Research Article

On the Basic $k$-nacci Sequences in Finite Groups

Ömür Deveci$^{1,2}$ and Erdal Karaduman$^{1,2}$

$^1$ Department of Mathematics, Faculty of Science and Letters, Kafkas University, 36100 Kars, Turkey
$^2$ Department of Mathematics, Faculty of Science, Atatürk University, 25240 Erzurum, Turkey

Correspondence should be addressed to Ömür Deveci, odeveci36@yahoo.com.tr

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We define the basic $k$-nacci sequences and the basic periods of these sequences in finite groups, then we obtain the basic periods of the basic $k$-nacci sequences and the periods of the $k$-nacci sequences in symmetric group $S_4$, its subgroups, and binary polyhedral groups which related with these groups.

1. Introduction

The study of Fibonacci sequences in groups began with the earlier work of Wall [1], where the ordinary Fibonacci sequences in cyclic groups were investigated. In the mid-eighties, Wilcox extended the problem to Abelian groups [2]. The theory is expanded to some finite simple groups by Campbell et al. [3]. There, they defined the Fibonacci length of the Fibonacci orbit and the basic Fibonacci length of the basic Fibonacci orbit in a 2-generator group. The concept of Fibonacci length for more than two generators has also been considered; see, for example, [4, 5]. Also, the theory has been expanded to the nilpotent groups; see, for example, [6, 7]. Other works on Fibonacci length are discussed in, for example, [8–10]. Knox proved that the periods of $k$-nacci ($k$-step Fibonacci) sequences in dihedral groups were equal to $2k + 2$ [11]. Deveci, Karaduman, and Campbell examined the period of the $k$-nacci sequences in some finite binary polyhedral groups in [12]. Recently, $k$-nacci sequences have been investigated; see, for example, [13, 14].

This paper defines the basic $k$-nacci sequences and the periods of these sequences in finite groups and discusses the basic periods of the basic $k$-nacci sequences and the periods of the $k$-nacci sequences in the symmetric group $S_4$, alternating group $A_4$, $D_2$ four-group, and binary polyhedral groups $(2,3,4)$ and $(2,3,3)$ with related $S_4$ and $A_4$, respectively. We
consider the groups $S_4, A_4, (2,3,4), \text{ and } (2,3,3)$ both as 2-generator and as 3-generator groups.

A $k$-nacci sequence in a finite group is a sequence of group elements $x_0, x_1, x_2, \ldots, x_n, \ldots$ for which, given an initial (seed) set $x_0, x_1, x_2, \ldots, x_{j-1}$, each element is defined by

$$x_n = \begin{cases} x_0x_1\cdots x_{n-1} & \text{for } j \leq n < k, \\ x_{n-k}x_{n-k+1}\cdots x_{n-1} & \text{for } n \geq k. \end{cases} \quad (1.1)$$

We also require that the initial elements of the sequence $x_0, x_1, x_2, \ldots, x_{j-1}$ generate the group, thus forcing the $k$-nacci sequence to reflect the structure of the group. The $k$-nacci sequence of a group $G$ generated by $x_0, x_1, x_2, \ldots, x_{j-1}$ is denoted by $F_k(G; x_0, x_1, \ldots, x_{j-1})$ [11].

A sequence of group elements is periodic if, after a certain point, it consists only of repetitions of a fixed subsequence. The number of elements in the repeating subsequence is called the period of the sequence. For example, the sequence $a, b, c, d, e, b, c, d, e, b, c, d, e, \ldots$ is periodic after the initial element $a$ and has period 4. A sequence of group elements is simply periodic with period $k$ if the first $k$ elements in the sequence form a repeating subsequence. For example, the sequence $a, b, c, d, e, f, a, b, c, d, e, f, a, b, c, d, e, \ldots$ is simply periodic with period 6. In [11], Knox had denoted the period of a $k$-nacci sequence $F_k(G; x_0, x_1, \ldots, x_{j-1})$ by $P_k(G; x_0, x_1, \ldots, x_{j-1})$.

**Definition 1.1.** For a finitely generated group $G = \langle A \rangle$, where $A = \{a_1, a_2, \ldots, a_n\}$, the sequence $x_i = a_{i+1}, 0 \leq i \leq n-1, x_{i+n} = \prod_{j=1}^{i} x_{i+j-1}, i \geq 0$ is called the Fibonacci orbit of $G$ with respect to the generating set $A$, denoted as $F_A(G)$ [4].

**Definition 1.2.** If $F_A(G)$ is simply periodic, then the period of the sequence is called the Fibonacci length of $G$ with respect to generating set $A$, written, $L_{EN_A}(G)$ [4].

Notice that the orbit of a $k$-generated group is a $k$-nacci sequence.

Let $G$ be a finite $j$-generator group, and let $X$ be the subset of $G \times G \times G \cdots \times G$ such that $(x_0, x_1, \ldots, x_{j-1}) \in X$ if and only if $G$ is generated by $x_0, x_1, \ldots, x_{j-1}$. We call $(x_0, x_1, \ldots, x_{j-1})$ a generating $j$-tuple for $G$.

### 2. Basic Period of Basic $k$-nacci Sequence

To examine the concept more fully, we study the action of automorphism group $\text{Aut}G$ of $G$ on $X$ and on the $k$-nacci sequences $F_k(G; x_0, x_1, \ldots, x_{j-1}), (x_0, x_1, \ldots, x_{j-1}) \in X$. Now, $\text{Aut}G$ consists of all isomorphism $\theta : G \rightarrow G$ and if $\theta \in \text{Aut}G$ and $(x_0, x_1, \ldots, x_{j-1}) \in X$, then $(x_0\theta, x_1\theta, \ldots, x_{j-1}\theta) \in X$.

For a subset $A \subseteq G$ and $\theta \in \text{Aut}G$, the image of $A$ under $\theta$ is

$$A\theta = \{a\theta : a \in A\}. \quad (2.1)$$

**Definition 2.1.** For a generating pair $(x, y) \in X$, the basic Fibonacci orbit $\overline{F}_{x,y}$ of the basic length $m$ is defined by the sequence $\{b_i\}$ of elements of $G$ such that

$$b_0 = x, \quad b_1 = y, \quad b_{i+2} = b_ib_{i+1}, \quad i \geq 0, \quad (2.2)$$
where \( m \geq 1 \) is the least integer with

\[
    b_0 = b_m \theta, \quad b_1 = b_{m+1} \theta, \quad (2.3)
\]

for some \( \theta \in \text{Aut}G \). Since \( b_m, b_{m+1} \) generate \( G \), it follows that \( \theta \) is uniquely determined. For more information, see [3].

**Lemma 2.2.** Let \((x_0, x_1, \ldots, x_{j-1}) \in X \) and let \( \theta \in \text{Aut}G \), then \((F_k(G : x_0, x_1, \ldots, x_{j-1})) \theta = F_k(G : x_0 \theta, x_1 \theta, \ldots, x_{j-1} \theta)\).

**Proof.** Let \( F_k(G : x_0, x_1, \ldots, x_{j-1}) = \{b_i\} \). The result is obvious since \( \{b_i \} \theta = \{b_i \theta \} \) and

\[
    b_{i+k} \theta = (b_1 b_{i+1} \cdots b_{i+k-1}) \theta = b_i \theta b_{i+1} \theta \cdots b_{i+k-1} \theta. \quad (2.4)
\]

Each generating \( j \)-tuple \((x_0, x_1, \ldots, x_{j-1}) \in X \) maps to \(|\text{Aut}G|\) distinct elements of \( X \) under the action of elements of \( \text{Aut}G \). Hence, there are

\[
    d_j(G) = |X|/|\text{Aut}G|, \quad (2.5)
\]

(where \(|X|\) means the number of elements of \( X \)) nonisomorphic generating \( j \)-tuples for \( G \). The notation \( d_j(G) \) was introduced in [15].

Suppose that \( \omega \) elements of \( \text{Aut}G \) map \( F_k(G : x_0, x_1, \ldots, x_{j-1}) \) into itself, then there are \(|\text{Aut}G|/\omega\) distinct \( k \)-nacci sequences \( F_k(G : x_0 \theta, x_1 \theta, \ldots, x_{j-1} \theta) \) for \( \theta \in \text{Aut}G \).

**Definition 2.3.** For a \( j \)-tuple \((x_0, x_1, \ldots, x_{j-1}) \in X \), the basic \( k \)-nacci sequence \( F_k(G : x_0, x_1, \ldots, x_{j-1}) \) of the basic period \( m \) is a sequence of group elements \( b_0, b_1, b_2, \ldots, b_n, \ldots \) for which, given an initial (seed) set \( b_0 = x_0, \ b_1 = x_1, \ b_2 = x_2, \ldots, b_{j-1} = x_{j-1} \), each element is defined by

\[
    b_n = \begin{cases} 
    b_0 b_1 \cdots b_{n-1} & \text{for } j \leq n < k, \\
    b_{n-k} b_{n-k+1} \cdots b_{n-1} & \text{for } n \geq k,
    \end{cases} \quad (2.6)
\]

where \( m \geq 1 \) is the least integer with

\[
    b_0 = b_m \theta, \ b_1 = b_{m+1} \theta, \ b_2 = b_{m+2} \theta, \ldots, \ b_{k-1} = b_{m+k-1} \theta, \quad (2.7)
\]

for some \( \theta \in \text{Aut}G \). Since \( G \) is a finite \( j \)-generator group and \( b_m, b_{m+1}, \ldots, b_{m+j-1} \) generate \( G \), it follows that \( \theta \) is uniquely determined. The basic \( k \)-nacci sequence \( F_k(G : x_0, x_1, \ldots, x_{j-1}) \) is finite containing \( m \) element.

In this paper, we denote the basic period of the basic \( k \)-nacci sequence \( F_k(G : x_0, x_1, \ldots, x_{j-1}) \) by \( BP_k(G, x_0, x_1, \ldots, x_{j-1}) \).

From the definitions, it is clear that the periods of the \( k \)-nacci sequences and the basic \( k \)-nacci sequences in a finite group depend on the chosen generating set and the order of the generating elements.
Definition 3.1. The polyhedral group $(l, m, n)$ for $l, m, n > 1$ is defined by the presentation

$$\langle x, y, z : x^l = y^m = z^n = xyz = e \rangle,$$

or

$$\langle x, y : x^l = y^m = (xy)^n = e \rangle.$$

The polyhedral group $(l, m, n)$ is finite if and only if the number

$$\mu = lmn\left(\frac{1}{l} + \frac{1}{m} + \frac{1}{n} - 1\right) = mn + nl + lm - lmn$$

is positive, that is, in the cases $(2, 2, n), (2, 3, 3), (2, 3, 4), \text{ and } (2, 3, 5).$ Its order is $2lmn/\mu.$ $A_4, S_4,$ and $A_5$ are the groups $(2, 3, 3), (2, 3, 4),$ and $(2, 3, 5),$ respectively. Also, the groups $A_4, S_4,$ and $A_5$ being isomorphic to the groups of rotations of the regular tetrahedron, octahedron, and icosahedron. Using Tietze transformations, we may show that $(l, m, n) \equiv (m, n, l) \equiv (n, l, m).$ For more information on these groups, see [16, 17, pp. 67-68].

Definition 3.2. The binary polyhedral group $(l, m, n)$ for $l, m, n > 1,$ is defined by the presentation

$$\langle x, y, z : x^l = y^m = z^n = xyz \rangle,$$

or

$$\langle x, y : x^l = y^m = (xy)^n \rangle.$$

The binary polyhedral group $(l, m, n)$ is finite if and only if the number $k = lmn(1/l + 1/m + 1/n - 1) = mn + nl + lm - lmn$ is positive. Its order is $4lmn/k.$
For more information on these groups, see [17, pp. 68–71].

**Definition 3.3.** Let \( f_n^{(k)} \) denote the \( n \)th member of the \( k \)-step Fibonacci sequence defined as

\[
 f_n^{(k)} = \sum_{j=1}^{k} f_{n-j}^{(k)} \quad \text{for } n > k, \tag{3.6}
\]

with boundary conditions \( f_i^{(k)} = 0 \) for \( 1 \leq i < k \) and \( f_k^{(k)} = 1 \). Reducing this sequence by a modulo \( m \), we can get a repeating sequence, which we denote by

\[
 f(k, m) = \left( f_1^{(k,m)}, f_2^{(k,m)}, \ldots, f_n^{(k,m)} \right), \tag{3.7}
\]

where \( f_i^{(k,m)} = f_i^{(k)} \pmod{m} \). We then have that \( (f_1^{(k,m)}, f_2^{(k,m)}, \ldots, f_k^{(k,m)}) = (0, 0, \ldots, 1) \), and it has the same recurrence relation as in (3.6) [18].

**Theorem 3.4** (\( f(k, m) \) is a periodic sequence [18]). Let \( h_k(m) \) denote the smallest period of \( f(k, m) \), called the period of \( f(k, m) \) or the wall number of the \( k \)-step Fibonacci sequence modulo \( m \).

**Theorem 3.5.** The periods of the \( k \)-nacci sequences and the basic periods of the basic \( k \)-nacci sequences in the group \( S_4 \) are as follows.

1. If the group is defined by the presentation \( S_4 = \langle x, y, z : x^2 = y^3 = z^4 = xyz = e \rangle \), then
   
   (i) if \( k = 2 \), \( P_2(S_4; y, z, x) = 18 \) and \( BP_2(S_4; y, z, x) = 9 \),
   
   (ii) if \( k > 2 \), \( P_k(S_4; x, y, z) = 6k + 6 \) and \( BP_k(S_4; x, y, z) = 3k + 3 \).

If \( S_4 \) has the presentation \( S_4 = \langle x, y : x^2 = y^3 = (xy)^4 = e \rangle \), then

1. (i') if \( k = 2 \), \( P_2(S_4; x, y) = 18 \) and \( BP_2(S_4; x, y) = 9 \),
   
   (ii') if \( k > 2 \), \( P_k(S_4; x, y) = 6k + 6 \) and \( BP_k(S_4; x, y) = 3k + 3 \).

**Proof.** Firstly, let us consider the 3-generator case. We first note that \( |x| = 2, |y| = 3, \) and \( |z| = 4 \) (where \( |x| \) means the order of \( x \)).

(i) If \( k = 2 \), we have the sequence for the generating triple \( (y, z, x) \),

\[
 y, z, x, y^2, xy^2, y^2xy^2, z^2y, z^2yz^3y, yxy, xyx, \\
 xy^2, xy^2x, y^2x, yxy, yxz, zy, y^2xy^2, y, z, x, \ldots, \tag{3.8}
\]

which has period 18 and the basic period 9 since \( x\theta = x, y\theta = yxy, \) and \( z\theta = xy^2 \), where \( \theta \) is the inner automorphism induced by conjugation by \( x \).

(ii) If \( k = 3 \), we have the sequence for the generating triple \( (x, y, z) \),

\[
 x, y, z, e, x, y^2, xy^2, xzxy^2, x, y, yxy^2, xzy^2, x, \\
 y^2, yx, e, x, y, x, z^2, x, y^2, zy, z^2, x, y, z \ldots, \tag{3.9}
\]
which has period 24 and the basic period 12 since \( x\theta = x, \ y\theta = y^2, \) and \( z\theta = yx \) where \( \theta \) is an outer automorphism of order 2.

If \( k \geq 4 \), the first \( k \) elements of sequence for the generating triple \((x, y, z)\) are

\[
x_0 = x, \ x_1 = y, \ x_2 = z, \ x_3 = xyz, \ x_4 = (xyz)^2, \ldots, \ x_{k-1} = (xyz)^{2^{k-4}}.
\] (3.10)

Thus, using the above information, sequence reduces to

\[
x_0 = x, \ x_1 = y, \ x_2 = z, \ x_3 = e, \ldots, e, \ x_{k-1} = e, \quad (3.11)
\]

where \( x_j = e \) for \( 3 \leq j \leq k - 1 \). Thus,

\[
x_k = e, \ x_{k+1} = x, \ x_{k+2} = y^2, \ x_{k+3} = xy^2, \ x_{k+4} = xyz^2, \ 
\]

\[
x_{k+5} = e, \ldots, e, x_{2k+1} = e, \ x_{2k+2} = x, x_{2k+3} = y, \ 
\]

\[
x_{2k+4} = yxy^2, \ x_{2k+5} = xzxy^2, \ x_{2k+6} = e, \ldots, e, x_{3k+2} = e, \ 
\]

\[
x_{3k+3} = x, x_{3k+4} = y^2, x_{3k+5} = xy, x_{3k+6} = e, \ldots, e, x_{4k+3} = e, \ 
\]

\[
x_{4k+4} = x, x_{4k+5} = y, x_{4k+6} = xy, x_{4k+7} = z^2, \ 
\]

\[
x_{4k+8} = e, \ldots, e, x_{5k+4} = e, x_{5k+5} = x, x_{5k+6} = y^2, \ 
\]

\[
x_{5k+7} = zy, x_{5k+8} = z^2, x_{5k+9} = e, \ldots, e, x_{6k+5} = e, \quad (3.12)
\]

where \( x_j = e \) for \( k + 5 \leq j \leq 2k + 1, 2k + 6 \leq j \leq 3k + 2, 3k + 6 \leq j \leq 4k + 3, 4k + 8 \leq j \leq 5k + 4, \) and \( 5k + 9 \leq j \leq 6k + 5 \).

We also have

\[
x_{6k+6} = \prod_{i=5k+6}^{6k+5} x_i = x, \quad x_{6k+7} = \prod_{i=5k+7}^{6k+6} x_i = y, \quad x_{6k+8} = \prod_{i=5k+8}^{6k+7} x_i = z. \quad (3.13)
\]

Since the elements succeeding \( x_{6k+6}, x_{6k+7}, \) and \( x_{6k+8} \) depend on \( x, y, \) and \( z \) for their values, the cycle begins again with the \( 6k+6^{th} \) element, that is, \( x_0 = x_{6k+6}, \ x_1 = x_{6k+7}, \ x_2 = x_{6k+8}, \ldots \)

Thus, \( P_k(S_4; x, y, z) = 6k + 6 \).

It is easy to see from the above sequence that

\[
x_{3k+3} = x, \ x_{3k+4} = y^2, \ x_{3k+5} = xy, \ x_{3k+6} = e, \ldots, e, \ x_{4k+2} = e. \quad (3.14)
\]

\( BP_k(S_4; x, y, z) = 3k + 3 \) since \( x\theta = x, \ y\theta = y^2, \) and \( z\theta = yx \) where \( \theta \) is an outer automorphism of order 2.

Secondly, let us consider the 2-generator case. We first note that \(|x| = 2, |y| = 3, \) and \(|xy| = 4\).

(i') If \( k = 2, P_2(S_4, x, y) = 18 \) and \( BP_2(S_4; x, y) = 9 \) since \( x\theta = x \) and \( y\theta = yxy \) where \( \theta \) is the inner automorphism induced by conjugation by \( x \).
The proofs are similar to above and are omitted.

\[\text{Theorem 3.6.} \quad \text{The periods of the } k\text{-nacci sequences and the basic periods of the basic } k\text{-nacci sequences in the binary polyhedral group } (2, 3, 4) \text{ are as follows.}\]

\[
\begin{align*}
\text{If the group is defined by the presentation } (2, 3, 4) &= \langle x, y, z : x^2 = y^3 = z^4 = xyz \rangle, \text{ then} \\
\text{(i) if } k &= 2, P_k((2, 3, 4); y, z, x) = 18 \text{ and } BP_k((2, 3, 4); y, z, x) = 9, \\
\text{(ii) if } k &= 2, P_k((2, 3, 4); x, y, z) = 6k + 6 \text{ and } BP_k((2, 3, 4); x, y, z) = 6k + 6. \\
\text{If the group is defined by the presentation } (2, 3, 4) &= \langle x, y : x^2 = y^3 = (xy)^4 \rangle, \text{ then} \\
\text{(i') if } k &= 2, P_k((2, 3, 4); x, y) = 18 \text{ and } BP_k((2, 3, 4); x, y) = 9, \\
\text{(ii') if } k &= 2, P_k((2, 3, 4); x, y) = 6k + 6 \text{ and } BP_k((2, 3, 4); x, y) = 6k + 6.
\end{align*}
\]

\[\text{Proof.} \quad \text{Firstly, let us consider the 2-generator case. We first note that } |x| = 4, |y| = 6, \text{ and } |xy| = 8.
\]

\[\begin{align*}
\text{(i') If } k &= 2, \text{ we have the sequence for the generating pair } (x, y), \\
&= x, y, xy, yxy, xyxy, yxxy, yxy^2, xy^2x, xy^2y, y^2x, y, x, y^3, yx^3, yx^2y, y^2x, y^2y, y^4x, y, x, y, \ldots,
\end{align*}
\]

which has period 18 and the basic period 9 since \(x^\theta = x^3\) and \(y^\theta = x^3y\) where \(\theta\) is an outer automorphism of order 2.

\[\begin{align*}
\text{(ii') If } k &= 3, \text{ we have the sequence for the generating pair } (x, y), \\
&= x, y, xy, (xy)^2, x, y^2, y^5xy, (xy)^2, x, y, (xy)^3, (xy)^4, x^3, \\
y^2, xy^2, (yx)^2, x^3, y, xy^2, (yx)^2, x^3, y^2, y^4x, e, x, y, xy, \ldots,
\end{align*}
\]

which has period 24 and the basic period 24 since \(x^\theta = x\) and \(y^\theta = y\) where \(\theta\) is an inner automorphism induced by conjugation by \(x^2\).

If \(k = 4\), we have the sequence for the generating pair \((x, y)\),

\[\begin{align*}
&= x, y, xy, (xy)^2, (xy)^4, x^3, y^2, y^5xy, (xy)^2, e, x, \\
y, (xy)^3, (xy)^4, e, x^3, y^2, xy^2, (yx)^2, x^2, x, y, \\
yxy^2, (yx)^2, e, x^3, y^2, y^4x, e, e, x, y, xy, (xy)^2, \ldots,
\end{align*}
\]

which has period 30 and the basic period 30 since \(x^\theta = x\) and \(y^\theta = y\) where \(\theta\) is an inner automorphism induced by conjugation by \(x^2\).
If \( k \geq 5 \), the first \( k \) elements of sequence for the generating pair \((x, y)\) are

\[
x_0 = x, \quad x_1 = y, \quad x_2 = xy, \quad x_3 = (xy)^2, \quad x_4 = (xy)^4, \quad x_5 = (xy)^8, \ldots, \quad x_{k-1} = (xy)^{2^{k-3}}.
\] (3.18)

Thus, using the above information, sequence reduces to

\[
x_0 = x, \quad x_1 = y, \quad x_2 = xy, \quad x_3 = (xy)^2, \quad x_4 = (xy)^4, \quad x_5 = e, \ldots, \quad e, \quad x_{k-1} = e,
\] (3.19)

where \( x_j = e \) for \( 5 \leq j \leq k - 1 \). Thus,

\[
x_k = e, \quad x_{k+1} = x^3, \quad x_{k+2} = y^2, \quad x_{k+3} = y^3xy,
\]

\[
x_{k+4} = (xy)^2, \quad x_{k+5} = e, \ldots, \quad e, \quad x_{2k+1} = e, \quad x_{2k+2} = x,
\]

\[
x_{2k+3} = y, \quad x_{2k+4} = (xy)^3, \quad x_{2k+5} = (xy)^4, \quad x_{2k+6} = e, \ldots, \quad e,
\]

\[
x_{3k+2} = e, \quad x_{3k+3} = x^3, \quad x_{3k+4} = y^2, \quad x_{3k+5} = xy^2,
\]

\[
x_{3k+6} = (yx)^2, \quad x_{3k+7} = x^2, \quad x_{3k+8} = e, \ldots, \quad e, \quad x_{4k+3} = e,
\]

\[
x_{4k+4} = x, \quad x_{4k+5} = y, \quad x_{4k+6} = yxy^2x_{4k+7} = (yx)^2,
\]

\[
x_{4k+8} = e, \ldots, \quad e, \quad x_{5k+4} = e, \quad x_{5k+5} = x^3, \quad x_{5k+6} = y^2,
\]

\[
x_{5k+7} = y^4x, \quad x_{5k+8} = e, \ldots, \quad e, \quad x_{6k+5} = e,
\]

where \( x_j = e \) for \( k + 5 \leq j \leq 2k + 1, 2k + 6 \leq j \leq 3k + 2, 3k + 8 \leq j \leq 4k + 3, 4k + 8 \leq j \leq 5k + 4, \) and \( 5k + 8 \leq j \leq 6k + 5 \).

We also have

\[
x_{6k+6} = \prod_{i=3k+6}^{6k+5} x_i = x, \quad x_{6k+7} = \prod_{i=5k+7}^{6k+6} x_i = y.
\] (3.22)

Since the elements succeeding \( x_{6k+6}, x_{6k+7} \) depend on \( x \) and \( y \) for their values, the cycle begins again with the \( 6k + 6 \)th element, that is, \( x_0 = x_{6k+6}, \quad x_1 = x_{6k+7}, \ldots \) Thus, \( P_k((2, 3, 4); x, y) = 6k + 6 \) and \( BP_k((2, 3, 4); x, y) = 6k + 6 \) since \( x^\theta = x \) and \( y^\theta = y \) where \( \theta \) is an inner automorphism induced by conjugation by \( x^2 \).

Secondly, let us consider the 3-generator case. We first note that \( |x| = 4, \ |y| = 6, \) and \( |z| = 8 \).

(i) If \( k = 2, P_2((2, 3, 4); y, z, x) = 18 \) and \( BP_2((2, 3, 4); y, z, x) = 9 \) since \( x^\theta = x^3, \ y^\theta = x^3y^x, \) and \( z^\theta = xy^2 \) where \( \theta \) is an outer automorphism of order 2.

(ii) If \( k > 2, P_k((2, 3, 4); x, y, z) = 6k + 6 \) and \( BP_k((2, 3, 4); x, y, z) = 6k + 6 \) since \( x^\theta = x \) and \( y^\theta = y \) where \( \theta \) is an inner automorphism induced by conjugation by \( x^2 \).

The proofs are similar to the proofs of Theorems 3.5(i) and 3.5(ii) and are omitted. \( \square \)
Theorem 3.7. The periods of the k-nacci sequences and the basic periods of the basic k-nacci sequences in the group $A_4$ are as follows.

If the group is defined by the presentation $A_4 = \langle x, y, z : x^2 = y^3 = z^3 = xyz = e \rangle$, then

(i) if $k = 2$, $P_2(A_4; y, z, x) = 16$ and $BP_2(A_4; y, z, x) = 4$,

(ii) if $k > 2$,

$$P_k(A_4; x, y, z) = \begin{cases} 
3BP_k(A_4; x, y, z), & k \equiv 0 \mod 4, \\
2BP_k(A_4; x, y, z), & k \equiv 2 \mod 4, \\
2BP_k(A_4; x, y, z), & \text{otherwise,}
\end{cases}$$

$$BP_k(A_4; x, y, z) = \begin{cases} 
3BP_k(A_4; x, y, z), & k \equiv 0 \mod 4, \\
2BP_k(A_4; x, y, z), & k \equiv 2 \mod 4, \\
3BP_k(A_4; x, y, z), & \text{otherwise,}
\end{cases}$$

(3.23)

where $u_1, u_2, u_3 \in \mathbb{N}$, and $h_k(3)$ denote the wall number of the k-step Fibonacci sequence modulo 3.

If the group is defined by the presentation $A_4 = \langle x, y : x^2 = y^3 = (xy)^3 = e \rangle$, then

(i') if $k = 2$, $P_2(A_4; x, y) = 16$ and $BP_2(A_4; x, y) = 4$,

(ii') if $k > 2$,

$$P_k(A_4; x, y) = \begin{cases} 
3BP_k(A_4; x, y), & k \equiv 0 \mod 4, \\
2BP_k(A_4; x, y), & k \equiv 2 \mod 4, \\
2BP_k(A_4; x, y), & \text{otherwise,}
\end{cases}$$

$$BP_k(A_4; x, y) = \begin{cases} 
u_1h_k(3), & k \equiv 0 \mod 4, \\
u_2h_k(3), & k \equiv 2 \mod 4, \\
u_3h_k(3), & \text{otherwise,}
\end{cases}$$

(3.24)

where $u_1, u_2, u_3 \in \mathbb{N}$.

Proof. Firstly, let us consider the 2-generator case. We process as similar to the proof of Theorem 3.6 We first note that $|x| = 2$, $|y| = 3$, and $|xy| = 3$.

(i') If $k = 2$, we have the sequence for the generating pair $(x, y)$,

$$x, y, xy, yxy, xxy, (xy)^2, xy^2, y, x,$$

$$yx, xyx, y^2x, xxy^2, yxy, y^2, yx, x, y, \ldots,$$ (3.25)
which has period 16 and the basic period 4 since \( x\theta = yxy^2 \) and \( y\theta = yxy \) where \( \theta \) is an outer automorphism of order 4.

(ii’) If \( k > 2 \),

let \( k \) be even, then the first \( k \) elements of sequence for the generating pair \((x, y)\) are

\[
x_0 = x, \ x_1 = y, \ x_2 = xy, \ x_3 = (xy)^2, \ x_4 = xy, \ x_5 = (xy)^2 \ldots, \ x_{k-2} = xy, \ x_{k-1} = (xy)^2.
\]

(3.26)

If \( k \equiv 0 \mod 4 \),

\[
x_{u_1h_4(3)-(k-2)} = e, \ x_{u_1h_4(3)-(k-1)} = e, \ldots, \ e,
\]

\[
x_{u_1h_4(3)-1} = e, \ x_{u_1h_4(3)} = y^2xy, \ x_{u_1h_4(3)+1} = yx, \ldots.
\]

(3.27)

\( P_k(A_4; x, y) = 3BP_k(A_4; x, y) \) and \( BP_k(A_4; x, y) = u_1h_k(3) \) since \( x\theta = yxy^2 \) and \( y\theta = xyx \) where \( \theta \) is an outer automorphism of order 3.

If \( k \equiv 2 \mod 4 \),

\[
x_{u_2h_4(3)-(k-2)} = e, \ x_{u_2h_4(3)-(k-1)} = e, \ldots, \ e,
\]

\[
x_{u_2h_4(3)-1} = e, \ x_{u_2h_4(3)} = x, \ x_{u_2h_4(3)+1} = xy, \ldots.
\]

(3.28)

\( P_k(A_4; x, y) = 2BP_k(A_4; x, y) \) and \( BP_k(A_4; x, y) = u_2h_k(3) \) since \( x\theta = x \) and \( y\theta = yx \) where \( \theta \) is an outer automorphism of order 2.

Let \( k \) be odd, then the first \( k \) elements of sequence are for the generating pair \((x, y)\),

\[
x_0 = x, \ x_1 = y, \ x_2 = xy, \ x_3 = (xy)^2, \ x_4 = xy, \ x_5 = (xy)^2 \ldots, \ x_{k-2} = (xy)^2, \ x_{k-1} = xy.
\]

(3.29)

Also,

\[
x_{u_3h_4(3)-(k-2)} = e, \ x_{u_3h_4(3)-(k-1)} = e, \ldots, \ e,
\]

\[
x_{u_3h_4(3)-1} = e, \ x_{u_3h_4(3)} = x, \ x_{u_3h_4(3)+1} = yx, \ldots.
\]

(3.30)

\( P_k(A_4; x, y) = 2BP_k(A_4; x, y) \) and \( BP_k(A_4; x, y) = u_3h_k(3) \) since \( x\theta = x \) and \( y\theta = xy \) where \( \theta \) is an outer automorphism of order 2.

Secondly, let us consider the 3-generator case. We first note that \(|x| = 2\), \(|y| = 3\), and \(|z| = 3\).

(i) If \( k = 2 \), \( P_2(A_4; y, z, x) = 16 \) and \( BP_2(A_4; y, z, x) = 4 \) since \( x\theta = y^2xy \), \( y\theta = yxy \), and \( z\theta = yx \) where \( \theta \) is an outer automorphism of order 4.

(ii) If \( k > 2 \),

let \( k \equiv 0 \mod 4 \), then \( P_k(A_4; x, y, z) = 3BP_k(A_4; x, y, z) \) and \( BP_k(A_4; x, y, z) = u_1h_k(3) \) since \( x\theta = y^2xy \), \( y\theta = yxy \), and \( z\theta = zx \) where \( \theta \) is an outer automorphism of order 4.
automorphism of order 3; let \( k \equiv 2 \mod 4 \), then \( P_k(A_4; x, y, z) = 2BP_k(A_4; x, y, z) \) and \( BP_k(A_4; x, y, z) = u_3h_k(3) \) since \( x^3 = x, y^2 = xy \), and \( z^2 = zy^2 \). where \( \theta \) is an outer automorphism of order 2; let \( k \) be odd; then \( P_k(A_4; x, y, z) = 2BP_k(A_4; x, y, z) \) and \( BP_k(A_4; x, y, z) = u_3h_k(3) \) since \( x^3 = x, y^2 = xy \), and \( z^2 = zy^2 \). where \( \theta \) is an outer automorphism of order 2.

The proofs are similar to the proofs of Theorems 3.5(i) and 3.5(ii) and are omitted.

**Theorem 3.8.** The periods of the \( k \)-nacci sequences and the basic periods of the basic \( k \)-nacci sequences in the binary polyhedral group \( (2, 3, 3) \) are as follows.

If the group is defined by the presentation \( (2, 3, 3) = \langle x, y, z : x^2 = y^3 = z^3 = xyz \rangle \), then

(i) if \( k = 2 \), \( P_2((2, 3, 3); y, z, x) = 48 \) and \( BP_2((2, 3, 3); y, z, x) = 12 \),

(ii) if \( k > 2 \),

\[
P_k((2, 3, 3); x, y, z) = \begin{cases} 
3BP_k((2, 3, 3); x, y, z), & k \equiv 0 \mod 4, \\
BP_k((2, 3, 3); x, y, z), & k \not\equiv 0 \mod 4,
\end{cases}
\]

(3.31)

\[
BP_k((2, 3, 3); x, y, z) = \begin{cases} 
v_1h_k(6), & k \equiv 0 \mod 4, \\
v_2h_k(6), & k \not\equiv 0 \mod 4,
\end{cases}
\]

(3.32)

where \( v_1, v_2 \in \mathbb{N} \), and \( h_k(6) \) denote the wall number of the \( k \)-step Fibonacci sequence modulo 6.

If the group is defined by the presentation \( (2, 3, 3) = \langle x, y : x^2 = y^3 = (xy)^3 \rangle \), then

(i') if \( k = 2 \), \( P_2((2, 3, 3); x, y) = 48 \) and \( BP_2((2, 3, 3); x, y) = 12 \),

(ii') if \( k > 2 \),

\[
P_k((2, 3, 3); x, y) = \begin{cases} 
3BP_k((2, 3, 3); x, y), & k \equiv 0 \mod 4, \\
BP_k((2, 3, 3); x, y), & k \not\equiv 0 \mod 4,
\end{cases}
\]

(3.33)

\[
BP_k((2, 3, 3); x, y) = \begin{cases} 
v_1h_k(6), & k \equiv 0 \mod 4, \\
v_2h_k(6), & k \not\equiv 0 \mod 4,
\end{cases}
\]

(3.34)

where \( v_1, v_2 \in \mathbb{N} \).

**Proof.** Firstly, let us consider the 3-generator case. We first note that \( |x| = 4, |y| = 6, \) and \( |z| = 6 \).

(i) If \( k = 2 \), \( P_2((2, 3, 3); y, z, x) = 48 \) and \( BP_2((2, 3, 3); y, z, x) = 12 \) since \( x^2 = y^3 \), \( y^2 = xy, \) \( y^3 = y^2xy \), and \( z^2 = zy^2 \). where \( \theta \) is an outer automorphism of order 4.
(ii) If $k > 2$,

let $k \equiv 0 \mod 4$, then $P_k((2,3,3);x,y,z) = 3BP_k((2,3,3);x,y,z)$ and $BP_k((2,3,3);x,y,z) = v_1h_k(6)$ since $x\theta = yxy^2$, $y\theta = z^3xy$, and $z\theta = xy^2x$ where $\theta$ is an inner automorphism induced by conjugation by $z^3yx$;

let $k \not\equiv 0 \mod 4$, then $P_k((2,3,3);x,y,z) = BP_k((2,3,3);x,y,z)$ and $BP_k((2,3,3);x,y,z) = v_2h_k(6)$ since $x\theta = x$, $y\theta = y$, and $z\theta = z$ where $\theta$ is an inner automorphism induced by conjugation by $x^2$.

The proofs are similar to the proofs of Theorems 3.5.(i) and 3.5.(ii) and are omitted.

Secondly, let us consider the 2-generator case. We first note that $|x| = 4$, $|y| = 6$, and $|xy| = 6$.

(i') If $k = 2$, $P_2((2,3,3);x,y) = 48$ and $BP_2((2,3,3);x,y) = 12$ since $x\theta = yxy^2$ and $y\theta = y^2x$ where $\theta$ is an outer automorphism of order 4.

(ii') If $k > 2$,

let $k \equiv 0 \mod 4$, then $P_k((2,3,3);x,y) = 3BP_k((2,3,3);x,y)$ and $BP_k((2,3,3);x,y) = v_1h_k(6)$ since $x\theta = yxy^2$, $y\theta = yx$, and $z\theta = xy^2x$ where $\theta$ is an inner automorphism induced by conjugation by $y^2x$;

let $k \not\equiv 0 \mod 4$, then $P_k((2,3,3);x,y) = BP_k((2,3,3);x,y)$ and $BP_k((2,3,3);x,y) = v_2h_k(6)$ since $x\theta = x$ and $y\theta = y$ where $\theta$ is an inner automorphism induced by conjugation by $x^2$.

The proofs are similar to the proofs of Theorem 3.6.(i') and Theorem 3.6.(ii') and are omitted.

\[ \text{Theorem 3.9.} \quad \text{The periods of the k-nacci sequences are } k + 1, \text{ and the basic period of the basic k-nacci sequences is } k + 1 \text{ in } D_2 \text{ four-group.} \]

Proof. We have the presentation $D_2 = \langle x, y : x^2 = y^2 = e, xy = yx \rangle$. $P_k(D_2; x, y) = k + 1$; see [14] for a proof and $BP_k(D_2; x, y) = k + 1$ since $x\theta = x$ and $y\theta = y$ where $\theta$ is an inner automorphism induced by conjugation by $x$.

\[ \square \]

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\section*{References}


