32] are much smaller than those of the present scheme (2.5). This is naturally expected since the order of accuracy of the scheme in [32] is $O(\tau^2 + h^4)$ which is better than that of (2.5), namely, $O(\tau^2 + h^2)$. This difference in the errors disappears at $t = 2.0$ on behalf of the scheme (2.5). In other words, as t increases to 2.0, the errors of the scheme (2.5) are decreasing quietly for all values of h, but the errors of the scheme in [32] remain almost at the same level when compared with those at $t = 1.0$.

Now, consider the main equation in (1.2) in two cases $\alpha = 4$, $\beta = 2$ and $\alpha = 6$, $\beta = 2$ with the same exact solution and initial-boundary conditions of the problem (3.7). The errors are tabulated in Table 8 for $h = 1/50$ and $\tau = 1/10000$ and compared with those found by using the radial basic function method in [34].

It is obvious that the scheme (2.5) is almost 5 and 10 times better in the sense of error size than the scheme in [34] for $\alpha = 4$, $\beta = 2$ and $\alpha = 6$, $\beta = 2$ respectively. However, the computation time of the solution in [34] is quite smaller than that of the present method.

Example 3. Third, consider the following equation [32, 33]:

$$\begin{aligned} &u_{tt}(t,x) + 2\alpha u_t(t,x) + \beta^2 u(t,x) \\ &\quad = u_{xx}(t,x) + \left(\beta^2 - 2\right)\sinh x \cos t - 2\alpha \sinh x \sin t, \quad \alpha > \beta \geq 0, \end{aligned} \tag{3.10}$$

over a region $\Omega = [0 < x < 1] \times [t > 0]$. The initial and boundary conditions are given by

$$\begin{aligned} u(0,x) &= \sin x, \qquad u_t(0,x) = 0, \quad 0 \leq x \leq 1, \\ u(t,0) &= 0, \qquad u(t,1) = \sinh 1 \cos t, \quad t \geq 0. \end{aligned} \tag{3.11}$$

The analytical solution given in [32, 33] is $u(t,x) = \sinh x \cos t$. The exact and numerical solutions obtained for $\alpha = 10$ and $\beta = 2$ with $h = 1/5$ and $\tau = 1/3$ are shown in Figures 5 and 6.

Table 11: RMS errors of scheme (2.5).

h	$\alpha = 10, \beta = 2$	$\alpha = 100, \beta = 2$	$\alpha = 10, \beta = 5$	$\alpha = 100, \beta = 5$
1/8	0.5577 (−02)	1.0809 (−02)	0.4685 (−01)	0.0735 (−01)
1/16	26.3103 (−04)	4.4087 (−03)	0.9412 (−02)	0.8668 (−02)
1/32	7.9742 (−04)	12.0832 (−04)	12.5707 (−04)	2.8841 (−03)
1/64	21.4628 (−05)	31.5917 (−05)	33.7970 (−05)	84.7773 (−05)

Table 12: RMS errors of scheme (2.5) and the scheme in [30].

	$t = 1.0$		$t = 2.0$	
h	The scheme (2.5)	[30]	The scheme (2.5)	[30]
1/16	0.0079 (−01)	0.1618 (−01)	0.0459 (−02)	0.9262 (−02)
1/32	0.0248 (−02)	0.4252 (−02)	0.0130 (−02)	0.1906 (−02)
1/64	0.0861 (−03)	0.9984 (−03)	0.0390 (−03)	0.4491 (−03)

The difference between the exact and numerical solutions for the given example is not obviously apparent though the largest step-size values are used again to graph them. Numerical solutions for different values of α and β when $h = 1/8.2^k$, for $k = 0, 1, 2, 3$, and $\tau = 3.2h$, obtained by using the scheme (2.5) are compared with those obtained by using the schemes in [32, 33]. The RMS errors in solutions are tabulated in Tables 9, 10 and 11 at $t = 2.0$.

The errors in Tables 9, 10 and 11 involved in the results obtained by using the schemes of [32, 33] are naturally much smaller than those in Table 11. The reason of this huge difference is based on that the order of accuracy of the schemes in [32, 33] is $O(\tau^2 + h^4)$, whilst it is $O(\tau^2 + h^2)$ in the present paper.

Example 4. Finally, the following equation [30]:

$$\begin{aligned} &u_{tt}(t,x) + 2e^{x+t}u_t(t,x) + \sin^2(x+t)u(t,x) \\ &\quad = \left(1+x^2\right)u_{xx}(t,x) + \left(3 - 4e^{x+t} - x^2 - \sin^2(x+t)\right)e^{-2t}\sinh x \end{aligned} \tag{3.12}$$

with time and space variable coefficients that are defined in the region $\Omega = [0 < x < 1] \times [t > 0]$ is solved numerically. The initial and boundary conditions of this problem are the same as the initial and boundary conditions of the first example because they have the same exact solution. RMS errors are tabulated at the points $t = 1.0$ and $t = 2.0$ for a fixed values $\lambda = 1.6$, and the results are listed in Table 12.

Apparently, for the given equation with both time and space variable coefficients, the results of the scheme (2.5) are much more satisfactory than the results of the scheme in [30] for all values of h shown in Table 12 although the order of accuracy for both schemes is $O(\tau^2 + h^2)$. On the other hand, when applied to solve an equation with constant coefficients, both schemes result with the same order of errors.

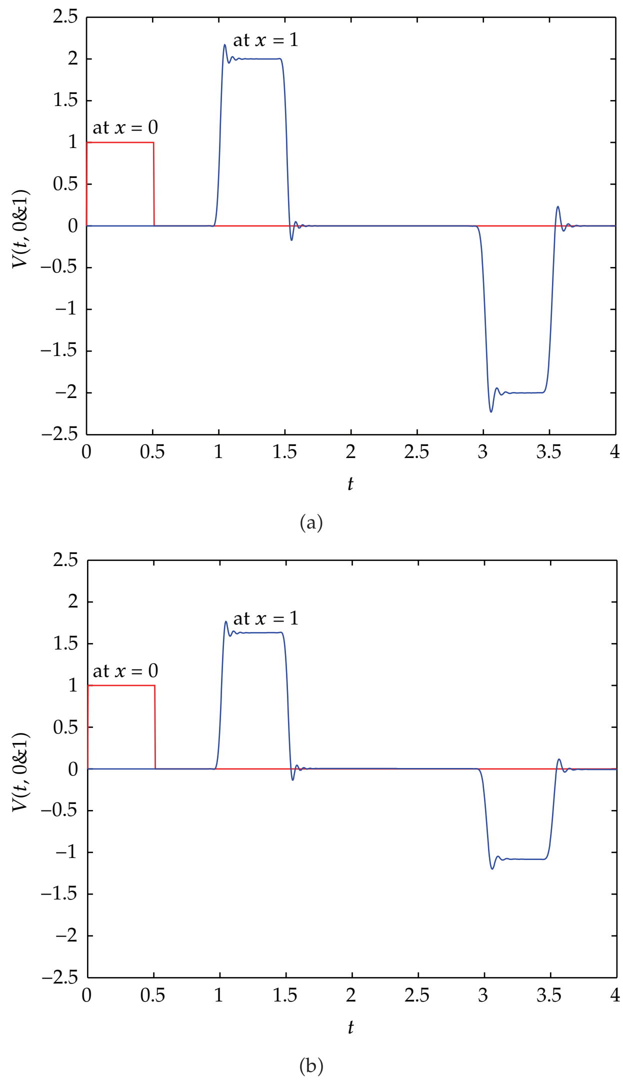

Figure 7: (a) The receiving- and sending-end voltages of the line in the case of slight loss. (b) The receiving- and sending-end voltages of the line in the case of strong loss.

4. Application

As an example, consider the power transmission line having a length of $d = 150\,\text{km}$ with the resistance, inductance, conductance, and capacitance parameters $R = 0.03\,\Omega/\text{km}$, $L = 1\,\text{mH/km}$, $G = 0.001\,\mu\text{S/km}$, and $C = 0.01\,\mu\text{F/km}$ in [8].

The line is represented by the following telegraphic equation:

$$LCV_{tt} + (LG + RC)V_t + \text{RGV} = V_{xx}. \tag{4.1}$$

When the line parameters are normalized with respect to the impedance $\sqrt{L/C}$, frequency $w = 1/(d\sqrt{LC})$, and distance d, then the above second-order partial differential equation becomes

$$V_{tt} + 4.110961 \times 10^{-7} V_t + 3 \times 10^{-14} V = V_{xx}. \tag{4.2}$$

Due to the slight losses of the line the coefficients of V_t and V in the above equation are quite small. When the relaxed line is excited at one end (sending end) by a unit pulse of duration $0.75/\sqrt{10}$ ms which corresponds to a normalized time of 0.5 s, the voltage at the receiving end $(x = 1)$ is computed by solving (4.2) with the boundary and initial conditions

$$V(t, 0) = \begin{cases} 1, & 0 \le t \le 0.5, \\ 0, & 0.5 < t, \end{cases} \qquad V_x(t, 1) = 0, \tag{4.3}$$

$$V(0, x) = V_t(0, x) = 0. \tag{4.4}$$

The problem is solved by the difference scheme (2.5) for the intervals $t \in [0, 4]$ for time and $x \in [0, 1]$ for space with number of grids 1000 and 100, respectively. The plot of the computed receiving-end voltage is shown in Figure 7(a) including the exciting voltage as well. It is obvious that the input pulse is reached at the end of the line at $t = 1$, and it is doubled in magnitude by the reflection at the open end. The wave travels back to the sending end, it gets inverted due to the short-circuit property of the sending end, and reflects back to the receiving end after inversion. Since the travelling time along the line is 1, the wave reaches the receiving end at $t = 3$ as the second time. Due to the very slight loss in the line parameters, the attenuation is not obvious in Figure 7(a). The same problem is solved by increasing the line losses (R and G parameters) by 10^7 times, though this is not a realistic case. The attenuation in the receiving-end voltage is now obvious as plotted in Figure 7(b).

5. Concluding Remarks

In this work, the second-order unconditionally absolutely stable difference scheme in [20] generated by integer powers of $A(t)$ is modified for solving the abstract Cauchy problem for the second-order linear hyperbolic differential equations containing the unbounded selfadjoint positive linear operators. The modified difference scheme is applied for solving various initial-boundary value problems and compared with other published papers in the literature. It is observed that this scheme is more accurate as compared to the other difference schemes whose order of accuracy is the same. Moreover, the present difference scheme (2.5) is compared with difference schemes whose order of accuracy is higher, and it is observed that the good results can be obtained using the difference scheme in the paper depending on the type of the problem and solution parameters of the problem. The difference scheme is applied to a linear hyperbolic equation with variable coefficients, and it is observed that it is much more satisfactory than the scheme obtained by radial basic function, having the same order of accuracy. The modified difference scheme (2.5) is shown to be well applied to analyze a real engineering problem involving a lossy power transmission line with constant coefficients. Being defined by an operator, it can be applied to multidimensional linear hyperbolic differential equations with time and space variables coefficients.

Acknowledgment

The author is very thankful to Professor Ashyralyev for his valuable comments.

References

[1] H. Lamb, *Hydrodynamics*, Cambridge Mathematical Library, Cambridge University Press, Cambridge, UK, 6th edition, 1993.
[2] J. Lighthill, *Waves in Fluids*, Cambridge University Press, Cambridge, UK, 1978.
[3] J. A. Hudson, *The Excitation and Propagation of Elastic Waves*, Cambridge Monographs on Mechanics and Applied Mathematic, Cambridge University Press, Cambridge, UK, 1980.
[4] D. S. Jones, *Acoustic and Electromagnetic Waves*, Oxford Science Publications, The Clarendon Press/Oxford University Press, New York, NY, USA, 1986.
[5] A. Taflove, *Computational Electrodynamics: The Finite-Difference Time-Domain Method*, Artech House, Boston, Mass, USA, 1995.
[6] S. Sieniutycz and R. S. Berry, "Variational theory for thermodynamics of thermal waves," *Physical Review E*, vol. 65, no. 4, Article ID 046132, 11 pages, 2002.
[7] M. S. Mamis and M. Koksal, "Numerical solutions of partial differential equations for transmission lines terminated by lumped components," in *Proceedings of the 5th International Colloquium on Numerical Analysis*, pp. 87–98, 1996.
[8] M. S. Mamis and M. Koksal, "Transient analysis of nonuniform lossy transmission lines with frequency dependent parameters," *Electric Power Systems Research*, vol. 52, no. 3, pp. 223–228, 1999.
[9] M. S. Mamiş and M. Köksal, "Remark on the lumped parameter modeling of transmission lines," *Electric Machines and Power Systems*, vol. 28, no. 6, pp. 565–575, 2000.
[10] S. Herdem and M. S. Mamiş, "Computation of corona effects in transmission lines using state-space techniques," *Computers and Electrical Engineering*, vol. 29, no. 5, pp. 603–611, 2003.
[11] I. Abu-Alshaikh and M. E. Koksal, "One-dimensional transient dynamic response in functionally gradient spherical multilayered media," *Proceedings of Dynamical Systems and Applications*, pp. 1–20, 2004.
[12] I. Abu-Alshaikh, "One-dimensional wave propagation in functionally graded cylindrical layered media," in *Mathematical Methods in Engineering*, pp. 111–121, Springer, New York, NY, USA, 2007.
[13] M. S. Mamiş, A. Kaygusuz, and M. Köksal, "State variable distributed-parameter representation of transmission line for transient simulations," *Turkish Journal of Electrical Engineering and Computer Sciences*, vol. 18, no. 1, pp. 31–42, 2010.
[14] P. E. Sobolevskiĭ and L. M. Čebotareva, "Approximate solution of the Cauchy problem for an abstract hyperbolic equation by the method of lines," *Izvestija Vysših Učebnyh Zavedeniĭ Matematika*, no. 5(180), pp. 103–116, 1977 (Russian).
[15] A. Ashyralyev and P. E. Sobolevskii, "A note on the difference schemes for hyperbolic equations," *Abstract and Applied Analysis*, vol. 6, no. 2, pp. 63–70, 2001.
[16] A. Ashyralyev and P. E. Sobolevskii, "Two new approaches for construction of the high order of accuracy difference schemes for hyperbolic differential equations," *Discrete Dynamics in Nature and Society*, vol. 2005, no. 2, pp. 183–213, 2005.
[17] A. Ashyralyev and M. E. Koksal, "On the second order of accuracy difference scheme for hyperbolic equations in a Hilbert space," *Numerical Functional Analysis and Optimization*, vol. 26, no. 7-8, pp. 739–772, 2005.
[18] A. Ashyralyev and M. E. Koksal, "Stability of a second order of accuracy difference scheme for hyperbolic equation in a Hilbert space," *Discrete Dynamics in Nature and Society*, vol. 2007, Article ID 57491, 25 pages, 2007.

[19] A. Ashyralyev, M. E. Koksal, and R. P. Agarwal, “A difference scheme for Cauchy problem for the hyperbolic equation with self-adjoint operator,” *Mathematical and Computer Modelling*, vol. 52, no. 1-2, pp. 409–424, 2010.

[20] A. Ashyralyev, M. E. Koksal, and R. P. Agarwal, “An operator-difference scheme for abstract Cauchy problems,” *Computers & Mathematics with Applications*, vol. 61, no. 7, pp. 1855–1872, 2011.

[21] M. Ciment and S. H. Leventhal, “A note on the operator compact implicit method for the wave equation,” *Mathematics of Computation*, vol. 32, no. 141, pp. 143–147, 1978.

[22] E. H. Twizell, “An explicit difference method for the wave equation with extended stability range,” *BIT*, vol. 19, no. 3, pp. 378–383, 1979.

[23] S. I. Piskarëv, “Stability of difference schemes in Cauchy problems with almost periodic solutions,” *Differentsial'nye Uravneniya*, vol. 20, no. 4, pp. 689–695, 1984.

[24] S. Piskarev, *Principles of Discretization Methods III. Report*, Acoustic Institute, Academy of Science USSR, 1986.

[25] S. Piskarev, “Approximation of holomorphic semigroups,” *Tartu Riikliku Ülikooli Toimetised*, no. 492, pp. 3–14, 1979.

[26] R. K. Mohanty, M. K. Jain, and K. George, “On the use of high order difference methods for the system of one space second order nonlinear hyperbolic equations with variable coefficients,” *Journal of Computational and Applied Mathematics*, vol. 72, no. 2, pp. 421–431, 1996.

[27] H. O. Fattorini, *Second Order Linear Differential Equations in Banach Spaces*, vol. 108 of *North-Holland Mathematics Studies*, North-Holland, Amsterdam, The Netherlands, 1985.

[28] S. G. Kreĭn, *Lineikhye differentsialnye uravneniya v Banakhovom prostranstve*, Nauka, Moscow, Russia, 1967.

[29] R. K. Mohanty, “An unconditionally stable difference scheme for the one-space-dimensional linear hyperbolic equation,” *Applied Mathematics Letters*, vol. 17, no. 1, pp. 101–105, 2004.

[30] R. K. Mohanty, “An unconditionally stable finite difference formula for a linear second order one space dimensional hyperbolic equation with variable coefficients,” *Applied Mathematics and Computation*, vol. 165, no. 1, pp. 229–236, 2005.

[31] F. Gao and C. Chi, “Unconditionally stable difference schemes for a one-space-dimensional linear hyperbolic equation,” *Applied Mathematics and Computation*, vol. 187, no. 2, pp. 1272–1276, 2007.

[32] H.-W. Liu and L.-B. Liu, “An unconditionally stable spline difference scheme of $O(k^2+h^2)$ for solving the second-order 1D linear hyperbolic equation,” *Mathematical and Computer Modelling*, vol. 49, no. 9-10, pp. 1985–1993, 2009.

[33] R. K. Mohanty, “New unconditionally stable difference schemes for the solution of multi-dimensional telegraphic equations,” *International Journal of Computer Mathematics*, vol. 86, no. 12, pp. 2061–2071, 2009.

[34] M. Dehghan and A. Shokri, “A numerical method for solving the hyperbolic telegraph equation,” *Numerical Methods for Partial Differential Equations*, vol. 24, no. 4, pp. 1080–1093, 2008.

[35] H.-W. Liu, L.-B. Liu, and Y. Chen, “A semi-discretization method based on quartic splines for solving one-space-dimensional hyperbolic equations,” *Applied Mathematics and Computation*, vol. 210, no. 2, pp. 508–514, 2009.

[36] L.-B. Liu and H.-W. Liu, “Quartic spline methods for solving one-dimensional telegraphic equations,” *Applied Mathematics and Computation*, vol. 216, no. 3, pp. 951–958, 2010.

[37] P. E. Sobolevskiĭ and V. A. Pogorelenko, “Hyperbolic equations in a Hilbert space,” *Sibirskiĭ Matematičeskiĭ Žurnal*, vol. 8, pp. 123–145, 1967 (Russian).

[38] P. E. Sobolevskii, *Difference Methods for the Approximate Solution of Differential Equations*, Izdat. Voronezh. Gosud. Univ., Voronezh, Russia, 1975.

[39] B. Bülbül and M. Sezer, “Taylor polynomial solution of hyperbolic type partial differential equations with constant coefficients,” *International Journal of Computer Mathematics*, vol. 88, no. 3, pp. 533–544, 2011.

[40] A. Ashyralyev and M. E. Koksal, “On the numerical solution of hyperbolic PDEs with variable space operator,” *Numerical Methods for Partial Differential Equations*, vol. 25, no. 5, pp. 1086–1099, 2009.