Research Article

Existence of Periodic Positive Solutions for Abstract Difference Equations

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We will consider the existence of multiple positive periodic solutions for a class of abstract difference equations by using the well-known fixed point theorem (due to Krasnoselskii).

In the past several years, the existence of periodic solutions for first-order functional differential equations

$$y'(t) = -a(t)y(t) + f(t, y(t - \tau(t))) \quad (1)$$

has been extensively investigated (see [1–3], and the references therein). In [4–6], the existence of periodic positive solutions for difference equations

$$x_{n+1} = a_n x_n + \lambda h_n f(x_{n-\tau(n)}) \quad (2)$$

has been considered. To the best of our knowledge, however, little has been done for the abstract difference equations (see [7–9]). In this note, we will consider this problem. To this end, let X be a real Banach space and let $K \subset X$ be a cone, then a Banach space X with a partial ordering $\leq$ induced by a cone K is called an ordered Banach space. On the other hand, we will denote the identity operator defined on X by I .

In [7–9], the authors considered the existence of periodic solutions for the abstract equation

$$x_{n+1} = A_n x_n + F_n(x_n). \quad (3)$$

In this note, we will consider the equation

$$x_{n+1} = A_n x_n + \lambda F_n(x_{n-\tau(n)}), \quad n \in \mathbb{Z}, \quad (4)$$

where $\{A_n\}_{n \in \mathbb{Z}}$ is a T -periodic sequence of bounded linear operator defined on X and satisfies $(\prod_{k=0}^{T-1} A_k^{-1} - I)^{-1} A_n (\prod_{k=0}^{T-1} A_k^{-1} - I) = A_n$ for $n \in \mathbb{Z}$, $(\prod_{k=0}^{T-1} A_k^{-1} - I)x \in K$ and $(\prod_{k=0}^{T-1} A_k^{-1} - I)^{-1} x \in K$ for any $x \in K$, $A_k x \in K$ and $A_k^{-1} x \in K$ for any $x \in K$ ($k = 0, 1, \dots, T-1$), $\{\tau(n)\}_{n \in \mathbb{Z}}$ is an integer valued T -periodic sequence, and $\{F_n\}_{n \in \mathbb{Z}}$ is a T -periodic sequence of bounded functions from X to K , and λ is a positive constant.

If (4) has a T -periodic solution in X , then we have

$$\prod_{k=0}^n A_k^{-1} x_{n+1} - \prod_{k=0}^{n-1} A_k^{-1} x_n = \prod_{k=0}^n A_k^{-1} (\lambda F_n(x_{n-\tau(n)})). \quad (5)$$

Summing the above equation from n to $n+T-1$, we have

$$\prod_{k=0}^{n-1} A_k^{-1} \left(\prod_{k=n}^{n+T-1} A_k^{-1} - I \right) x_n = \sum_{s=n}^{n+T-1} \prod_{k=0}^s A_k^{-1} (\lambda F_s(x_{s-\tau(s)})). \quad (6)$$

That is,

$$x_n = \lambda \sum_{s=n}^{n+T-1} G(n, s) F_s(x_{s-\tau(s)}), \quad n \in \mathbb{Z}, \quad (7)$$

where

$$G(n, s) = \left(\prod_{k=0}^{T-1} A_k^{-1} - I \right)^{-1} \prod_{k=n}^s A_k^{-1}. \quad (8)$$

If (7) has a T -periodic solution in X , then we have

$$\begin{aligned} x_{n+1} - x_n &= \left(\prod_{k=0}^{T-1} A_k^{-1} - I \right)^{-1} \sum_{s=n+1}^{n+T} \prod_{k=n+1}^s A_k^{-1} (\lambda F_s(x_{s-\tau(s)})) \\ &\quad - \left(\prod_{k=0}^{T-1} A_k^{-1} - I \right)^{-1} \sum_{s=n}^{n+T-1} \prod_{k=n}^s A_k^{-1} (\lambda F_s(x_{s-\tau(s)})) \end{aligned}$$

$$\begin{aligned}
&= \left(\left(\prod_{k=0}^{T-1} A_k^{-1} - I \right)^{-1} A_n \left(\prod_{k=0}^{T-1} A_k^{-1} - I \right) - I \right) \sum_{s=n}^{n+T-1} G(n, s) (\lambda F_s(x_{s-\tau(s)})) \\
&\quad + \left(\prod_{k=0}^{T-1} A_k^{-1} - I \right)^{-1} \left(\prod_{k=n+1}^{n+T} A_k^{-1} - I \right) (\lambda F_n(x_{n-\tau(n)})) \\
&= A_n x_n - x_n + \lambda F_n(x_{n-\tau(n)}).
\end{aligned} \tag{9}$$

This equation is equivalent to (4). Thus, we have the following result.

Theorem 1. Assume that $A_0, A_1, \dots, A_{T-1}$ and $(\prod_{k=0}^{T-1} A_k^{-1} - I)$ are invertible and $A_{n+1}^{-1} A_{n+2}^{-1} \dots A_{n+T}^{-1} = A_0^{-1} A_1^{-1} \dots A_{T-1}^{-1}$ ($n \in \mathbb{Z}$). Then $\{x_n\}_{n \in \mathbb{Z}}$ ($x_n \in X$) is a T -periodic solution of (4) if and only if it is a T -periodic solution of (7).

We now assume that $0 < N \leq \|G(n, s)\| \leq M < +\infty$ for $n \in \mathbb{Z}$ and $n \leq s \leq n + T - 1$ and that $\sigma = N/M$. To obtain our main results, we firstly give a lemma. The proof of that lemma can be found in [10].

Lemma 1. Let E be a Banach space, and let $P \subset E$ be a cone. Assume Ω_1, Ω_2 are bounded open subsets of E such that $0 \in \Omega_1 \subset \Omega_2$. Suppose that $T : P \cap (\overline{\Omega_2} \setminus \Omega_1) \rightarrow P$ is a completely continuous operator such that

- (1) $\|Tu\| \leq \|u\|$ for $u \in P \cap \partial\Omega_1$ and $\|Tu\| \geq \|u\|$ for $u \in P \cap \partial\Omega_2$ or that
- (2) $\|Tu\| \geq \|u\|$ for $u \in P \cap \partial\Omega_1$ and $\|Tu\| \leq \|u\|$ for $u \in P \cap \partial\Omega_2$.

Then T has a fixed point in $P \cap (\overline{\Omega_2} \setminus \Omega_1)$.

For the sake of convenience, the conditions needed for our criteria are listed as follows.

- (H₁) $F_n \in C(X, X)$, and there exists $\{u_k\} \subset X$ with $\|u_k\| \rightarrow 0$ such that $F_n(u_k) > \theta$ ($u_k \geq \theta$) for $n = 1, 2, \dots, T$ and $k = 1, 2, \dots$.
- (H₂) $F_n \in C(X, X)$ and $F_n(u) > \theta$ for $u > \theta$ and $n = 1, 2, \dots, T$.
- (L₁) $\lim_{\|u\| \rightarrow 0} \|F_n(u)\| / \|u\| = \infty$ for $n = 1, 2, \dots, T$.
- (L₂) $\lim_{\|u\| \rightarrow \infty} \|F_n(u)\| / \|u\| = \infty$ for $n = 1, 2, \dots, T$.
- (L₃) $\lim_{\|u\| \rightarrow 0} \|F_n(u)\| / \|u\| = 0$ for $n = 1, 2, \dots, T$.
- (L₄) $\lim_{\|u\| \rightarrow \infty} \|F_n(u)\| / \|u\| = 0$ for $n = 1, 2, \dots, T$.
- (L₅) $\lim_{\|u\| \rightarrow 0} \|F_n(u)\| / \|u\| = l$ for $n = 1, 2, \dots, T$ and $0 < l < \infty$.
- (L₆) $\lim_{\|u\| \rightarrow \infty} \|F_n(u)\| / \|u\| = L$ for $n = 1, 2, \dots, T$ and $0 < L < \infty$.

Now let $\hat{Y}$ be the set of all T -periodic sequences in X , endowed with the usual linear structure and the norm

$$\|u\| = \max_{0 \leq n \leq T-1} \|u_n\|. \tag{10}$$

Then $\hat{Y}$ is a Banach space with cone

$$\Omega = \left\{ u = \{u_n\} \in \hat{Y} : u_n \geq \theta, \|u_n\| \geq \sigma \|u\|, n \in \mathbb{Z} \right\}. \quad (11)$$

Define a mapping $H : \hat{Y} \rightarrow \hat{Y}$ by

$$(Hu)_n = \lambda \sum_{s=n}^{n+T-1} G(n, s) (F_s(u_{s-\tau(s)})), \quad n \in \mathbb{Z}. \quad (12)$$

Then it is easily seen that H is completely continuous on (bounded) subset of Ω , and for $u \in \Omega$,

$$\begin{aligned} \|(Hu)_n\| &\leq \lambda \sum_{s=n}^{n+T-1} \|G(n, s)\| \cdot \|F_s(u_{s-\tau(s)})\| \\ &\leq \lambda M \sum_{s=n}^{n+T-1} \|F_s(u_{s-\tau(s)})\| \end{aligned} \quad (13)$$

so that

$$\|(Hu)_n\| \geq \lambda N \sum_{s=n}^{n+T-1} \|F_s(u_{s-\tau(s)})\| \geq \sigma \|Hu\| \quad (14)$$

That is, $H\Omega$ is contained in Ω .

Lemma 2. Assume that there exist two positive numbers a and b such that $a \neq b$,

$$\max_{0 \leq \|x\| \leq a, 0 \leq n \leq T-1} \|F_n(x)\| \leq \frac{a}{\lambda A}, \quad (15)$$

$$\min_{\sigma b \leq \|x\| \leq b, 0 \leq n \leq T-1} \|F_n(x)\| \geq \frac{b}{\lambda B}, \quad (16)$$

where

$$A = \max_{0 \leq n \leq T-1} \sum_{s=n}^{n+T-1} \|G(n, s)\|, \quad (17)$$

$$B = \min_{0 \leq n \leq T-1} \sum_{s=n}^{n+T-1} \|G(n, s)\|. \quad (18)$$

Then there exists $\bar{u} \in \Omega$ which is a fixed point of H and satisfies $\min\{a, b\} \leq \|\bar{u}\| \leq \max\{a, b\}$.

Proof. Let $\Omega_\xi = \{w \in \Omega \mid \|w\| < \xi\}$. Assume that $a < b$, then, for any $u \in \Omega$ which satisfies $\|u\| = a$, in view of (15), we have

$$\|(Hu)_n\| \leq \left\{ \lambda \sum_{s=n}^{n+T-1} \|G(n, s)\| \right\} \cdot \frac{a}{\lambda A} \leq \lambda A \cdot \frac{a}{\lambda A} = a. \quad (19)$$

That is, $\|Hu\| \leq \|u\|$ for $u \in \partial\Omega_a$. For any $u \in \Omega$ which satisfies $\|u\| = b$, we have

$$\|(Hu)_n\| \geq \left\{ \lambda \sum_{s=n}^{n+T-1} \|G(n, s)\| \right\} \cdot \frac{b}{\lambda B} \geq \lambda B \cdot \frac{b}{\lambda B} = b. \quad (20)$$

That is, we have $\|Hu\| \geq \|u\|$ for $u \in \partial\Omega_b$. In view of Theorem 1, there exists $\bar{u} \in \Omega$, which satisfies $a \leq \|\bar{u}\| \leq b$ such that $H\bar{u} = \bar{u}$. If $a > b$, (19) is replaced by $\|(Hu)_n\| \geq b$ in view of (16) and (20) is replaced by $\|(Hu)_n\| \leq a$ in view of (15). The same conclusion is proved. The proof is complete. $\square$

Theorem 2. Suppose (H_1) , (L_1) , and (L_2) hold. Then for any $\lambda \in (0, \lambda^*)$, (4) has at least two positive periodic solutions, where

$$\lambda^* = \frac{1}{A} \sup_{r>0} \frac{r}{\max_{0 \leq \|u\| \leq r, 0 \leq n \leq T-1} \|F_n(u)\|}. \quad (21)$$

Proof. In view of (H_1) , we can let $q(r) = r / (A \max_{0 \leq \|u\| \leq r, 0 \leq n \leq T-1} \|F_n(u)\|)$. By (L_1) and (L_2) , we see further that $\lim_{r \rightarrow 0} q(r) = \lim_{r \rightarrow \infty} q(r) = 0$. Thus, there exists $r_0 > 0$ such that $q(r_0) = \max_{r>0} q(r) = \lambda^*$. For any $\lambda \in (0, \lambda^*)$, by the intermediate value theorem, there exist $a_1 \in (0, r_0)$ and $a_2 \in (r_0, \infty)$ such that $q(a_1) = q(a_2) = \lambda$. Thus, we have $\|F_n(u)\| \leq a_1 / (\lambda A)$ for $\|u\| \in [0, a_1]$ and $n = 0, 1, 2, \dots, T-1$, and $\|F_n(u)\| \leq a_2 / (\lambda A)$ for $\|u\| \in [0, a_2]$ and $n = 0, 1, 2, \dots, T-1$. On the other hand, in view of (L_1) and (L_2) , we see that there exist $b_1 \in (0, a_1)$ and $b_2 \in (a_2, \infty)$ such that $\|F_n(u)\| / \|u\| \geq 1 / (\lambda \sigma B)$ for $\|u\| \in (0, b_1] \cup [b_2 \sigma, \infty)$. That is, $\|F_n(u)\| \geq b_1 / (\lambda B)$ for $\|u\| \in [b_1 \sigma, b_1]$ and $\|F_n(u)\| \geq b_2 / (\lambda B)$ for $\|u\| \in [b_2 \sigma, b_2]$. An application of Lemma 2 leads to two distinct solutions of (4). $\square$

Theorem 3. Suppose (H_2) , (L_3) , and (L_4) hold. Then for any $\lambda > \lambda^{**}$, (4) has at least two positive periodic solutions, where

$$\lambda^{**} = \frac{1}{B} \inf_{r>0} \frac{r}{\min_{\sigma r \leq \|u\| \leq r, 0 \leq n \leq T-1} \|F_n(u)\|}, \quad (22)$$

and B is defined by (18).

Proof. Let $p(r) = r / (B \min_{\sigma r \leq \|u\| \leq r, 0 \leq n \leq T-1} \|F_n(u)\|)$. Clearly, $p \in C((0, \infty), (0, \infty))$. From (L_3) and (L_4) , we see that $\lim_{r \rightarrow 0} p(r) = \lim_{r \rightarrow \infty} p(r) = \infty$. Thus, there exists $r_0 > 0$ such that $p(r_0) = \min_{r>0} p(r) = \lambda^{**}$. For any $\lambda > \lambda^{**}$, there exist $b_1 \in (0, r_0)$ and $b_2 \in (r_0, \infty)$ such that $p(b_1) = p(b_2) = \lambda$. Thus we have $\|F_n(u)\| \geq b_1 / (\lambda B)$ for $\|u\| \in [\sigma b_1, b_1]$ and $n = 0, 1, \dots, T-1$, and $\|F_n(u)\| \geq b_2 / (\lambda B)$ for $\|u\| \in [\sigma b_2, b_2]$ and $n = 0, 1, \dots, T-1$. On the other hand, in view of (L_3) , we see that there exists $a_1 \in (0, b_1)$ such that $\|F_n(u)\| / \|u\| \leq 1 / (\lambda A)$ for $\|u\| \in (0, a_1]$ and

$n = 0, 1, \dots, T-1$. Thus we have $\|F_n(u)\| \leq a_1/(\lambda A)$ for $0 \leq \|u\| \leq a_1$ and $n = 0, 1, \dots, T-1$. In view of (L_4) , we see that there exists $a \in (b_2, \infty)$ such that $\|F_n(u)\|/\|u\| \leq 1/(\lambda A)$ for $\|u\| \in (a, \infty)$ and $n = 0, 1, \dots, T-1$. Let $\delta = \max_{0 \leq \|u\| \leq a, 0 \leq n \leq T-1} \|F_n(u)\|$. Then we have $\|F_n(u)\| \leq a_2/(\lambda A)$ for $\|u\| \in [0, a_2]$ and $n = 0, 1, \dots, T-1$, where $a_2 > a$ and $a_2 \geq \lambda \delta A$. An application of Lemma 2 leads to two distinct solutions of (4). $\square$

Theorem 4. Assume that (H_2) , (L_5) , and (L_6) hold. Then, for each λ satisfying

$$\frac{1}{\sigma B L} < \lambda < \frac{1}{A l} \quad (23)$$

or

$$\frac{1}{\sigma B l} < \lambda < \frac{1}{A L}, \quad (24)$$

equation (4) has a positive periodic solution.

Proof. Suppose (23) holds. Let $\varepsilon > 0$ such that

$$\frac{1}{\sigma B(L - \varepsilon)} \leq \lambda \leq \frac{1}{A(l + \varepsilon)}. \quad (25)$$

Note that $l > 0$, then there exists $H_1 > 0$ such that $\|F_n(u)\| \leq (l + \varepsilon)\|u\|$ for $0 < \|u\| \leq H_1$ and $n = 0, 1, \dots, T-1$. So, for $u \in \Omega$ with $\|u\| = H_1$, we have

$$\begin{aligned} \|(Hu)_n\| &\leq \lambda(l + \varepsilon) \sum_{s=n}^{n+T-1} \|G(n, s)\| \cdot \|u_{s-\tau(s)}\| \\ &\leq \lambda(l + \varepsilon)\|u\| \sum_{s=n}^{n+T-1} \|G(n, s)\| \\ &\leq \lambda A(l + \varepsilon)\|u\| \leq \|u\|. \end{aligned} \quad (26)$$

Next, since $L > 0$, there exists a $\overline{H}_2 > 0$ such that $\|F_n(u)\| \geq (L - \varepsilon)\|u\|$ for $\|u\| \geq \overline{H}_2$ and $n = 0, 1, \dots, T-1$. Let $H_2 = \max\{2H_1, \overline{H}_2\}$. Then for $u \in \Omega$ with $\|u\| = H_2$,

$$\begin{aligned} \|(Hu)_n\| &\geq \lambda(L - \varepsilon) \sum_{s=n}^{n+T-1} \|G(n, s)\| \cdot \|u_{s-\tau(s)}\| \\ &\geq \lambda(L - \varepsilon)\sigma\|u\| \sum_{s=n}^{n+T-1} \|G(n, s)\| \\ &\geq \lambda(L - \varepsilon)\sigma B\|u\| \geq \|u\|. \end{aligned} \quad (27)$$

In view of Lemma 1, we see that (4) has a positive periodic solution.

The other case is similarly proved. $\square$

Our Theorems 1–4 generalize the main results from [5, 6].

If $T = 2$, X is a Hilbert space, A_0, A_1 , and $A_0^{-1}A_1^{-1} - I$ are invertible self-conjugate operator defined on X , $A_0A_1, (A_0^{-1}A_1^{-1} - I)A_0, (A_0^{-1}A_1^{-1} - I)A_1$ are self-conjugate operator defined on X , then A_0, A_1 satisfy conditions of this paper.

As an example, let both $\{\lambda_n\}$ and $\{\lambda'_n\}$ be real bounded sequence, $\{\mu_n\}$ and $\{\mu'_n\}$ are also real bounded sequence, where

$$\mu_n = \begin{cases} \frac{1}{\lambda_n}, & \lambda_n \neq 0, \\ 0, & \lambda_n = 0, \end{cases} \quad \mu'_n = \begin{cases} \frac{1}{\lambda'_n}, & \lambda'_n \neq 0, \\ 0, & \lambda'_n = 0. \end{cases} \quad (28)$$

$\{e_n\}$ is complete orthonormal set of space $l^2 : e_n = \{0, \dots, 0, \overset{(n)}{1}, 0, \dots, 0\}$ ($n = 1, 2, \dots$). Let

$$A_0x = \sum_{n=1}^{\infty} \xi_n \lambda_n e_n, \quad A_1x = \sum_{n=1}^{\infty} \xi_n \lambda'_n e_n \quad (29)$$

for any $x = \sum_{n=1}^{\infty} \xi_n e_n$, then A_0 and A_1 are both self-conjugate operator, and satisfy all of above conditions.

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