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Research Article

Almost Periodic Functions on Time Scales and Applications

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We first propose the concept of almost periodic time scales and then give the definition of almost periodic functions on almost periodic time scales, then by using the theory of calculus on time scales and some mathematical methods, some basic results about almost periodic differential equations on almost periodic time scales are established. Based on these results, a class of high-order Hopfield neural networks with variable delays are studied on almost periodic time scales, and some sufficient conditions are established for the existence and global asymptotic stability of the almost periodic solution. Finally, two examples and numerical simulations are presented to illustrate the feasibility and effectiveness of the results.

1. Introduction

It is well known that in celestial mechanics, almost periodic solutions and stable solutions to differential equations or difference equations are intimately related. In the same way, stable electronic circuits, ecological systems, neural networks, and so forth exhibit almost periodic behavior. A vast amount of researches have been directed toward studying these phenomena (see [1–6]). Also, the theory of calculus on time scales (see [7] and references cited therein) was initiated by Stefan Hilger in his Ph.D. thesis in 1988 [8] in order to unify continuous and discrete analysis, and it has a tremendous potential for applications and has recently received much attention since his foundational work. Therefore, it is meaningful to study that on time scales which can unify the continuous and discrete situations. However, there are no concepts of almost periodic time scales and almost periodic functions on time scales, so that it is impossible for us to study almost periodic solutions to differential equations on time scales.

Motivated by the above, the main purpose of this paper is to propose the concept of almost periodic time scales and then give the definition of almost periodic functions on almost periodic time scales, then establish some basic results about almost periodic

differential equations on almost periodic time scales by using the theory of calculus on time scales and some mathematical methods. Furthermore, based on these results, as an application, we consider the following high-order Hopfield neural networks with variable delays on time scales:

$$\begin{aligned} x_i^{\Delta}(t) = {} & -c_i(t)x_i(t) + \sum_{j=1}^{n} a_{ij}(t) f_j\big(x_j(t-\gamma_{ij}(t))\big) \\ & + \sum_{j=1}^{n}\sum_{l=1}^{n} b_{ijl}(t) g_j\big(x_j(t-\sigma_{ijl}(t))\big) g_l\big(x_l(t-v_{ijl}(t))\big) + I_i(t), \qquad i = 1,2,\ldots,n, \end{aligned} \tag{1.1}$$

where n corresponds to the number of units in a neural network, $x_i(t)$ corresponds to the state vector of the ith unit at the time t, $c_i(t)$ represents the rate with which the ith unit will reset its potential to the resting state in isolation when disconnected from the network and external inputs, $a_{ij}(t)$ and $b_{ijl}(t)$ are the first- and second-order connection weights of the neural network, $\gamma_{ij}(t) \geq 0$, $\sigma_{ijl}(t) \geq 0$ and $v_{ijl}(t) \geq 0$ correspond to the transmission delays, $I_i(t)$ denote the external inputs at time t, and f_j and g_j are the activation functions of signal transmission.

2. Almost Periodic Differential Equations on Time Scales

In this section, we will establish some basic results about almost periodic differential equations on almost periodic time scales.

Let $\mathbb{T}$ be a nonempty closed subset (time scale) of $\mathbb{R}$. The forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ and the graininess $\mu : \mathbb{T} \to \mathbb{R}^+$ are defined, respectively, by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \qquad \rho(t) = \sup\{s \in \mathbb{T} : s < t\}, \qquad \mu(t) = \sigma(t) - t. \tag{2.1}$$

A point $t \in \mathbb{T}$ is called left-dense if $t > inf\,\mathbb{T}$ and $\rho(t) = t$, left-scattered if $\rho(t) < t$, right-dense if $t < sup\,\mathbb{T}$ and $\sigma(t) = t$, and right-scattered if $\sigma(t) > t$. If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T} \setminus \{m\}$, otherwise $\mathbb{T}^k = \mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_k = \mathbb{T} \setminus \{m\}$, otherwise $\mathbb{T}_k = \mathbb{T}$.

A function $f : \mathbb{T} \to \mathbb{R}$ is right-dense continuous provided it is continuous at right-dense point in $\mathbb{T}$ and its left-side limits exist at left-dense points in $\mathbb{T}$. If f is continuous at each right-dense point and each left-dense point, then f is said to be a continuous function on $\mathbb{T}$.

For $y : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$, we define the delta derivative of $y(t)$, $y^{\Delta}(t)$, to be the number (if it exists) with the property that for a given $\varepsilon > 0$, there exists a neighborhood U of t such that

$$\Big|\big[y(\sigma(t)) - y(s)\big] - y^{\Delta}(t)[\sigma(t) - s]\Big| < \varepsilon|\sigma(t) - s|, \tag{2.2}$$

for all $s \in U$.

Let y be right-dense continuous, if $Y^{\Delta}(t) = y(t)$, then we define the delta integral by

$$\int_a^t y(s)\Delta s = Y(t) - Y(a). \tag{2.3}$$

A function $p : \mathbb{T} \to \mathbb{R}$ is called regressive provided $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}^k$. The set of all regressive and rd-continuous functions $p : \mathbb{T} \to \mathbb{R}$ will be denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R})$. We define the set $\mathcal{R}^+ = \mathcal{R}^+(\mathbb{T}, \mathbb{R}) = \{p \in \mathcal{R} : 1 + \mu(t)p(t) > 0, \forall t \in \mathbb{T}\}$.

If r is a regressive function, then the generalized exponential function e_r is defined by

$$e_r(t,s) = \exp\left\{\int_s^t \xi_{\mu(\tau)}(r(\tau))\Delta\tau\right\}, \tag{2.4}$$

for all $s, t \in \mathbb{T}$, with the cylinder transformation

$$\xi_h(z) = \begin{cases} \dfrac{\operatorname{Log}(1+\mathrm{hz})}{h}, & \text{if } \ h \neq 0, \\ z, & \text{if } \ h = 0. \end{cases} \tag{2.5}$$

Definition 2.1 (see [7]). Let $p, q : \mathbb{T} \to \mathbb{R}$ are two regressive functions, define

$$p \oplus q = p + q + \mu pq, \qquad \ominus p = -\frac{p}{1+\mu p}, \qquad p \ominus q = p \oplus (\ominus q). \tag{2.6}$$

Lemma 2.2 (see [7]). *Assume that $p, q : \mathbb{T} \to R$ be two regressive functions, then*

(i) $e_0(t,s) \equiv 1$ *and* $e_p(t,t) \equiv 1$;

(ii) $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t,s)$;

(iii) $e_p(t,s) = 1/e_p(s,t) = e_{\ominus p}(s,t)$;

(iv) $e_p(t,s)e_p(s,r) = e_p(t,r)$;

(v) $(e_{\ominus p}(t,s))^{\Delta} = (\ominus p)(t)e_{\ominus p}(t,s)$;

(vi) *if* $a, b, c \in \mathbb{T}$, *then* $\int_a^b p(t)e_p(c,\sigma(t))\Delta t = e_p(c,a) - e_p(c,b)$.

Definition 2.3. A subset S of $\mathbb{T}$ is called relatively dense if there exists a positive number L such that $[a, a+L] \cap s \neq \emptyset$ for all $a \in \mathbb{T}$. The number L is called the inclusion length.

Definition 2.4. Let $\mathcal{C}$ be a collection of sets which is constructed by subsets of $\mathbb{R}$. A time scale $\mathbb{T}$ is called an almost periodic time scale with respect to $\mathcal{C}$, if

$$\mathcal{C}^* = \left\{\pm\tau \in \bigcap_{c\in\mathcal{C}} c : t \pm \tau \in \mathbb{T}, \ \forall t \in \mathbb{T}\right\} \neq \emptyset, \tag{2.7}$$

and $\mathcal{C}^*$ is called the smallest almost periodic set of $\mathbb{T}$.

Remark 2.5. If $\mathcal{C} = \{\mathbb{R}\}$, then $\mathcal{C}^* = \{\pm\tau \in \mathbb{R} : t \pm \tau \in \mathbb{T}, \forall t \in \mathbb{T}\}$, $\mathcal{C}^*$ is called the smallest almost periodic set of $\mathbb{T}$. If $\mathcal{B}$ is a set which is constructed by absolute values of all the elements in $\mathcal{C}^*$, that is $\mathcal{B} = \{|\tau| : \tau \in \mathcal{C}^*\}$, obviously, $\mathcal{B}$ is the smallest positive almost periodic set of the time scale $\mathbb{T}$. Let $p = \inf_{\tau\in\mathcal{B}}\{\tau\} > 0$, p is called the smallest positive period of a time scale $\mathbb{T}$ with respect to $\mathcal{C}$. It is easy to see that this definition includes the concept of periodic time scale and is proper (see [9]).

Throughout this paper, we always restrict our discussion on almost periodic time scales. In this section, we use the notation $|\cdot|$ to denote a norm of $\mathbb{R}^n$.

Definition 2.6. Let $\mathbb{T}$ be an almost periodic time scale with respect to $\mathcal{C}$. A function $f(t) \in C(\mathbb{T}, \mathbb{E}^n)$ is called almost periodic if for any given $\varepsilon > 0$, the set

$$E(\varepsilon, f) = \left\{\tau \in \mathcal{C}^* : \left|f(t+\tau) - f(t)\right| < \varepsilon,\ \forall t \in \mathbb{T}\right\} \tag{2.8}$$

is relatively dense in $\mathbb{T}$; that is, for any given $\varepsilon > 0$, there exists an $l = l(\varepsilon) > 0$ such that each interval of length l contains at least one $\tau = \tau(\varepsilon) \in T(f, \varepsilon, \mathbb{T})$ satisfying

$$\left|f(t+\tau) - f(t)\right| < \varepsilon,\ \forall t \in \mathbb{T}. \tag{2.9}$$

The set $E(\varepsilon, f)$ is called ε-translation set of $f(t)$, τ is called ε-translation number of $f(t)$, and $l(\varepsilon)$ is called contain interval length of $E(\varepsilon, f)$.

Obviously, $E\{\varepsilon, f\} \subseteq \mathcal{C}^*$. So if $E\{\varepsilon, f\} \neq \emptyset$, then we can discuss almost periodic problems on an almost periodic time scale and it is meaningful. We denote $AP(\mathbb{T})$ as a set constructed by all almost periodic functions on an almost time scale $\mathbb{T}$.

Remark 2.7. If $\mathcal{C} = \{\mathbb{R}\}$ and $\mathbb{T} = \mathbb{R}$, then $\mathcal{C}^* = \mathbb{R}$, in this case, Definition 2.6 is equivalent to Definition 1.1 in [10]. If $\mathcal{C} = \{\mathbb{Z}\}$ and $\mathbb{T} = \mathbb{Z}$, then $\mathcal{C}^* = \mathbb{Z}$, in this case, Definition 2.6 is equivalent to the definition of the almost periodic sequences in [11, 12].

Lemma 2.8. *Let $f \in C(\mathbb{T}, \mathbb{R}^n)$ be an almost periodic function, then $f(t)$ is bounded on $\mathbb{T}$.*

Proof. For given $\varepsilon \leq 1$, there exists a constant l, such that in any interval of length $l(\varepsilon)$, there exists $\tau \in E\{\varepsilon, f\}$, such that the inequality $|f(t+\tau) - f(t)| < \varepsilon$, $\forall t \in \mathbb{T}$ holds. And noticing that $f \in C(\mathbb{T}, \mathbb{R}^n)$, then in the limited interval $[0, l]_{\mathbb{T}}$, there exists a number $M > 0$, such that $|f(t)| < M$. For any given $t \in \mathbb{T}$, we can take $\tau \in E\{\varepsilon, f\} \cap [-t, -t+l]_{\mathbb{T}}$, then we have $t + \tau \in [0, l]_{\mathbb{T}}$. Hence, we can obtain $|f(t+\tau)| < M$ and $|f(t+\tau) - f(t)| < 1$. So for all $t \in \mathbb{T}$, we have $|f(t)| < M + 1$. This completes the proof. □

Similar to the case of $\mathbb{T} = \mathbb{R}$, one can easily show the following theorems.

Theorem 2.9. *If $f, g \in C(\mathbb{T}, \mathbb{R}^n)$ are almost periodic, then $f + g$, fg are almost periodic.*

Theorem 2.10. *If $f(t) \in C(\mathbb{T}, \mathbb{R}^n)$ is almost periodic, then $F(t, x)$ is almost periodic if and only if $F(t)$ is bounded on $\mathbb{T}$, where $F(t) = \int_0^t f(s)\Delta s$.*

Theorem 2.11. *If $f(t)$ is almost periodic, $F(\cdot)$ is uniformly continuous on the value field of $f(t)$, then $F \circ f$ is almost periodic.*

Definition 2.12. Let $x \in \mathbb{R}^n$, and $A(t)$ be an $n \times n$ rd-continuous matrix on $\mathbb{T}$, the linear system

$$x^{\Delta}(t) = A(t)x(t), \quad t \in \mathbb{T} \tag{2.10}$$

is said to admit an exponential dichotomy on $\mathbb{T}$ if there exist positive constant k, α, projection P, and the fundamental solution matrix $X(t)$ of (2.10), satisfying

$$\begin{aligned} \left|X(t)PX^{-1}(\sigma(s))\right|_0 &\le k e_{\ominus\alpha}(t,\sigma(s)), \quad s,t \in \mathbb{T},\ t \ge \sigma(s), \\ \left|X(t)(I-P)X^{-1}(\sigma(s))\right|_0 &\le k e_{\ominus\alpha}(\sigma(s),t), \quad s,t \in \mathbb{T},\ t \le \sigma(s), \end{aligned} \tag{2.11}$$

where $|\cdot|_0$ is a matrix norm on $\mathbb{T}$, (say, e.g., if $A = (a_{ij})_{n\times m}$, then we can take $|A|_0 = (\sum_{i=1}^{n}\sum_{j=1}^{m}|a_{ij}|^2)^{1/2}$).

Consider the following almost periodic system

$$x^{\Delta}(t) = A(t)x(t) + f(t), \quad t \in \mathbb{T}, \tag{2.12}$$

where $A(t)$ is an almost periodic matrix function, $f(t)$ is an almost periodic vector function.

Lemma 2.13. *If the linear system* (2.10) *admits exponential dichotomy, then system* (2.12) *has a bounded solution* $x(t)$ *as follows:*

$$x(t) = \int_{-\infty}^{t} X(t)PX^{-1}(\sigma(s))f(s)\Delta s - \int_{t}^{+\infty} X(t)(I-P)X^{-1}(\sigma(s))f(s)\Delta s, \tag{2.13}$$

where $X(t)$ *is the fundamental solution matrix of* (2.10).

Proof. In fact,

$$\begin{aligned} &x^{\Delta}(t) - A(t)x(t) \\ &\quad = X^{\Delta}(t)\int_{-\infty}^{t} PX^{-1}(\sigma(s))f(s)\Delta s + X(\sigma(t))PX^{-1}(\sigma(t))f(t) \\ &\qquad - X^{\Delta}(t)\int_{t}^{+\infty}(I-P)X^{-1}(\sigma(s))f(s)\Delta s + X(\sigma(t))(I-P)X^{-1}(\sigma(t))f(t) \\ &\qquad - A(t)X(t)\int_{-\infty}^{t} PX^{-1}(\sigma(s))f(s)\Delta s + A(t)X(t)\int_{t}^{+\infty}(I-P)X^{-1}(\sigma(s))f(s)\Delta s \\ &\quad = X(\sigma(t))(P+I-P)X^{-1}(\sigma(t))f(t) \\ &\quad = f(t), \end{aligned}$$

$$\begin{aligned}\|x\| &= \sup_{t\in\mathbb{T}}\left|\int_{-\infty}^{t} X(t)PX^{-1}(\sigma(s))f(s)\Delta s - \int_{t}^{+\infty} X(t)(I-P)X^{-1}(\sigma(s))f(s)\Delta s\right| \\ &\le \sup_{t\in\mathbb{T}}\left(\left|\int_{-\infty}^{t} e_{\ominus\alpha}(t,\sigma(s))\Delta s\right| + \left|\int_{t}^{+\infty} e_{\ominus\alpha}(\sigma(s),t)\Delta s\right|\right)k\|f\| \\ &\le \left(\frac{1}{\alpha}-\frac{1}{\ominus\alpha}\right)k\|f\| = \frac{2+\mu\alpha}{\alpha}k\|f\|,\end{aligned} \tag{2.14}$$

where $\|\cdot\| = \sup_{t\in\mathbb{T}}|\cdot|$. So, by Lemma 2.8, $x(t)$ is a bounded solution of system (2.12). The proof is complete. □

Lemma 2.14 (see [7]). *Let A be a regressive $n\times n$-matrix-valued function on $\mathbb{T}$. Let $t_0\in\mathbb{T}$ and $y_0\in\mathbb{R}^n$. Then the initial value problem*

$$y^{\Delta}(t) = A(t)y(t), \qquad y(t_0) = y_0 \tag{2.15}$$

has a unique solution $y:\mathbb{T}\to\mathbb{R}^n$. Moreover, the solution is given by

$$y(t) = e_A(t,t_0)y_0. \tag{2.16}$$

Lemma 2.15. *Let $c_i(t)$ be an almost periodic function on $\mathbb{T}$, where $c_i(t)>0$, $-c_i(t)\in\mathcal{R}^+$, $\forall t\in\mathbb{T}$ and*

$$\min_{1\le i\le n}\left\{\inf_{t\in\mathbb{T}} c_i(t)\right\} = \tilde{m} > 0, \tag{2.17}$$

then the linear system

$$x^{\Delta}(t) = \operatorname{diag}(-c_1(t),-c_2(t),\ldots,-c_n(t))x(t) \tag{2.18}$$

admits an exponential dichotomy on $\mathbb{T}$.

Proof. According to Lemma 2.14, one can see that

$$X(t) = e_{-c}(t,t_0), \tag{2.19}$$

where $-c = \operatorname{diag}(-c_1(t),-c_2(t),\ldots,-c_n(t))$, is a fundamental solution matrix of (2.18).

Now, we prove that $X(t)$ admits an exponential dichotomy on $\mathbb{T}$. In fact, noticing that $-c_i(t)\in\mathcal{R}^+$, then for $t\ge\sigma(s), t,s\in\mathbb{T}$, $\forall i=1,2,\ldots,n$.

If $\mu(\theta)>0, \theta\in[\sigma(s),t)_{\mathbb{T}}, s,t\in\mathbb{T}$, we have

$$1-\frac{\mu(\theta)(\tilde{m}/2)}{1+\mu(\theta)(\tilde{m}/2)} > 1-\mu(\theta)\frac{\tilde{m}}{2} > 1-\mu(\theta)c_i(\theta) > 0, \tag{2.20}$$

then

$$\int_{\sigma(s)}^{t} \frac{\operatorname{Log}(1-\mu(\theta)c_i(\theta))}{\mu(\theta)}\Delta\theta \le \int_{\sigma(s)}^{t} \frac{\operatorname{Log}(1-(\mu(\theta)(\tilde{m}/2)/1+\mu(\theta)(\tilde{m}/2)))}{\mu(\theta)}\Delta\theta, \tag{2.21}$$

therefore

$$\exp\left\{\int_{\sigma(s)}^{t} \frac{\operatorname{Log}(1-\mu(\theta)c_i(\theta))}{\mu(\theta)}\Delta\theta\right\} \le \exp\left\{\int_{\sigma(s)}^{t} \frac{\operatorname{Log}(1-(\mu(\theta)(\tilde{m}/2)/1+\mu(\theta)(\tilde{m}/2)))}{\mu(\theta)}\Delta\theta\right\}, \tag{2.22}$$

then, we can get

$$e_{-c_i}(t,\sigma(s)) \le e_{\ominus(\tilde{m}/2)}(t,\sigma(s)). \tag{2.23}$$

If $\mu(\theta)=0, \theta\in[\sigma(s),t)_{\mathbb{T}}, s,t\in\mathbb{T}$, we can get

$$e_{-c_i}(t,\sigma(s)) = \exp\left\{\int_{\sigma(s)}^{t} -c_i(\theta)\Delta\theta\right\} \le \exp\left\{\int_{\sigma(s)}^{t} -\frac{\tilde{m}}{2}\Delta\theta\right\} = e_{\ominus(\tilde{m}/2)}(t,\sigma(s)). \tag{2.24}$$

Hence, set $P=I$, then

$$\begin{aligned}\left|X(t)PX^{-1}(\sigma(s))\right|_0 &= \left|e_{-c}(t,t_0)Ie_{-c}^{-1}(\sigma(s),t_0)\right|_0 \\ &= \left|e_{-c}(t,t_0)e_{-c}(t_0,\sigma(s))\right|_0 \\ &= \left|e_{-c}(t,\sigma(s))\right|_0 \\ &\le n^{1/2}e_{\ominus(\tilde{m}/2)}(t,\sigma(s)),\end{aligned} \tag{2.25}$$

where $\tilde{m}=\min_{1\le i\le n}\{\inf_{t\in\mathbb{T}}c_i(t)\}$. We can take $k=n^{1/2}, \alpha=\tilde{m}/2$, therefore, $X(t)$ admits an exponential dichotomy on $\mathbb{T}$ with $P=I$. This completes the proof. □

3. An Application

It is well known that high-order Hopfield neural networks (HHNNs) have been extensively applied in psychophysics, speech, perception, robotics, adaptive pattern recognition, vision, and image processing. There exist many results on the existence and stability of periodic and almost periodic solutions for the neural networks with delays. We refer the reader to [13–27] and references therein.

In fact, both continuous and discrete systems are very important in implementing and applications. But it is troublesome to study the existence and stability of almost periodic

solutions for continuous and discrete systems, respectively. Therefore, it is meaningful to study that on time scales which can unify the continuous and discrete situations (see [28, 29]). In this section, by using the concepts and results developed in previous sections, we will study the existence and global asymptotic stability of almost periodic solution of (1.1).

The system (1.1) is supplemented with initial values given by

$$\begin{gathered} x_i(s) = \varphi_i(s), \quad s \in [-\theta, 0]_{\mathbb{T}}, \quad \theta = \max\{\gamma, \sigma, v\}, \\ \gamma = \max_{1\le i,j\le n}\{\gamma_{ij}\}, \quad \sigma = \max_{1\le i,j,l\le n}\{\sigma_{ijl}\}, \quad v = \max_{1\le i,j,l\le n}\{v_{ijl}\}, \quad i,j,l = 1,2,\ldots,n, \\ \gamma_{ij} = \sup_{t\in\mathbb{T}}\gamma_{ij}(t), \qquad \sigma_{ijl} = \sup_{t\in\mathbb{T}}\sigma_{ijl}(t), \qquad v_{ijl} = \sup_{t\in\mathbb{T}} v_{ijl}(t), \end{gathered} \tag{3.1}$$

where $\varphi_i(\cdot) \in C([-\theta, 0]_{\mathbb{T}}, \mathbb{R})$.

For the sake of convenience, we introduce the following notations:

$$\begin{gathered} \underline{c_i} = \inf_{t\in\mathbb{T}}|c_i(t)|, \qquad \overline{c_i} = \sup_{t\in\mathbb{T}}|c_i(t)|, \qquad \overline{a_{ij}} = \sup_{t\in\mathbb{T}}\left|a_{ij}(t)\right|, \\ \overline{b_{ijl}} = \sup_{t\in\mathbb{T}}\left|b_{ijl}(t)\right|, \qquad \overline{I_i} = \sup_{t\in\mathbb{T}}|I_i(t)|. \end{gathered} \tag{3.2}$$

In this section, we assume the follwing.

(H_1) $c_i,\ a_{ij},\ b_{ijl},\ I_i,\ t-\gamma_{ij},\ t-\sigma_{ijl},\ t-v_{ijl} \in C(\mathbb{T}, \mathbb{R})$ are almost periodic, $-c_i \in \mathcal{R}^+$ and $\underline{c_i} > 0$, for $i, j, l = 1, 2, \ldots, n$.

(H_2) There exist positive constants $M_j,\ N_j,\ j = 1, 2, \ldots, n$ such that $|f_j(x)| \le M_j$, $|g_j(x)| \le N_j$ for $j = 1, 2, \ldots, n,\ x \in \mathbb{R}$.

(H_3) Functions $f_j(u),\ g_j(u)(j = 1, 2, \ldots, n)$ satisfy the Lipschitz condition, that is, there exist constants $L_j, H_j > 0$ such that $|f_j(u_1) - f_j(u_2)| \le L_j|u_1 - u_2|,\ |g_j(u_1) - g_j(u_2)| \le H_j|u_1 - u_2|,\ j = 1, 2, \ldots, n$.

Let $X = \{\varphi \mid \varphi : \mathbb{T} \to \mathbb{R}^n,$ is a continuous almost periodic function$\}$ with the norm $\|\varphi\|_X = \sup_{t\in\mathbb{T}}\|\varphi(s)\|$. Clearly, X is a Banach space.

Definition 3.1. The almost periodic solution $x^*(t)$ of system (1.1) is said to be globally asymptotically stable if for any $\varepsilon > 0$ and $t_0 \in [-\theta, +\infty)_{\mathbb{T}}$, there exists $\delta(\varepsilon) > 0$ and $\sigma = \sigma(t_0, \varepsilon, \varphi) > 0$ such that $\|\varphi(t) - x^*(t)\| < \delta$ for $t \in [-\theta, 0]_{\mathbb{T}}$ implies $\|x(t, \varphi) - x^*(t)\| < \varepsilon$ for all $t \in [t_0 + \sigma, +\infty)_{\mathbb{T}}$.

Theorem 3.2. *Assume that* (H_1)–(H_3) *hold, and suppose that*

(H_4) $\max_{1\le i\le n}\{\sum_{j=1}^n \overline{a_{ij}}L_j + \sum_{j=1}^n\sum_{l=1}^n \overline{b_{ijl}}N_jH_l + \sum_{j=1}^n\sum_{l=1}^n \overline{b_{ijl}}N_lH_j/\underline{c_i}\} < 1$,

then (1.1) *has a unique almost periodic solution.*

Proof. For any given $\varphi \in X$, we consider the following almost periodic differential system:

$$\begin{aligned} x_i^{\Delta}(t) &= -c_i(t)x_i(t) + \sum_{j=1}^{n} a_{ij}(t) f_j(\varphi_j(t-\gamma_{ij}(t))) \\ &\quad + \sum_{j=1}^{n}\sum_{l=1}^{n} b_{ijl}(t) g_j(\varphi_j(t-\sigma_{ijl}(t))) g_l(\varphi_l(t-v_{ijl}(t))) + I_i(t), \quad i = 1,2,\ldots,n. \end{aligned} \tag{3.3}$$

Since $\min_{1\le i\le n}\{\inf c_i(t)\} > 0,\ i = 1,2,\ldots,n, t \in \mathbb{T}$, it follows from Lemma 2.15 that the linear system

$$x_i^{\Delta}(t) = -c_i(t)x_i(t), \quad i = 1,2,\ldots,n \tag{3.4}$$

admits an exponential dichotomy on $\mathbb{T}$. Thus, by Lemmas 2.13 and 2.15, we obtain that system (1.1) has a bounded solution:

$$\begin{aligned} &x_{\varphi i}(t) \\ &= \int_{-\infty}^{t} e_{-c_i}(t,\sigma(s)) \\ &\quad \times \left(\sum_{j=1}^{n} a_{ij}(t) f_j(\varphi_j(t-\gamma_{ij}(t))) + \sum_{j=1}^{n}\sum_{l=1}^{n} b_{ijl}(t) g_j(\varphi_j(t-\sigma_{ijl}(t))) g_l(\varphi_l(t-v_{ijl}(t))) + I_i(t) \right) \Delta s, \\ &\qquad i = 1,2,\ldots,n \end{aligned} \tag{3.5}$$

and it follows from Theorems 2.9–2.11 and $e_{-c_i}(t,\sigma(s))$ being almost periodic that x_φ is also almost periodic.

Denote

$$\max_i \left\{ \frac{\sum_{j=1}^{n} \overline{a_{ij}} M_j + \sum_{j=1}^{n}\sum_{l=1}^{n} \overline{b_{ijl}} N_j N_l}{\underline{c_i}} \right\} + \max_i \left\{ \frac{\overline{I_i}}{\underline{c_i}} \right\} := L \tag{3.6}$$

and define a mapping $T : X \to X, T\varphi(t) = x_\varphi(t), \forall \varphi \in X$. Set

$$X^* = \{ \varphi \in X \mid \|\varphi\|_X \le L \}. \tag{3.7}$$

Next, let us check that $T\varphi \in X^*$. For any given $\varphi \in X^*$, it suffices to prove that $\|T(\varphi)\| \le L$;

$$\begin{aligned} \|T(\varphi)\|_X = \sup_{t\in\mathbb{T}} \max_i \Bigg\{ \Bigg| \int_{-\infty}^{t} e_{-c_i}(t,\sigma(s)) \Bigg(&\sum_{j=1}^{n} a_{ij}(t) f_j(\varphi_j(t-\gamma_{ij}(t))) \\ &+ \sum_{j=1}^{n}\sum_{l=1}^{n} b_{ijl}(t) g_j(\varphi_j(t-\sigma_{ijl}(t))) g_l(\varphi_l(t-v_{ijl}(t))) + I_i(t) \Bigg) \Delta s \Bigg| \Bigg\} \end{aligned}$$

$$
\begin{aligned}
&\leq \sup_{t\in\mathbb{T}} \max_i \left\{ \left| \int_{-\infty}^{t} e_{-\underline{c_i}}(t,\sigma(s)) \left(\sum_{j=1}^{n} \overline{a_{ij}} f_j(\varphi_j(t-\gamma_{ij}(t))) \right.\right.\right. \\
&\qquad \left.\left.\left. + \sum_{j=1}^{n}\sum_{l=1}^{n} \overline{b_{ijl}} g_j(\varphi_j(t-\sigma_{ijl}(t))) g_l(\varphi_l(t-v_{ijl}(t))) \right) \right| \right\} + \max_i \left\{ \frac{\overline{I_i}}{\underline{c_i}} \right\} \Delta s \\
&\leq \sup_{t\in\mathbb{T}} \max_i \left\{ \left| \int_{-\infty}^{t} e_{-\underline{c_i}}(t,\sigma(s)) \left(\sum_{j=1}^{n} \overline{a_{ij}} M_j + \sum_{j=1}^{n}\sum_{l=1}^{n} \overline{b_{ijl}} N_j N_l \right) \right| \right\} + \max_i \left\{ \frac{\overline{I_i}}{\underline{c_i}} \right\} \Delta s \\
&\leq \max_i \left\{ \frac{\sum_{j=1}^{n} \overline{a_{ij}} M_j + \sum_{j=1}^{n}\sum_{l=1}^{n} \overline{b_{ijl}} N_j N_l}{\underline{c_i}} \right\} + \max_i \left\{ \frac{\overline{I_i}}{\underline{c_i}} \right\} = L,
\end{aligned}
\tag{3.8}
$$

which shows that $T\varphi \in X^*$. So T is a self-mapping from X^* to X^*.

Next, we shall prove that T is a contraction of X^*.

For any $\varphi, \psi \in X^*$,

$$
\begin{aligned}
&\|T(\varphi) - T(\psi)\|_X \\
&\quad = \sup_{t\in\mathbb{T}} \|T(\varphi)(t) - T(\psi)(t)\| \\
&\quad \leq \sup_{t\in\mathbb{T}} \max_i \left\{ \left| \int_{-\infty}^{t} e_{-c_i}(t,\sigma(s)) \left(\sum_{j=1}^{n} a_{ij}(t) (f_j(\varphi_j(t-\gamma_{ij}(t))) - f_j(\psi_j(t-\gamma_{ij}(t)))) \right.\right.\right. \\
&\qquad + \sum_{j=1}^{n}\sum_{l=1}^{n} b_{ijl}(t) (g_j(\varphi_j(t-\sigma_{ijl}(t))) g_l(\varphi_l(t-v_{ijl}(t))) \\
&\qquad \left.\left.\left. - g_j(\psi_j(t-\sigma_{ijl}(t))) g_l(\psi_l(t-v_{ijl}(t)))) \right) \Delta s \right| \right\} \\
&\quad \leq \sup_{t\in\mathbb{T}} \max_i \left\{ \left| \int_{-\infty}^{t} e_{-c_i}(t,\sigma(s)) \left(\sum_{j=1}^{n} a_{ij}(t) (f_j(\varphi_j(t-\gamma_{ij}(t))) - f_j(\psi_j(t-\gamma_{ij}(t)))) \right.\right.\right. \\
&\qquad + \sum_{j=1}^{n}\sum_{l=1}^{n} b_{ijl}(t) (g_j(\varphi_j(t-\sigma_{ijl}(t))) |g_l(\varphi_l(t-v_{ijl}(t))) - g_l(\psi_l(t-v_{ijl}(t)))| \\
&\qquad \left.\left.\left. + |g_j(\varphi_j(t-\sigma_{ijl}(t))) - g_j(\psi_j(t-\sigma_{ijl}(t)))| g_l(\psi_l(t-v_{ijl}(t)))) \right) \Delta s \right| \right\} \\
&\quad \leq \sup_{t\in\mathbb{T}} \max_i \left\{ \left| \left(\sum_{j=1}^{n} \overline{a_{ij}} L_j + \sum_{j=1}^{n}\sum_{l=1}^{n} \overline{b_{ijl}} N_j H_l + \sum_{j=1}^{n}\sum_{l=1}^{n} \overline{b_{ijl}} N_l H_j \right) \Delta s \right| \right\} \|\varphi - \psi\|_X
\end{aligned}
$$

$$\leq \max_i \left\{ \frac{\sum_{j=1}^n \overline{a_{ij}} L_j + \sum_{j=1}^n \sum_{l=1}^n \overline{b_{ijl}} N_j H_l + \sum_{j=1}^n \sum_{l=1}^n \overline{b_{ijl}} N_l H_j}{\underline{c_i}} \right\} \|\varphi - \psi\|_X$$
$$= \rho \|\varphi - \psi\|_X. \tag{3.9}$$

Because $\rho < 1$, so T is a contraction of X^*.

By the fixed point theorem of Banach space, T has a unique fixed point φ in X^* such that $T\varphi = \varphi$, φ is an almost periodic solution of system (1.1) in X^*. The proof is complete. □

Theorem 3.3. *Assume that* (H_1)–(H_4) *hold. Suppose further that* (H_5). *Let*

$$\delta_i = \frac{2\underline{c_i} - 2\sum_{j=1}^n \overline{a_{ij}} L_j - 2\sum_{j=1}^n \sum_{l=1}^n \overline{b_{ijl}} (N_l H_j + N_j H_l)}{\left(\overline{c_i} + \sum_{j=1}^n \overline{a_{ij}} L_j + \sum_{j=1}^n \sum_{l=1}^n \overline{b_{ijl}} (N_l H_j + N_j H_l)\right)^2}, \tag{3.10}$$

and for any $t_0 \in [-\theta, +\infty)_{\mathbb{T}}$, *the following holds*

$$\int_{t_0}^t (\delta_i - \mu(s)) \Delta s \longrightarrow +\infty, \quad t \longrightarrow +\infty, \ i = 1, 2, \ldots, n. \tag{3.11}$$

Then the almost periodic solution of system (1.1) *is globally asymptotically stable.*

Proof. According to Theorem 3.2, we know that (1.1) has an almost periodic solution $x^*(t) = (x_1^*(t), x_2^*(t), \ldots, x_n^*(t))^T$. Suppose that $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$ is an arbitrary solution of (1.1). Then it follows from system (1.1) that

$$\begin{aligned}
(x_i(t) - x_i^*(t))^\Delta = {} & -c_i(t)(x_i(t) - x_i^*(t)) + \sum_{j=1}^n a_{ij}(t)\Big(f_j(x_j(t - \gamma_{ij}(t))) - f_j\Big(x_j^*(t - \gamma_{ij}(t))\Big)\Big) \\
& + \sum_{j=1}^n \sum_{l=1}^n b_{ijl}(t)\Big(g_j(x_j(t - \sigma_{ijl}(t))) g_l(x_l(t - v_{ijl}(t))) \\
& - g_j\Big(x_j^*(t - \sigma_{ijl}(t))\Big) g_l(x_l^*(t - v_{ijl}(t)))\Big),
\end{aligned} \tag{3.12}$$

for $i = 1, 2, \ldots, n$, the initial condition of (3.12) is

$$\varphi_i(s) = \varphi_i(s) - x_i^*(s), \quad s \in [-\theta, 0]_{\mathbb{T}}, \ i = 1, 2, \ldots, n. \tag{3.13}$$

Let $y_i(t) = x_i(t) - x_i^*(t)$, we use the Lyapunov function $V(y) = y^T y$, where $y(t) = (y_1(t), y_2(t), \ldots, y_n(t))^T$, from (3.12) we have

$$\begin{aligned}
\left(y_i^2(t)\right)^{\Delta} &= 2y_i(t)y_i^{\Delta}(t) + \mu(t)\left(y_i^{\Delta}(t)\right)^2 \\
&\le 2y_i(t)\left[-c_i y_i(t) + \sum_{j=1}^{n}\overline{a_{ij}}L_j\left|y_j(t-\gamma_{ij}(t))\right|\right. \\
&\qquad \left. + \sum_{j=1}^{n}\sum_{l=1}^{n}\overline{b_{ijl}}\left(N_l H_j\left|y_j(t-\sigma_{ijl}(t))\right| + H_l N_j\left|y_l(t-v_{ijl}(t))\right|\right)\right] \\
&\qquad + \mu(t)\left[-c_i y_i(t) + \sum_{j=1}^{n}\overline{a_{ij}}L_j\left|y_j(t-\gamma_{ij}(t))\right|\right. \\
&\qquad \left. + \sum_{j=1}^{n}\sum_{l=1}^{n}\overline{b_{ijl}}\left(N_l H_j\left|y_j(t-\sigma_{ijl}(t))\right| + H_l N_j\left|y_l(t-v_{ijl}(t))\right|\right)\right]^2 \\
&\le \left(-2\underline{c_i} + 2\sum_{j=1}^{n}\overline{a_{ij}}L_j + 2\sum_{j=1}^{n}\sum_{l=1}^{n}\overline{b_{ijl}}(N_l H_j + N_j H_l)\right)\|y\|^2 \\
&\quad + \mu(t)\left(\overline{c_i} + \sum_{j=1}^{n}\overline{a_{ij}}L_j + \sum_{j=1}^{n}\sum_{l=1}^{n}\overline{b_{ijl}}(N_l H_j + N_j H_l)\right)^2\|y\|^2 \\
&= -\left[2\underline{c_i} - 2\sum_{j=1}^{n}\overline{a_{ij}}L_j - 2\sum_{j=1}^{n}\sum_{l=1}^{n}\overline{b_{ijl}}(N_l H_j + N_j H_l)\right. \\
&\qquad \left. -\mu(t)\left(\overline{c_i} + \sum_{j=1}^{n}\overline{a_{ij}}L_j + \sum_{j=1}^{n}\sum_{l=1}^{n}\overline{b_{ijl}}(N_l H_j + N_j H_l)\right)^2\right]\|y\|^2 \\
&= -\left(\overline{c_i} + \sum_{j=1}^{n}\overline{a_{ij}}L_j + \sum_{j=1}^{n}\sum_{l=1}^{n}\overline{b_{ijl}}(N_l H_j + N_j H_l)\right)^2 \\
&\quad \times \left(\frac{2\underline{c_i} - 2\sum_{j=1}^{n}\overline{a_{ij}}L_j - 2\sum_{j=1}^{n}\sum_{l=1}^{n}\overline{b_{ijl}}(N_l H_j + N_j H_l)}{\left(\overline{c_i} + \sum_{j=1}^{n}\overline{a_{ij}}L_j + \sum_{j=1}^{n}\sum_{l=1}^{n}\overline{b_{ijl}}(N_l H_j + N_j H_l)\right)^2} - \mu(t)\right)\|y\|^2 \\
&= -\left(\overline{c_i} + \sum_{j=1}^{n}\overline{a_{ij}}L_j + \sum_{j=1}^{n}\sum_{l=1}^{n}\overline{b_{ijl}}(N_l H_j + N_j H_l)\right)^2(\delta_i - \mu(t))\|y\|^2.
\end{aligned} \tag{3.14}$$

Then we can easily get

$$V^{\Delta}(y(t)) = \sum_{i=1}^{n}\left(y_i^2(t)\right)^{\Delta} \le -\sum_{i=1}^{n}\left(\overline{c_i} + \sum_{j=1}^{n}\overline{a_{ij}}L_j + \sum_{j=1}^{n}\sum_{l=1}^{n}\overline{b_{ijl}}(N_lH_j + N_jH_l)\right)^2 (\delta_i - \mu(t))\|y\|^2$$
$$\le -\xi(t)\phi(\|y\|), \tag{3.15}$$

where $\xi(t) = \min_{1\le i\le n}\{\delta_i\} - \mu(t)$, $\phi(s) = \sum_{i=1}^{n}(\overline{c_i} + \sum_{j=1}^{n}\overline{a_{ij}}L_j + \sum_{j=1}^{n}\sum_{l=1}^{n}\overline{b_{ijl}}(N_lH_j + N_jH_l))^2 s^2$.

Let $a(s) = (1/n)s^2$ and $b = s^2$ where $a, b \in C([0,\infty), \mathbb{R}^+)$, it is easy to see that

(i) $a(\|y\|) \le V(y(t)) \le b(\|y\|)$;

(ii) $V^{\Delta}(y(t))|_{(1\cdot 1)} \le -\xi(t)\phi(\|y\|)$.

Let Ω be a domain in the space $\mathbb{R}^n$ that contains the origin of coordinates. We choose a constant $\alpha > 0$ and set $\Gamma = \{y \in \Omega : V(y(t)) \le a(\alpha)\}$. Assume that statement S_0 has the following form: for any $\varphi \in \Gamma$ and $\varepsilon > 0$, $t_0 \in [-\theta, +\infty)_{\mathbb{T}}$, there exists a constant $\varsigma = \varsigma(t_0, \varepsilon, \varphi) > 0$ such that $\|y(t_0 + \varsigma, \varphi)\| < \varepsilon$.

Assume that statement S_0 is not true, that is, there exist $\varphi' \in \Gamma$ and $\varepsilon_1 > 0$ such that for any constant $\varsigma > 0$, one has $\|y(t_0 + \varsigma, \varphi')\| \ge \varepsilon_1$. Since

$$\int_{t_0}^{t} \xi(s)\Delta s \longrightarrow +\infty, \quad t \longrightarrow +\infty, \tag{3.16}$$

we conclude that, for $\eta > a(\alpha)/\phi(\varepsilon_1)$, there exists a constant $\varsigma_1 = \varsigma_1(t_0, \varepsilon_1, \varphi')$ such that

$$\int_{t_0}^{t_0+\varsigma_1} \xi(s)\Delta s > \eta. \tag{3.17}$$

Every $t \in [t_0, t_0 + \varsigma_1)_{\mathbb{T}}$ can be represented in the form $t = t_0 + \varsigma_2$. Hence, for all $t \in [t_0, t_0 + \varsigma_1)_{\mathbb{T}}$, we have $\|y(t, \varphi')\| \ge \varepsilon_1$. Integrating the inequality from (ii), we get

$$V(y(t_0 + \varsigma_1))$$
$$\le V(y(t_0)) - \int_{t_0}^{t_0+\varsigma_1} \xi(t)\phi(\|y\|)\Delta s < a(\alpha) - \phi(\varepsilon_1)\int_{t_0}^{t_0+\varsigma_1} \xi(t)\Delta s \le a(\alpha) - \phi(\varepsilon_1)\eta < 0, \tag{3.18}$$

which is impossible. Thus, S_0 is true.

We choose an arbitrary $\varepsilon > 0$ and set $\varepsilon_1 = b^{-1}(a(\varepsilon)) > 0$. Since statement S_0 is true, for any $\varphi \in \Gamma$ and $\varepsilon > 0$ there exists a constant $\varsigma = \varsigma(t_0, \varepsilon, \varphi) > 0$ such that $\|y(t_0 + \varsigma, \varphi)\| < \varepsilon_1$.

Since $t > t_0 + \varsigma$ for all $t \in [t_0 + \varsigma, +\infty)_{\mathbb{T}}$, we obtain the following inequality for the solution $y(t) = y(t, \varphi)$:

$$a(\|y\|) \le V(y(t)) \le V(y(t_0 + \varsigma)) \le b(\|y(t_0 + \varsigma)\|) < b(\varepsilon_1). \tag{3.19}$$

Hence, $\|y(t)\| \le a^{-1}b(\varepsilon_1) = \varepsilon$ for all $t \in [t_0 + \varsigma, +\infty)_{\mathbb{T}}$.

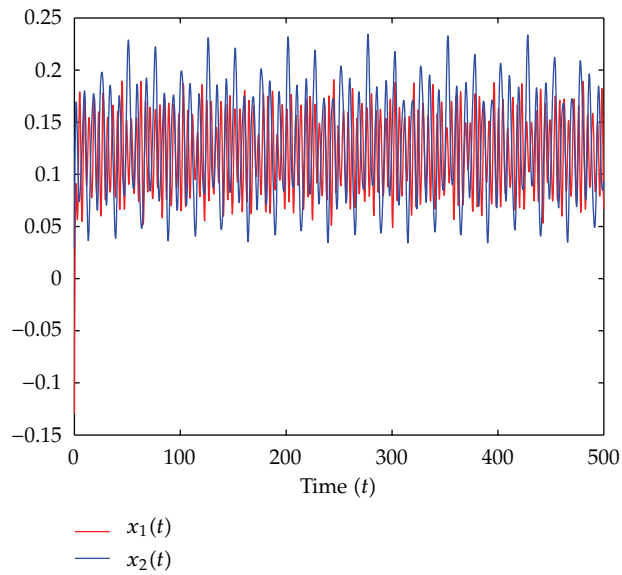

Figure 1: Transient response of state x_1, x_2 in Example 4.1.

Thus, for any $\varphi \in \Gamma$ and $\varepsilon > 0$, there is a constant $\varsigma = \varsigma(t_0, \varepsilon, \varphi) > 0$ such that $\|y(t,\varphi)\| < \varepsilon$ for all $t \in [t_0 + \varsigma, +\infty)_{\mathbb{T}}$. Choosing a constant $\delta > 0$ so that $b(\delta) \leq a(\alpha)$, we obtain $B_\delta := \{y \in \mathbb{R}^n : \|y\| < \delta\} \subset \Gamma$. Therefore, the almost periodic solution $x^*(t)$ of system (1.1) is globally asymptotically stable. This completes the proof. □

4. Numerical Examples and Simulations

Consider the following neural networks system on time scales:

$$\begin{aligned} x_i^{\Delta}(t) &= -c_i(t)x_i(t) + \sum_{j=1}^{2} a_{ij}(t) f_j\big(x_j(t-\gamma_{ij}(t))\big) \\ &\quad + \sum_{j=1}^{2}\sum_{l=1}^{2} b_{ijl}(t) g_j\big(x_j(t-\sigma_{ijl}(t))\big) g_l\big(x_l(t-v_{ijl}(t))\big) + I_i(t), \qquad i = 1,2,\ t > 0, \end{aligned} \tag{4.1}$$

where

$$f_1(x_1) = g_1(x_1) = \sin\left(\frac{3}{4}x_1\right), \qquad f_2(x_2) = g_2(x_2) = \cos\left(\frac{2}{5}x_2\right). \tag{4.2}$$

Obviously, $f_i(x_i), g_i(x_i) (i = 1,2)$ satisfy (H_2) and (H_3), and

$$L_1 = L_2 = H_1 = H_2 = M_1 = M_2 = N_1 = N_2 = 1. \tag{4.3}$$

Example 4.1. $\mathbb{T} = \mathbb{R}$, $\mu(t) \equiv 0$,

$$a_{11}(t) = \frac{1}{20} + \frac{1}{20}\cos t, \qquad a_{12}(t) = \frac{1}{40} + \frac{3}{40}\cos\left(\sqrt{2}t\right), \qquad a_{21}(t) = \frac{3}{20} + \frac{1}{20}\cos\left(\frac{4}{3}t\right),$$
$$a_{22}(t) = \frac{3}{40} + \frac{1}{40}\cos\left(\frac{1}{4}t\right), \qquad c_1(t) = 1 + \frac{1}{20}\sin\left(\frac{6}{5}t\right), \qquad c_2(t) = 1 + \frac{1}{25}\sin\left(\frac{1}{3}t\right),$$
$$I_1(t) = \frac{3}{40} + \frac{1}{40}\sin\left(\sqrt{3}t\right), \qquad I_2(t) = \frac{1}{40} + \frac{3}{40}\cos\left(\frac{3}{4}t\right),$$
$$b_{111}(t) = b_{222}(t) = \frac{1}{80} + \frac{1}{80}\sin\left(\sqrt{2}t\right), \qquad b_{112}(t) = b_{212}(t) = \frac{1}{40} + \frac{1}{40}\cos\left(\frac{4}{3}t\right),$$
$$b_{121}(t) = b_{221}(t) = \frac{1}{80} + \frac{1}{80}\cos\left(\sqrt{3}t\right), \qquad b_{122}(t) = b_{211}(t) = \frac{1}{80} + \frac{1}{80}\sin\left(\frac{3}{4}t\right). \tag{4.4}$$

We get that (H_1) is satisfied, and

$$\overline{c_1} = \frac{21}{20}, \qquad \overline{c_2} = \frac{26}{25}, \qquad \underline{c_1} = \frac{19}{20}, \qquad \underline{c_2} = \frac{24}{25},$$
$$\overline{a_{11}} = \frac{1}{10}, \qquad \overline{a_{12}} = \frac{1}{10}, \qquad \overline{a_{21}} = \frac{1}{5}, \qquad \overline{a_{22}} = \frac{1}{10},$$
$$\overline{b_{111}} = \overline{b_{222}} = \frac{1}{40}, \qquad \overline{b_{112}} = \overline{b_{212}} = \frac{1}{20},$$
$$\overline{b_{121}} = \overline{b_{221}} = \frac{1}{40}, \qquad \overline{b_{122}} = \overline{b_{211}} = \frac{1}{40}, \tag{4.5}$$

so, we have

$$\frac{\sum_{j=1}^{2}\overline{a_{1j}}L_j + \sum_{j=1}^{2}\sum_{l=1}^{2}\overline{b_{1jl}}N_jH_l + \sum_{j=1}^{2}\sum_{l=1}^{2}\overline{b_{1jl}}N_lH_j}{\underline{c_1}} = \frac{9}{19} < 1,$$
$$\frac{\sum_{j=1}^{2}\overline{a_{2j}}L_j + \sum_{j=1}^{2}\sum_{l=1}^{2}\overline{b_{2jl}}N_jH_l + \sum_{j=1}^{2}\sum_{l=1}^{2}\overline{b_{2jl}}N_lH_j}{\underline{c_2}} = \frac{55}{96} < 1,$$
$$\delta_1 = \frac{2\underline{c_1} - 2\sum_{j=1}^{n}\overline{a_{1j}}L_j - 2\sum_{j=1}^{n}\sum_{l=1}^{n}\overline{b_{1jl}}\left(N_lH_j + N_jH_l\right)}{\left(\overline{c_1} + \sum_{j=1}^{n}\overline{a_{1j}}L_j + \sum_{j=1}^{n}\sum_{l=1}^{n}\overline{b_{1jl}}\left(N_lH_j + N_jH_l\right)\right)^2} = 0.444 > 0,$$
$$\delta_2 = \frac{2\underline{c_2} - 2\sum_{j=1}^{n}\overline{a_{2j}}L_j - 2\sum_{j=1}^{n}\sum_{l=1}^{n}\overline{b_{2jl}}\left(N_lH_j + N_jH_l\right)}{\left(\overline{c_2} + \sum_{j=1}^{n}\overline{a_{2j}}L_j + \sum_{j=1}^{n}\sum_{l=1}^{n}\overline{b_{2jl}}\left(N_lH_j + N_jH_l\right)\right)^2} = 0.3164 > 0. \tag{4.6}$$

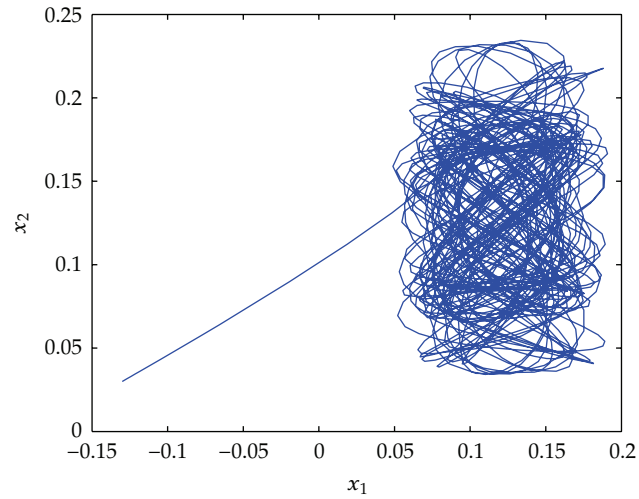

Figure 2: Phase response of state variables x_1, x_2 in Example 4.1.

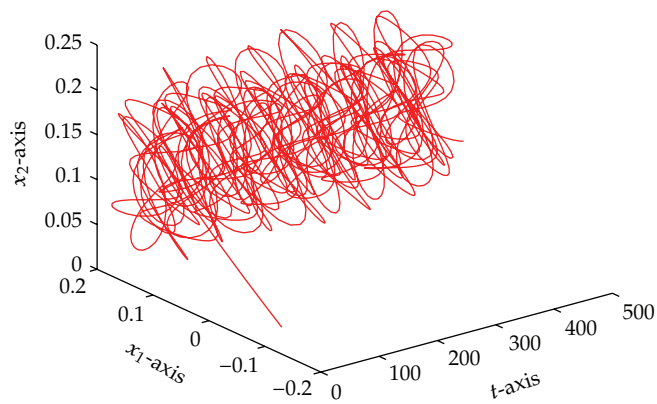

Figure 3: Phase response of state variables x_1, x_2 with time t in Example 4.1.

Thus, it is easy to see that

$$\int_{t_0}^{t} (\delta_i - \mu(s))\,\Delta s \longrightarrow +\infty, \quad t \longrightarrow +\infty, \ i = 1, 2. \tag{4.7}$$

The conditions of Theorems 3.2 and 3.3 is satisfied. Hence, we know that system (4.1) has an almost periodic solution, which is asymptotically stable.

We take $\gamma_{ij} = \sigma_{ijl} = v_{ijl} = 0.1, i, j, l = 1, 2$, and the initial condition $\varphi_1(\theta) = -0.13, \varphi_2(\theta) = 0.03, \theta \in [-0.1, 0]$, we can give the following numerical simulation figures to show our results are plausible and effective on time scales (see Figures 1, 2, and 3).

The numerical simulations of Figures 1, 2, and 3 in Example 4.1 show that the unique almost periodic solution is asymptotically stable, our results are effective on time scales.

Example 4.2. $\mathbb{T} = \mathbb{Z}$, $\mu(t) \equiv 1$,

$$\begin{gathered}
a_{11}(t) = 0.003 + 0.002\cos t, \qquad a_{12}(t) = 0.002 + 0.001\cos\left(\sqrt{2}t\right), \\
a_{21}(t) = 0.001 + 0.003\cos\left(\frac{4}{3}t\right), \qquad a_{22}(t) = 0.003 + 0.001\cos\left(\frac{1}{4}t\right), \\
c_1(t) = 0.7 + 0.05\sin\left(\frac{6}{5}t\right), \qquad c_2(t) = 0.8 + 0.02\sin\left(\frac{1}{3}t\right), \\
I_1(t) = 0.001 + 0.002\sin\left(\sqrt{3}t\right), \qquad I_2(t) = 0.002 + 0.001\cos\left(\frac{3}{4}t\right), \\
b_{111}(t) = b_{222}(t) = 0.001 + 0.002\sin\left(\sqrt{2}t\right), \qquad b_{112}(t) = b_{212}(t) = 0.002 + 0.003\cos\left(\frac{4}{3}t\right), \\
b_{121}(t) = b_{221}(t) = 0.001 + 0.001\cos\left(\sqrt{3}t\right), \qquad b_{122}(t) = b_{211}(t) = 0.003 + 0.002\sin\left(\frac{3}{4}t\right).
\end{gathered} \tag{4.8}$$

We get that (H_1) is satisfied, and

$$\begin{gathered}
\overline{c_1} = 0.75, \qquad \overline{c_2} = 0.82, \qquad \underline{c_1} = 0.65, \qquad \underline{c_2} = 0.78, \qquad \overline{a_{11}} = 0.005, \\
\overline{a_{12}} = 0.003, \qquad \overline{a_{21}} = 0.004, \qquad \overline{a_{22}} = 0.004, \\
\overline{b_{111}} = \overline{b_{222}} = 0.003, \qquad \overline{b_{112}} = \overline{b_{212}} = 0.005, \\
\overline{b_{121}} = \overline{b_{221}} = 0.002, \qquad \overline{b_{122}} = \overline{b_{211}} = 0.005,
\end{gathered} \tag{4.9}$$

so, we have

$$\begin{gathered}
\frac{\sum_{j=1}^{2}\overline{a_{1j}}L_j + \sum_{j=1}^{2}\sum_{l=1}^{2}\overline{b_{1jl}}N_jH_l + \sum_{j=1}^{2}\sum_{l=1}^{2}\overline{b_{1jl}}N_lH_j}{\underline{c_1}} = 0.058 < 1, \\
\frac{\sum_{j=1}^{2}\overline{a_{2j}}L_j + \sum_{j=1}^{2}\sum_{l=1}^{2}\overline{b_{2jl}}N_jH_l + \sum_{j=1}^{2}\sum_{l=1}^{2}\overline{b_{2jl}}N_lH_j}{\underline{c_2}} = 0.049 < 1, \\
\delta_1 = \frac{2\underline{c_1} - 2\sum_{j=1}^{n}\overline{a_{1j}}L_j - 2\sum_{j=1}^{n}\sum_{l=1}^{n}\overline{b_{1jl}}\left(N_lH_j + N_jH_l\right)}{\left(\overline{c_1} + \sum_{j=1}^{n}\overline{a_{1j}}L_j + \sum_{j=1}^{n}\sum_{l=1}^{n}\overline{b_{1jl}}\left(N_lH_j + N_jH_l\right)\right)^2} = 1.9712 > 1, \\
\delta_2 = \frac{2\underline{c_2} - 2\sum_{j=1}^{n}\overline{a_{2j}}L_j - 2\sum_{j=1}^{n}\sum_{l=1}^{n}\overline{b_{2jl}}\left(N_lH_j + N_jH_l\right)}{\left(\overline{c_2} + \sum_{j=1}^{n}\overline{a_{2j}}L_j + \sum_{j=1}^{n}\sum_{l=1}^{n}\overline{b_{2jl}}\left(N_lH_j + N_jH_l\right)\right)^2} = 1.8202 > 1.
\end{gathered} \tag{4.10}$$

Thus, it is easy to see that

$$\int_{t_0}^{t}\left(\delta_i - \mu(s)\right)\Delta s \longrightarrow +\infty, \quad t \longrightarrow +\infty, \ i = 1, 2. \tag{4.11}$$

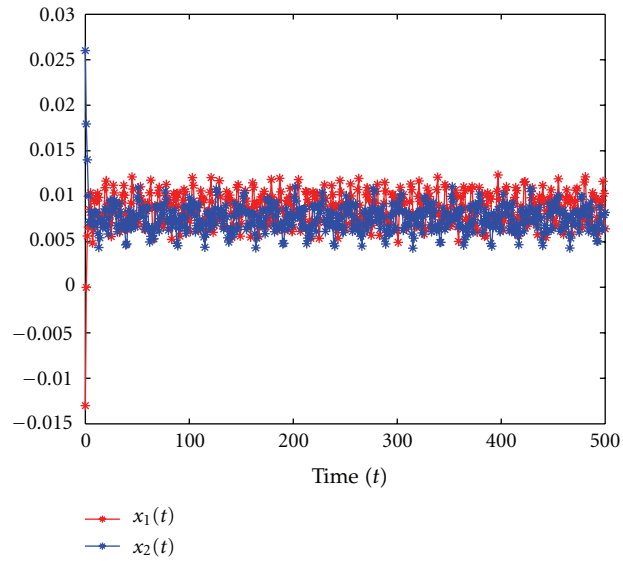

Figure 4: Transient response of state x_1, x_2 in Example 4.2.

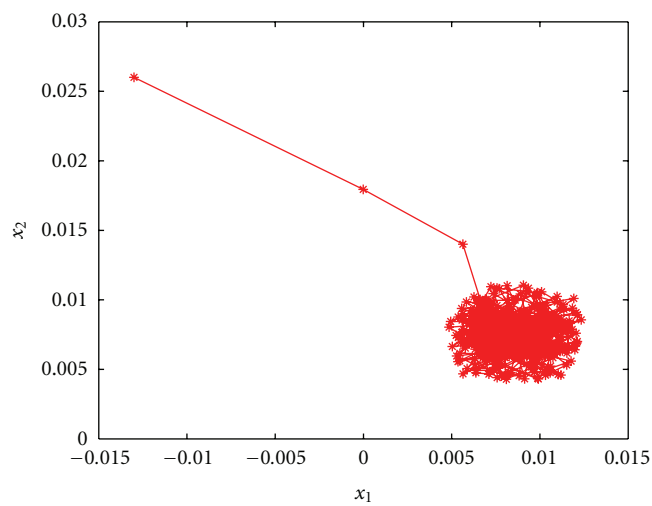

Figure 5: Phase response of state variables x_1, x_2 in Example 4.2.

The conditions of Theorems 3.2 and 3.3 is satisfied. Hence, we know that system (4.1) has an almost periodic solution, which is asymptotically stable.

We take $\gamma_{ij} = \sigma_{ijl} = v_{ijl} = 0.01, i, j, l = 1, 2$, and the initial condition $\varphi_1(\theta) = -0.013, \varphi_2(\theta) = 0.026, \theta \in [-0.01, 0]$, we can give the following numerical simulation figures to show our results are plausible and effective on time scales (see Figures 3, 4, and 5).

The numerical simulations of Figures 4, 5, and 6 in Example 4.2 show that the unique almost periodic solution is asymptotically stable, our results are effective on time scales.

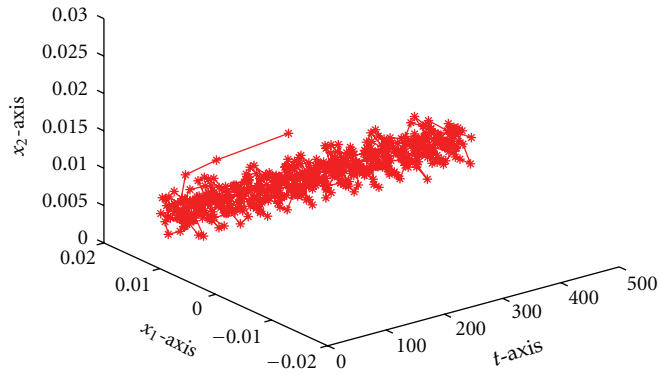

Figure 6: Phase response of state variables x_1, x_2 with time t in Example 4.2.

5. Conclusion

In this paper, some basic results about almost periodic differential equations on almost periodic time scales are established, and the existence and global asymptotic stability of an almost periodic solution for a class of high-order Hopfield neural networks on almost periodic time scales is investigated. The results derived in this paper are meaningful.

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