Research Article

Positive Solutions to a Second-Order Discrete Boundary Value Problem

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We are concerned with second-order discrete boundary value problems and obtain some sufficient conditions for the existence of at least one positive solution by using the fixed point theorem due to Krasnosel'skii on a cone.

1. Introduction

Boundary value problems for difference equations have been studied extensively by many authors, for example, [1–10] to name a few. Many techniques arose in the studies of this kind of problem. For example, Agarwal et al. [1] employed the critical point theory to establish the existence of multiple solutions of some regular as well as singular discrete boundary value problems. Cai and Yu [2] applied the Linking Theorem and the Mountain Pass Lemma in the critical point theory to study second-order discrete boundary value problems and obtained some new results for the existence of solutions. Li and Sun [3, 4] were concerned with discrete system boundary value problems and gave some sufficient conditions for the existence of one or two positive solutions by using a nonlinear alternative of Leray-Schauder type and Krasnosel'skii's fixed point theorem in a cone. Pang et al. [5] provided sufficient conditions for the existence of at least three positive solutions for quasilinear boundary value problems for finite difference equations by using a generalization of the Leggett-Williams fixed point theorem due to Avery and Peterson. Du [6], Lin and Liu [7] discussed triple positive solutions of some second-order discrete boundary value problems by making use of the Leggett-Williams fixed-point theorem, respectively.

This paper deals with the following three-point boundary value problem for second-order difference equation of the form

$$\begin{aligned}\Delta^2 y(k-1) + h(k)f(y(k)) &= 0, \quad k \in \{1, \dots, T\}, \\ y(0) - \alpha \Delta y(0) &= 0, \quad y(T+1) = \beta y(n),\end{aligned}\tag{1.1}$$

where $\Delta y(k-1) = y(k) - y(k-1)$, $\Delta^2 y(k-1) = y(k+1) - 2y(k) + y(k-1)$, $k \in \{1, \dots, T\}$.

Throughout this paper, we will assume that the following conditions are satisfied:

- (A1) $T \geq 3$ is a fixed positive integer, $n \in \{2, \dots, T-1\}$, constant $\alpha, \beta > 0$ such that $H := T+1 - \beta n + \alpha(1-\beta) > 0$ and $T+1 - \beta n > 0$;
- (A2) $f \in C([0, +\infty), [0, +\infty))$, f is either superlinear or sublinear, that is, either $f_0 = 0$, $f_\infty = \infty$ or $f_0 = \infty$, $f_\infty = 0$, where

$$f_0 = \lim_{u \rightarrow 0^+} \frac{f(u)}{u}, \quad f_\infty = \lim_{u \rightarrow \infty} \frac{f(u)}{u};\tag{1.2}$$

- (A3) $h(k)$ is nonnegative on $\{1, \dots, T\}$ and $h(k) \equiv 0$ does not hold on $\{n, \dots, T\}$.

In the paper, we show the existence of positive solutions of (1.1) under some assumptions. We also establish the associate Green's function. Readers may find that it is useful to define a cone on which a positive operator was defined, and a fixed point theorem due to Krasnosel'skii [11] will be applied to yield the existence of at least one positive solution.

2. Preliminary and Green's Function

Let $\mathbf{N}$ be the nonnegative integers; we let $\mathbf{N}_{i,j} = \{k \in \mathbf{N} : i \leq k \leq j\}$ and $\mathbf{N}_p = \mathbf{N}_{0,p}$.

By a *positive solution* y of problem (1.1), we mean $y : \mathbf{N}_{T+1} \rightarrow \mathbf{R}$, y satisfies the first equation of (1.1) on $\mathbf{N}_{1,T}$, y fulfills $y(0) - \alpha \Delta y(0) = 0$, $y(T+1) = \beta y(n)$, and y is nonnegative on $\mathbf{N}_{T+1}$ and positive on $\mathbf{N}_{1,T}$.

We shall need the following fixed point theorem due to Krasnosel'skii [8, 11].

Theorem A. *Let E be a Banach space, and let $K \subset E$ be a cone in E . Assume that Ω_1 and Ω_2 are open subsets of E with $0 \in \Omega_1$, $\overline{\Omega_1} \subset \Omega_2$, and let $A : K \cap (\overline{\Omega_2} \setminus \Omega_1) \rightarrow K$ be a completely continuous operator such that either*

- (1) $\|Au\| \leq \|u\|$, $u \in K \cap \partial\Omega_1$ and $\|Au\| \geq \|u\|$, $u \in K \cap \partial\Omega_2$ or
- (2) $\|Au\| \geq \|u\|$, $u \in K \cap \partial\Omega_1$ and $\|Au\| \leq \|u\|$, $u \in K \cap \partial\Omega_2$.

Then A has a fixed point in $K \cap (\overline{\Omega_2} \setminus \Omega_1)$.

Lemma 2.1 (see [7]). *The function*

$$G(k, l) = \frac{1}{H} \begin{cases} (l + \alpha)[T + 1 - k - \beta(n - k)], & l \in \mathbf{N}_{1, k-1} \cap \mathbf{N}_{1, n-1}, \\ (l + \alpha)(T + 1 - k) + \beta(n + \alpha)(k - l), & l \in \mathbf{N}_{n, k-1}, \\ (k + \alpha)[T + 1 - l - \beta(n - l)], & l \in \mathbf{N}_{k, n-1}, \\ (k + \alpha)(T + 1 - l), & l \in \mathbf{N}_{k, T} \cap \mathbf{N}_{n, T}, \end{cases} \quad (2.1)$$

is the Green's function of the problem

$$\begin{aligned} -\Delta^2 y(k-1) &= 0, & k \in \mathbf{N}_{1, T}, \\ y(0) - \alpha \Delta y(0) &= 0, & y(T+1) = \beta y(n). \end{aligned} \quad (2.2)$$

Remark 2.2. We observe that the condition $H > 0$ and $T + 1 - \beta n > 0$ implies $G(k, l)$ is positive on $\mathbf{N}_{T+1} \times \mathbf{N}_{1, T}$, which means that the finite set

$$\left\{ \frac{G(k, l)}{G(k, k)} : k \in \mathbf{N}_{T+1}, l \in \mathbf{N}_{1, T} \right\} \quad (2.3)$$

takes positive values. Then we let

$$M_1 = \min \left\{ \frac{G(k, l)}{G(k, k)} : k \in \mathbf{N}_{T+1}, l \in \mathbf{N}_{1, T} \right\}, \quad (2.4)$$

$$M_2 = \max \left\{ \frac{G(k, l)}{G(k, k)} : k \in \mathbf{N}_{T+1}, l \in \mathbf{N}_{1, T} \right\}. \quad (2.5)$$

3. Main Results

Theorem 3.1. *Assume that (A1)–(A3) hold, then problem (1.1) has at least one positive solution.*

Proof. In the following, we denote

$$m = \min_{k \in \mathbf{N}_{n, T}} G(k, k), \quad M = \max_{k \in \mathbf{N}_{T+1}} G(k, k). \quad (3.1)$$

Then $0 < m < M$.

Let E be the Banach space defined by $E = \{y : \mathbf{N}_{T+1} \rightarrow \mathbb{R}\}$. Define

$$K = \left\{ y \in E : y(k) \geq 0, \text{ for } k \in \mathbf{N}_{T+1} \text{ and } \min_{k \in \mathbf{N}_{n, T}} y(k) \geq \sigma \|y\| \right\}, \quad (3.2)$$

where $\sigma = M_1 m / M_2 M \in (0, 1)$, $\|y\| = \max_{k \in \mathbf{N}_{T+1}} |y(k)|$. It is clear that K is a cone in E .

We define the operator $S : K \rightarrow E$ by

$$(Sy)(k) = \sum_{l=1}^T G(k, l)h(l)f(y(l)), \quad k \in \mathbf{N}_{T+1}. \quad (3.3)$$

It is clear that problem (1.1) has a solution y if and only if $y \in E$ is a solution of the operator equation $y(k) = (Sy)(k)$. We shall now show that the operator S maps K into itself. For this, let $y \in K$; from (A2), (A3), we find

$$(Sy)(k) = \sum_{l=1}^T G(k, l)h(l)f(y(l)) \geq 0, \quad \text{for } k \in \mathbf{N}_{T+1}. \quad (3.4)$$

From (2.5), we obtain

$$\begin{aligned} (Sy)(k) &= \sum_{l=1}^T G(k, l)h(l)f(y(l)) \leq M_2 \sum_{l=1}^T G(k, k)h(l)f(y(l)) \\ &\leq M_2 M \sum_{l=1}^T h(l)f(y(l)), \quad \text{for } k \in \mathbf{N}_{T+1}. \end{aligned} \quad (3.5)$$

Therefore

$$\|Sy\| \leq M_2 M \sum_{l=1}^T h(l)f(y(l)). \quad (3.6)$$

Now from (A2), (A3), (2.4), and (3.6), for $k \in \mathbf{N}_{n,T}$, we have

$$\begin{aligned} (Sy)(k) &\geq M_1 \sum_{l=1}^T G(k, k)h(l)f(y(l)) \geq M_1 m \sum_{l=1}^T h(l)f(y(l)) \\ &\geq \frac{M_1 m}{M_2 M} \|Sy\| = \sigma \|y\|. \end{aligned} \quad (3.7)$$

Then

$$\min_{k \in \mathbf{N}_{n,T}} (Sy)(k) \geq \sigma \|Sy\|. \quad (3.8)$$

From (3.4) and (3.6), we obtain $Sy \in K$. Hence $S(K) \subseteq K$. Also standard arguments yield that $S : K \rightarrow K$ is completely continuous. $\square$

Case 1. Suppose f is superlinear. Now since $f_0 = 0$, we may choose $C_1 > 0$ such that $f(u) \leq \delta_1 u$, for $0 < u \leq C_1$, where δ_1 satisfies

$$\delta_1 M_2 M \sum_{l=1}^T h(l) \leq 1. \quad (3.9)$$

Let $y \in K$ be such that $\|y\| = C_1$; by using (2.5) and (3.9), we have

$$\begin{aligned} (Sy)(k) &\leq M_2 \sum_{l=1}^T G(k, l) h(l) f(y(l)) \leq \delta_1 M_2 M \sum_{l=1}^T h(l) y(l) \\ &\leq \delta_1 M_2 M \sum_{l=1}^T h(l) \|y\| \leq \|y\|. \end{aligned} \quad (3.10)$$

Now if we let

$$\Omega_1 = \{y \in E : \|y\| < C_1\}, \quad (3.11)$$

then

$$\|Sy\| \leq \|y\|, \quad \text{for } y \in K \cap \partial\Omega_1. \quad (3.12)$$

Next since $f_\infty = \infty$, there exists $\overline{C}_2 > 0$, such that $f(u) \geq \delta_2 u$, for $u \geq \overline{C}_2$, where $\delta_2 > 0$ satisfying

$$\delta_2 M_1 \sigma \sum_{l=n}^T G(n, l) h(l) \geq 1. \quad (3.13)$$

Let $C_2 = \max\{2C_1, \overline{C}_2/\sigma\}$ and $\Omega_2 = \{y \in E : \|y\| < C_2\}$, and let $y \in K$ and $\|y\| = C_2$, then

$$\min_{k \in \mathbb{N}_{n,T}} y(k) \geq \sigma \|y\| \geq \overline{C}_2. \quad (3.14)$$

Applying (2.4) and (3.13), one has

$$\begin{aligned} (Sy)(n) &= M_1 \sum_{l=1}^T G(n, l) h(l) f(y(l)) \geq M_1 \sum_{l=n}^T G(n, l) h(l) f(y(l)) \\ &\geq \delta_2 M_1 \sum_{l=n}^T G(n, l) h(l) y(l) \geq \delta_2 M_1 \sigma \sum_{l=n}^T G(n, l) h(l) \|y\| \\ &\geq \|y\|. \end{aligned} \quad (3.15)$$

Thus

$$\|Sy\| \geq \|y\|, \quad y \in K \cap \partial\Omega_2. \quad (3.16)$$

In view of (3.12) and (3.16), it follows from Theorem A that S has a fixed point $y \in K \cap (\overline{\Omega_2} \setminus \Omega_1)$ such that $C_1 \leq \|y\| \leq C_2$.

Case 2. Suppose f is sublinear case. Since $f_0 = \infty$, we may choose $C_3 > 0$ such that $f(u) \geq \delta_3 u$ for $0 < u \leq C_3$, where $\delta_3 > 0$ satisfying

$$\delta_3 M_1 \sigma \sum_{l=n}^T G(n, n) h(l) \geq 1, \quad (3.17)$$

$\Omega_3 = \{y \in E : \|y\| < C_3\}$; let $y \in K$ and $\|y\| = C_3$. Using (2.4) and (3.17), one has

$$\begin{aligned} (Sy)(n) &\geq M_1 \sum_{l=n}^T G(n, n) h(l) f(y(l)) \geq \delta_3 M_1 \sum_{l=n}^T G(n, n) h(l) y(l) \\ &\geq \delta_3 M_1 \sigma \sum_{l=n}^T G(n, n) h(l) \|y\| \geq \|y\|. \end{aligned} \quad (3.18)$$

Then $\|Sy\| \geq \|y\|$, $y \in K \cap \partial\Omega_3$.

In view of $f_\infty = 0$, there exists $\overline{C}_4 > 0$ such that $f(u) \leq \delta_4 u$ for $u \geq \overline{C}_4$, where $\delta_4 > 0$ satisfying

$$\delta_4 M_2 M \sum_{l=n}^T h(l) \leq 1. \quad (3.19)$$

There are two subcases to consider, that is, f is bounded and f is unbounded.

Subcase 2.1. Suppose f is bounded, that is, $f(y) \leq L$ for all $y \in [0, \infty)$ for some $L > 0$. Let

$$C_4 = \max \left\{ 2C_3, LM_2 M \sum_{l=1}^T h(l) \right\}. \quad (3.20)$$

Then, for $y \in K$ and $\|y\| = C_4$, one has

$$\begin{aligned} (Sy)(k) &\leq M_2 \sum_{l=1}^T G(k, k) h(l) f(y(l)) \leq LM_2 M \sum_{l=1}^T h(l) \\ &\leq C_4 = \|y\|. \end{aligned} \quad (3.21)$$

Hence $\|Sy\| \leq \|y\|$.

Subcase 2.2. Suppose f is unbounded, that is, there exists $C_4 > \max\{2C_3, \overline{C}_4/\sigma\}$ such that $f(u) \leq f(C_4)$ for all $0 < u \leq C_4$. Then, for $y \in K$ with $\|y\| = C_4$, from (2.5) and (3.19), we have

$$\begin{aligned} (Sy)(k) &\leq M_2 \sum_{l=1}^T G(k, l) h(l) f(y(l)) \leq M_2 M \sum_{l=1}^T h(l) f(C_4) \\ &\leq \delta_4 M_2 M \sum_{l=1}^T h(l) C_4 \leq C_4 = \|y\|. \end{aligned} \quad (3.22)$$

Thus in both Subcases 2.1 and 2.2, we may put $\Omega_4 = \{y \in E : \|y\| < C_4\}$. Then

$$\|Sy\| \leq \|y\|, \quad y \in K \cap \partial\Omega_4. \quad (3.23)$$

By using the fixed point Theorem A, it follows that problem (1.1) has at least one positive solution, such that $C_3 \leq \|y\| \leq C_4$. The proof is finished.

Finally, we give an example to demonstrate our main result.

Example 3.2. Consider the following three-point boundary value problem:

$$\begin{aligned} \Delta^2 y(k-1) + \frac{2}{(-k^2 + 10k + 33)^{1.5}} (y+20)^{1.5} &= 0, \quad k \in \mathbf{N}_{1,8}, \\ y(0) - \frac{13}{9} \Delta y(0) &= 0, \quad y(9) = \frac{22}{37} y(4), \end{aligned} \quad (3.24)$$

where $T = 8$, $n = 4$, $\alpha = 13/9$, $\beta = 22/37$, $T+1-\beta n + \alpha(1-\beta) = 800/111 > 0$, $T+1-\beta n = 245/37 > 0$, $h(k) = 2/(-k^2 + 10k + 33)^{1.5}$, $k \in \mathbf{N}_{1,8}$, $f(y) = (y+20)^{1.5}$, then f is superlinear. Conditions of Theorem 3.1 are all satisfied. Then problem (3.24) has at least one positive solution y . Indeed $y = -k^2 + 10k + 13$ is one such positive solution.

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