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Research Article

Market Dynamics When Agents Anticipate Correlation Breakdown

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The aim of this paper is to analyse the effect introduced in the dynamics of a financial market when agents anticipate the occurrence of a correlation breakdown. What emerges is that correlation breakdowns can act both as a consequence and as a triggering factor in the emergence of financial crises rational bubbles. We propose a market with two kinds of agents: speculators and rational investors. Rational agents use excess demand information to estimate the variance-covariance structure of assets returns, and their investment decisions are represented as a Markowitz optimal portfolio allocation. Speculators are uninformed agents and form their expectations by imitative behavior, depending on market excess demand. Several market equilibria result, depending on the prevalence of one of the two types of agents. Differing from previous results in the literature on the interaction between market dynamics and speculative behavior, rational agents can generate financial crises, even without the speculator contribution.

1. Introduction

This paper is concerned with a dynamic model of market behavior. Several authors have analyzed market dynamics focusing on different frameworks such as agent utility, herding or asymmetries in the information set (see, e.g., the review in [1]). In many examples such models can explain how markets can collapse and then eventually revert to normal conditions. During financial crises an often debated issue is the one known as "correlation breakdown," that is, a sudden change in the correlation of the structure of financial assets returns resulting in a dramatic loss of the original diversification properties of portfolios. This topic is therefore remarkably relevant to the industry of managed funds.

Evidence on varying correlation between asset returns has been reported and analyzed in different studies. Examples of this literature are the works of the authors of [2, 3], who found evidence of an increase in the correlation of stock returns at the time of the 1987 crash. Also, the work in [4] reports correlation shifts during the Mexican crisis while, [5] finds

significant increases in correlation for several East Asian markets and currencies during the East Asian crisis. In [6] the origin of the Russian default in August 1998 has been identified in the "breakdowns of historical correlations." Factors influencing joint movements in the US-Japan markets are identified by [7] using regression methods.

Early analysis on crisis and correlation breakdown include also [8–10] who studied models based on extreme value theory while others, like [11–13], explored Markov switching models. To accommodate structural breaks in the variance of asset returns, in [14] the authors examine the potential for extreme comovements via a direct test of the underlying dependence structure.

In this paper we analyze a market with two kinds of agents: uninformed speculators and informed rational investors. We model rational investment decision as an optimal portfolio allocation in a Markowitz sense. However differently from usual CAPM assumptions, rational agents use excess demand information to estimate next period variance-covariance structure of traded assets returns. We show how such a (rational) anticipatory stance can drive the market to conditions where correlation breakdown even self-reinforces. Our model can explain several market dynamics, including market crashes, creation of rational bubbles, or cycles of diverse periods. These different results will depend on the initial conditions and some market characteristics, such as the percentage composition of the market between rational and irrational agents or their attitude to respond more or less aggressively to shocks in the excess demand.

Differing from previous results which appeared in the literature on the interaction between market dynamics and speculative behavior, we show that rational agents can generate financial crises, even without the "help" of speculators.

Financial research has already tried to address the origin of financial crises to "contagion" mechanisms (see, e.g., [15]). While this paradigm helps to explain important dynamics of financial markets such as financial crises and speculative bubbles, it tends (with some exceptions, e.g., [16]) to interpret these two phenomena as symmetric results of the same price formation process. Indeed Lux [17] defines the probabilities of becoming optimistic from a pessimistic stance in a symmetric way, and consequently also the switching from bear to bull market follow a symmetric contagion process; in [18] the authors model a financial market where both bubbles or crises emerge as a consequence of different initial conditions through the same price formation process. In a market composed by band-wagon speculators and fundamentalists, [19] also develops a market where the investment attitude waves symmetrically from bear to bull market. However, there are well known reasons evidencing that such a symmetry is not realistic. Risk aversion theory as well as several results in behavioral finance (e.g., [20, 21]) show that investment decisions are affected asymmetrically by losses and gains opportunities. Empirical researches exist (e.g., [22–24]) showing that bear market periods tend to follow different dynamics than bull market periods.

In this work we model speculators of both "momentum" and "contrarian" types. They are subject to contagion mechanism, as their demand depends on market excess demand. However, also rational agents are somehow subject to contagion in this model, as they use information on excess demand to update their estimation of the variance-covariance structure of traded asset. They do not use it to update the returns expectations.

In the setting of this work we also obtain a symmetric origin for crisis and booming market when speculators dominate the market. However, when rational agents are prominent, we show how they can generate a stable nonfundamental equilibrium, with prices steady below their "true" values, which is asymmetric in the sense that it does not have a mirroring bubble as a counterpart.

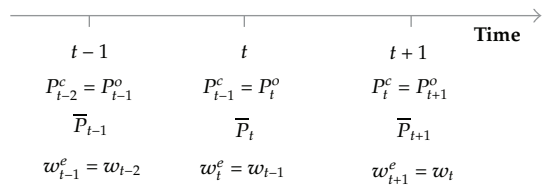


Figure 1: Frame of the discrete model.

The paper is organized as follows. Section 2 introduces the dynamic model of a two-assets financial market. Section 3 solves the optimization problem for a Markowitz portfolio where the variance-covariance matrix depends on time $t-1$ excess demand. Section 4 discusses the fundamental equilibrium of the system as well the non fundamental solutions for three market scenarios: all agents are speculators, all agents are informed rational investors, and the market is composed by a mix of these types of agents. Section 5 concludes the paper.

2. Market Description

We consider a market composed by two kinds of agents: informed rational investors and uninformed speculators. The relevant difference between the two kinds is that uninformed agents base their investment decision through an imitative behavior (also called herding) while informed ones follow a rational portfolio strategy based on an updated information of the fundamental value of assets and of the variance-covariance structure of asset returns.

Only two risky assets are traded on the market, a stock (s) and a bond (f), where the former shows more return volatility than the latter. We assume that the bond is available in unbounded quantity, so no excess demand applies to it. Since it cannot generate excess demand, the dynamics of this market will be analyzed observing only the riskier asset. We consider a discrete time version of the model (see Figure 1).

Following this frame at time $t-1$, the closing price of stock (P^c_{t-1}) coincides with opening price at time t (P^o_t). However, to simplify the notation we will use P_t as a shorthand for P^o_t. The fundamental value of the stock ($\overline{P}_t$) is revealed to informed agents at the beginning of each period t. $\overline{P}_t$ can be any process, possibly depending on time. Let $r_{f,t}$ be the expected bond rate of return and $r_{s,t}$ the expected rate of return of the stock. Both rates are expressed per unit time period. Rational investors observe P_t and use their information on $\overline{P}_t$ to update their (conditional) return expectation:

$$r_{s,t} = \ln\left(\frac{P_t + k\left(\overline{P}_t - P_t\right)}{P_t}\right). \tag{2.1}$$

Equation (2.1) describes a mean reverting attitude of informed agents. Their expected returns is positive when current price is less than its fundamental value, and vice versa when it is higher. In the development of this work we let $\overline{P}_t = \overline{P}$. This restriction reduces the generality of the results, in particular it eliminates the random component from the model. However, in this analysis the variety of the initial settings can be taken as the "surprise component" which will trigger different market dynamics and equilibria. The restriction does not alter significantly the main economic features of this model and it simplifies the analytical treatment. Coherently with a world where the fundamental value of the riskier

asset is constant, we can set expected return of the bond equal to zero ($r_{f,t} = 0$). We express as $\Upsilon \in [0,1]$ the market fraction composed of uninformed agents (the complement to unity will consist of informed agents) and $k \in (0,1)$ is a mean reversion speed coefficient. The excess demand for the stock which occurred in period $t-1$ (i.e., w_{t-1}) is taken as the expected excess demand for period t:

$$w_t^e = w_{t-1}. \tag{2.2}$$

Such expectation is relevant to speculative purposes. Technical analysis, through its large variety of rules, is substantially as an attempt to infer excess demand (along with its sign) from the statistical analysis of past prices. Indeed in the real world, financial markets can be expected to take precise directions (either bull or bear) if a significant volume in the excess demand grows (taking one of the two possible signs). Such are the occasions where speculators can profit. We assume that uninformed demand for the riskier asset is driven by speculative motivation and is defined as

$$w_t^{\Upsilon} = \Upsilon \chi_1 \frac{w_t^e}{1 + |w_t^e|}, \tag{2.3}$$

where $\chi_1 \in \mathbb{R} - \{0\}$ is a sensitivity parameter. Linking current excess demand to its expectation is a classical way to model a contagion mechanism (e.g., [18]). We do not specify how uninformed agents obtain an estimate of w_t^e (it can be argued that some popular methods based on the observation of past prices such as chart or technical analysis are adopted to this purpose), nor dos we give details on the mechanism translating those estimates into an investment decision. However, the overall result of such process is synthesized through (2.3), where the higher the expected excess demand, the higher (in absolute value) the excess demand which really occurs. Depending on the sign and the value of χ_1 we can classify the overall population of uniformed agents as momentum ($\chi_1 > 0$) or contrarian ($\chi_1 < 0$).

Turning to informed agents, we develop in what follows a model for their portfolio optimization. Letting q_t^R and $1 - q_t^R$ the time t weights of the stock and the bonds, respectively, we specify the following equation for the rational excess demand of the risky asset:

$$w_t^R = (1 - \Upsilon)\chi_2 \left(q_t^R(w_t^e, r_{s,t-1}) - q_t^R(0, r_{s,t-1}) \right), \tag{2.4}$$

where $\chi_2 \in \mathbb{R} - \{0\}$ is a sensitivity parameter for the rational demand and the expression $q_t^R(w_t^e, r_{s,t-1})$ shows the dependence on the expected return and the excess demand of the equity. Equation (2.4) tells us that the excess demand generated by rational agents is a (linear) function of the difference between $q_t^R(w_t^e, r_{s,t-1})$ and $q_t^R(0, r_{s,t-1})$, where the latter is the quantity held by a rational investor in the absence of any excess demand.

Summing up w_t^{Υ} and w_t^R we obtain the expression of the market excess demand:

$$\begin{aligned} w_t &= w_t^{\Upsilon} + w_t^R \\ &= \Upsilon \chi_1 \frac{w_t^e}{1 + |w_t^e|} + (1 - \Upsilon)\chi_2 \left(q_t^R(w_t^e, r_{s,t-1}) - q_t^R(0, r_{s,t-1}) \right). \end{aligned} \tag{2.5}$$

We can now discuss price dynamics. Time t actual return of the risky asset is modeled as

$$\Delta p_t = \ln\left(\frac{P_{t-1} + k\left(\overline{P} - P_{t-1}\right)}{P_{t-1}}\right) + \lambda w_{t-1}, \tag{2.6}$$

where $\Delta p_t = \ln(P_t/P_{t-1})$ represents the logarithmic return of the price, $\lambda > 0$ is a reaction coefficient of price to excess demand. Equation (2.6) models price dynamics as a combination of two components: the first is linked to the fundamental value of the stock and it is driven by expectation of rational informed agents, the second is the influence of excess demand. The case $\lambda = 0$ implies that excess demand does not affect (future) prices. The informational driver and the herd behavior driver in (2.6) will dominate one over the other depending not only on the direct effect of the coefficients k and λ. Consider, for example, a market condition where at a given time t we observe high prices ($P_t > \overline{P}$) and positive excess demand ($w_t > 0$). If the irrational investors dominates the market (i.e., Y tends to 1) and they applies aggressive momentum strategies ($\chi_1 > 1$), the second component in (2.6) will sustain inflation of P, and it will possibly dominate over the information driver which always acts as a mean reverting of the stock price.

Notice that time t expected return of the stock is calculated by rational agents through (2.1) leveraging on the information of the fundamental value $\overline{P}$. Such expectation (2.1) will not (in general) be equal to actual time t return (2.6). In other words rational investors cannot be perfect price forecasters.

Denoting $q_t^R(0, r_{s,t-1})$ as $\tilde{q}_t$, we are now able to specify a dynamic model for the price of the risky asset:

$$\begin{aligned} w_t &= Y\chi_1 \frac{w_{t-1}}{1 + |w_{t-1}|} + (1 - Y)\chi_2\left(q_t^* - \tilde{q}_t\right), \\ q_t^* &= \arg\max g(w_{t-1}, r_{s,t-1}), \\ P_t &= \left[P_{t-1} + k\left(\overline{P} - P_{t-1}\right)\right] \exp \lambda w_{t-1}, \\ r_{s,t} &= \ln\left(\frac{P_t + k\left(\overline{P} - P_t\right)}{P_t}\right), \end{aligned} \tag{2.7}$$

where g is a function depending on Markowitz efficient portfolios.

3. Rational Agent Optimization

Rational agents form their portfolio at time t optimizing the following performance indicator, which is closely related to the Sharpe ratio:

$$g = \frac{E[r_\pi]}{\mathrm{Var}[r_\pi]}, \tag{3.1}$$

where r_π is the return of a portfolio. The equivalence of the performance indicator g to the Sharpe ratio [25] is clear: the variance of portfolio π is used instead of its standard deviation. As it is known in the literature (e.g., [26, page 626]), the indicators of the type as g in (3.1) show larger values for portfolios which are mean-variance efficient. It can be shown that an optimal Sharpe ratio portfolio is also Markowitz efficient. To simplify the notation, next we will denote the one period expected rate of return of the riskier asset $r_{s,t-1}$ as r_s and r_f instead of $r_{f,t-1}$ for that of the bond whenever this will not generate confusion.

Based on standard portfolio theory, such objective can be expressed as the search of an optimal weight vector q_t^* satisfying:

$$q_t^* = \arg\max g = \arg\max \frac{\mathbf{q}_t^T \mathbf{r}}{\mathbf{q}_t^T V_{t-1} \mathbf{q}_t}, \tag{3.2}$$

where $\mathbf{r}^T = [r_s \quad r_f]$ is the vector of stock and bond portfolio expected returns, $\mathbf{q}_t^T = [q_t \quad 1-q_t]$ the vector of their percentage weights, V_{t-1} is the variance-covariance estimated matrix at time $t-1$.

This paper proposes contagion as the baseline factor to generate a correlation breakdown of assets returns. However, contagion is at the origin of other local changes in the behavior of prices of financial assets, as it has been variously discussed in the literature [8–10]. A first impact is the emergence of rational bubbles (or crashes) in the markets, where prices follow evident climbing (or downhill) trends, which can been explained by a growing "blind" consensus about the continuation of the going pattern. As long as such common belief extends to other investors it self-realizes, as new buying (or selling) orders will extend in time the bull (or the bear) phase. A second potential impact, which has received less attention in the literature, is that on the variance of returns. Following logical arguments, growing consensus is equivalent of a spreading common vision in the market. If a natural explanation for the variance of returns is nonhomogeneity of agents' beliefs, then markets are expected to show decreasing variance of returns when consensus is spreading, such as during marked bull (or bear) periods. Indeed, also from a mathematical point of view, given two sequences of returns with the same absolute values, they will show a lower variance when they have the same sign than in the case where their sign changes randomly. A persistent prevalence of a sign in the returns is exactly what can be observed during bull or bear periods.

To summarize these facts, in this paper we assume that rational agents expect that when the excess demand (positive or negative) increases:

(i) the variance of returns decreases;

(ii) the correlation of the two assets tends to unity.

In particular they use the following functions to estimate the variance (v) and the correlation (ρ) as functions of the excess demand estimated at time $t-1$:

$$\begin{aligned} v_{t-1} &= \alpha^2 e^{-2\mu w_{t-1}^2}, \\ \rho_{t-1} &= -e^{-\mu w_{t-1}^2} + 1, \end{aligned} \tag{3.3}$$

where $\alpha \geq 0$ is a scale parameter of variance and $\mu \geq 0$ is a sensitivity parameter mitigating or reinforcing the relevance of a contagion mechanism in a given market. When $\mu = 0$ the correlation does not depend anymore on the excess demand and it takes its natural value

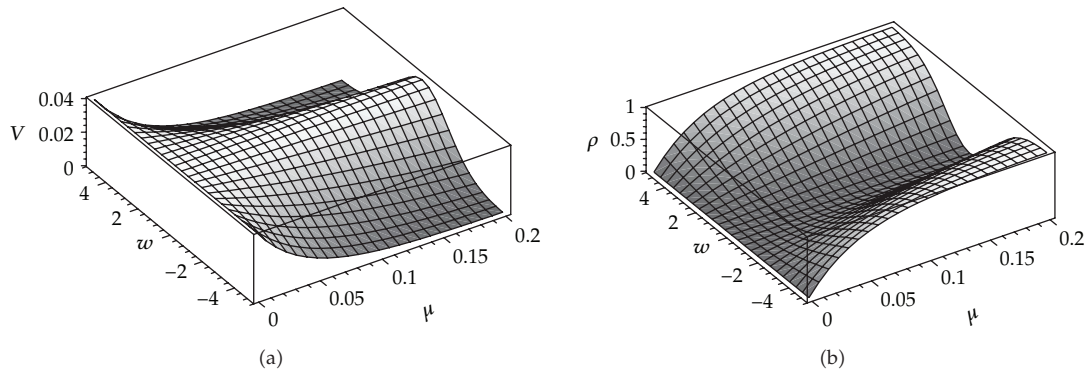


Figure 2: Graph of the variance (a) and of the correlation (b) as a function of excess demand and parameter μ.

(i.e., the one in force under normal regime), which we assume to be zero for the two assets of our model (see Figure 2).

We obtain the following model for V_{t-1}:

$$V_{t-1} = \begin{bmatrix} \alpha_1^2 e^{-2\mu w_{t-1}^2} & \alpha_1\alpha_2 e^{-2\mu w_{t-1}^2}\rho_{t-1} \\ \alpha_1\alpha_2 e^{-2\mu w_{t-1}^2}\rho_{t-1} & \alpha_2^2 e^{-2\mu w_{t-1}^2} \end{bmatrix}. \tag{3.4}$$

In general the portfolio variance in (3.2) is a risk measure depending negatively on the absolute value of excess demand.

The optimization problem in (3.2), where V_{t-1} is specified as in (3.4), has an explicit solution (details are given in the appendices) depending on r_s and w_{t-1}:

$$q_t^*(w_{t-1}, r_s) = \begin{cases} -\dfrac{r_f}{r_s - r_f} + \dfrac{\sqrt{\left(\alpha_1^2 + \alpha_2^2 + 2\alpha_1\alpha_2\left(e^{-\mu w_{t-1}^2} - 1\right)\right)\left(\left(r_s\alpha_2 - r_f\alpha_1\right)^2 + 2\alpha_1\alpha_2 r_f r_s e^{-\mu w_{t-1}^2}\right)}}{\left(r_s - r_f\right)\left(\alpha_1^2 + \alpha_2^2 - 2\alpha_1\alpha_2\left(-e^{-\mu w_{t-1}^2} + 1\right)\right)} & \text{if } r_s \neq r_f, \\ \dfrac{\alpha_1\alpha_2\left(-e^{-\mu w_{t-1}^2} + 1\right) - \alpha_2^2}{2\alpha_1\alpha_2\left(-e^{-\mu w_{t-1}^2} + 1\right) - \alpha_1^2 - \alpha_2^2} & \text{if } r_s = r_f. \end{cases} \tag{3.5}$$

Figure 3 plots the solution (3.5) for some values of the other parameters.

Recalling that $\tilde{q}_t = q_t^R(0, r_s)$, from the expression (3.5) the value of $\tilde{q}_t$ can be easily obtained as

$$\tilde{q}_t = -\frac{r_f}{r_s - r_f} + \frac{\sqrt{r_s^2\alpha_2^2 + r_f^2\alpha_1^2}}{\left(r_s - r_f\right)\sqrt{\alpha_1^2 + \alpha_2^2}}. \tag{3.6}$$

In Appendix A, we give the mathematical details of the solution to the optimization problem in (3.2) and briefly discuss its properties.

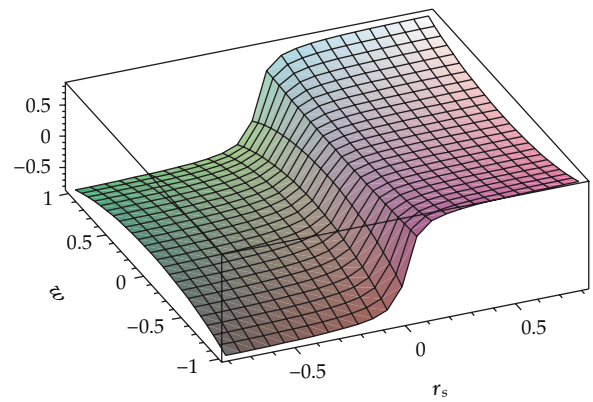


Figure 3: Plot of the solution $q_t^*(w_{t-1},\ r_s)$ with $r_f = 0.025, \alpha_1 = 0.15,$ and $\alpha_2 = 0.1$.

4. Dynamic System

We now observe that the third equation in (2.7) can be written in a more useful expression in terms of $r_{s,t}$. Indeed $r_{s,t} = \ln((P_t + k(\overline{P} - P_t))/P_t)$ implies that $P_t = k\overline{P}/(\exp r_{s,t} + k - 1)$ provided that $\exp r_{s,t} + k - 1 \neq 0$. Such exclusion is economically justified, as it is equivalent to excluding that $\overline{P} = 0$, as it can be easily seen letting $r_{s,t} = \ln(1 - k)$ in (2.1).

The third equation can be rewritten as

$$\frac{k\overline{P}}{\exp r_{s,t} + k - 1} = \left(\frac{k\overline{P}}{\exp r_{s,t-1} + k - 1} + k\left(\overline{P} - \frac{k\overline{P}}{\exp r_{s,t-1} + k - 1}\right)\right)\exp \lambda w_{t-1}, \tag{4.1}$$

yielding

$$\frac{1}{\exp r_{s,t} + k - 1} = \frac{\exp(r_{s,t-1} + \lambda w_{t-1})}{\exp r_{s,t-1} + k - 1}, \tag{4.2}$$

and finally

$$r_{s,t} = \ln\left(\frac{\exp r_{s,t-1} + k - 1}{\exp(r_{s,t-1} + \lambda w_{t-1})} - (k - 1)\right). \tag{4.3}$$

Putting together (2.5), (3.5), and (4.3), the evolution of the dynamic variables $w_t, r_{s,t}$ and q_t^* is described by a three-dimensional discrete dynamic system:

$$\begin{aligned} w_t &= \Upsilon \chi_1 \frac{w_{t-1}}{1 + |w_{t-1}|} + (1 - \Upsilon)\chi_2(q_t^* - \tilde{q}_t), \\ q_t^* &= \arg\max g(w_{t-1}, r_{s,t-1}), \\ r_{s,t} &= \ln\left(\frac{\exp r_{s,t-1} + k - 1}{\exp(r_{s,t-1} + \lambda w_{t-1})} - (k - 1)\right). \end{aligned} \tag{4.4}$$

At time t, starting from $r_{s,t-1}$ and w_{t-1}, the third equation supplies the return expected by rational agents at time t for the risky asset whereas the second one gives the optimal rational holdings q_t^*. Finally we find w_t by the first equation. Given the new values w_t and $r_{s,t}$ the system can be iterated. Since q_t^* is known given $r_{s,t-1}$ and w_{t-1}, we can eliminate the second equation (which we analytically solve in Section 3) and finally consider a two-dimensional map $(w_{t-1}, r_{s,t-1}) \to (w_t, r_{s,t})$ defined as

$$\begin{aligned} w_t &= \Upsilon \chi_1 \frac{w_{t-1}}{1+|w_{t-1}|} + (1-\Upsilon)\chi_2\left(q_t^* - \tilde{q}_t\right), \\ r_{s,t} &= \ln\left(\frac{\exp r_{s,t-1} + k - 1}{\exp(r_{s,t-1} + \lambda w_{t-1})} - (k-1)\right), \end{aligned} \tag{4.5}$$

which will generate different evolution of the system depending both on the coefficients and the initial condition $(w_0, r_{s,0})$. The coefficients of the system (4.5) are $\chi_1, \chi_2, \Upsilon, k$, and λ, whose possible values have been already discussed, and, next to this, the coefficient μ influencing the optimal portfolio q_t^*.

Our discussion will focus on the influence of several coefficients on the behavior of system (4.5). Besides this we will try to show how some initial conditions (excess demand in particular) will influence the emergence of fundamental and non fundamental equilibria, as well as price orbits.

4.1. Fixed Point Analysis

To simplify notations, let us introduce the unit time advancement operator "′" to reexpress (4.5):

$$\begin{aligned} w' &= \Upsilon \chi_1 \frac{w}{1+|w|} + (1-\Upsilon)\chi_2\left(q^* - \tilde{q}\right), \\ r_s' &= \ln\left(\frac{\exp r_s + k - 1}{\exp(r_s + \lambda w)} - (k-1)\right). \end{aligned} \tag{4.6}$$

The fixed points (w^*, r_s^*) of system (4.6) will be named fundamental equilibria when the condition $P^* = \overline{P}$ is verified, where P^* is the corresponding price to r_s^*. Other equilibria will be named non fundamental.

In the following proposition we show the existence of at least one equilibrium point (fundamental solution) for the system (4.6), given by the fixed points of the map (4.6).

Proposition 4.1. *The point $Q_0 = (w^*, r_s^*) = (0, 0)$ is an equilibrium for the model* (4.6) *for all values of the parameters.*

Proof. The following system is satisfied at the equilibrium:

$$\begin{aligned} w^* &= \Upsilon \chi_1 \frac{w^*}{1+|w^*|} + (1-\Upsilon)\chi_2\left(q^* - \tilde{q}\right), \\ r_s^* &= \ln\left(\frac{\exp r_s^* + k - 1}{\exp(r_s^* + \lambda w^*)} - (k-1)\right), \end{aligned} \tag{4.7}$$

rearranging terms, the second equation becomes:

$$\exp r_s^* + k - 1 = \frac{\exp r_s^* + k - 1}{\exp(r_s^* + \lambda w^*)}, \tag{4.8}$$

yielding

$$\exp(r_s^* + \lambda w^*) = 1. \tag{4.9}$$

Solving such equality, system (4.7) is equivalent to

$$\begin{gathered} w^* = \Upsilon \chi_1 \frac{w^*}{1 + |w^*|} + (1 - \Upsilon)\chi_2(q^* - \tilde{q}), \\ r_s^* = -\lambda w^*. \end{gathered} \tag{4.10}$$

When $w^* = 0$, the first equation is satisfied, since $\tilde{q} = q^R(0, r_s)$ by definition and the second equation yields the solution $r_s^* = 0$ for every $\lambda \neq 0$. This completes the proof. □

Observe that when $P_t = \overline{P}$ then $r_s = 0$, as it can be verified by inspection of (2.1). Rational agents fix to zero the expected return of the risky asset when current price is equal to its fundamental value. So $P_t = \overline{P}$ coupled with $w_t = 0$ and $r_s = 0$ is a fundamental equilibrium solution for (2.7).

When $r_s = r^* \neq 0$ eventual other equilibria have the form $(w^*, -\lambda w^*)$, where the expression of w^* is implicitly described by the first equation in (4.10).

Given that r_s is obtained through a monotone transformation of price P_t, we do not risk losing possible solutions of the original system.

4.2. Local Stability Analysis of the Fundamental Solution $(0, 0)$

4.2.1. The Contagion Effect

In the previous paragraph we have shown that $Q_0 = (w^*, r_s^*) = (0, 0)$ is an equilibrium point for the model (4.6); now we want to study the existence of other equilibria and their stability when all agents act following the market demand ($\Upsilon = 1$). In this case the system (4.6) becomes

$$\begin{gathered} w' = \chi_1 \frac{w}{1 + |w|}, \\ r_s' = \ln\left(\frac{\exp r_s + k - 1}{\exp(r_s + \lambda w)} - (k - 1)\right), \end{gathered} \tag{4.11}$$

as the rational component vanishes. Following the standard dynamic systems theory (see [27]), the local stability analysis of the fixed point is based on the location, in the complex plane, of the eigenvalues of the Jacobian matrix (for this and the other cases we refer to

Appendix B for detailed calculations needed to construct the Jacobian matrix):

$$J(0,0) = \begin{bmatrix} \chi_1 & 0 \\ -k\lambda & 1-k \end{bmatrix}. \tag{4.12}$$

The eigenvalues are $\lambda_1 = \chi_1$ and $\lambda_2 = 1 - k$; observe that λ_2 is always less than 1 in absolute value, given that $k \in (0,1)$ under the hypothesis of our model; then the stability analysis of $(0,0)$ depends on the λ_1 eigenvalue (i.e., on the value of the parameter χ_1).

More precisely, $Q_0 = (0,0)$ is a stable equilibrium if $|\chi_1| < 1$. $\chi_1 = 1$ and $\chi_1 = -1$ are bifurcation values. When $|\chi_1| > 1$ the point Q_0 becomes unstable and different situations can occur depending on cases $\chi_1 > 1$ and $\chi_1 < -1$. When $\chi_1 > 1$, two new equilibria $w_1^* = \chi_1 - 1$ and $w_2^* = -\chi_1 + 1$ appear corresponding to the points $Q_1 = (\chi_1 - 1, -\lambda(\chi_1 - 1))$ and $Q_2 = (-\chi_1 + 1, -\lambda(-\chi_1 + 1))$ in the phase plane (w, r_s), as a consequence of the bifurcation occurring when $\chi_1 = 1$. The nature of this bifurcation can be examined by studying the one-dimensional family of maps $w' = f(w, \chi_1)$, where $f(w, \chi_1) = \chi_1(w/(1 + |w|))$ depending on the parameter χ_1.

At $(0,1)$ we obtain

$$\frac{\partial f(0,1)}{\partial w} = 1, \qquad \frac{\partial^2 f(0,1)}{\partial w \partial \chi_1} = 1, \qquad \frac{\partial^3 f(0,1)}{\partial w^3} = 6, \tag{4.13}$$

so the conditions (4.13) guarantee that $(0,1)$ determines a supercritical pitchfork bifurcation.

A value of $\chi_1 > 1$ shows the tendency of the uninformed agents to overreact to signals about excess demand. When w is just greater than 0, they respond raising next period demand further until level $w_1^* = \chi_1 - 1$ is reached. At that point excess demand stabilizes. At the same time the equity price grows to $P_1^* = k/(e^{-\lambda(\chi_1 - 1)} - 1 + k)\overline{P}$. Since $P_1^* > \overline{P}$, if excess demand is positive the factor $k/(e^{-\lambda(\chi_1 - 1)} - 1 + k)$ is greater than 1; this implies $1 < \chi_1 < ((\lambda - \ln(1-k))/\lambda)$. So a further condition $\chi_1 < ((\lambda - \ln(1-k))/\lambda)$ is also required to guarantee market consistency. Mean reverting expectation of rational investors $r_s^* = -\lambda(\chi_1 - 1)$ is negative when $P_1^* > \overline{P}$. However, price remains high when $Y = 1$ no rational investors are present in the market to balance the positive excess demand generated by the speculators.

On the contrary when excess demand is negative, uninformed agents respond selling even more until the level $w_2^* = 1 - \chi_1$ is reached. Following similar arguments when $w < 0$ an equilibrium price is reached at price $P_2^* = k/(e^{-\lambda(1-\chi_1)} - 1 + k)\overline{P}$ which must be less than $\overline{P}$ under standard market conditions. The inequality $\chi_1 > (\lambda - \ln(1-k))/\lambda$ must be satisfied and corresponding rational expected return $r_s^* = -\lambda(1 - \chi_1)$ is positive.

In order to study the local stability of the new fixed points $Q_1 = (\chi_1 - 1, -\lambda(\chi_1 - 1))$ and $Q_2 = (-\chi_1 + 1, -\lambda(-\chi_1 + 1))$ we compute the eigenvalues of the Jacobian matrix in Q_1 and Q_2.

Being

$$J(Q_1) = \begin{bmatrix} \dfrac{1}{\chi_1^2} & 0 \\ -\lambda + \lambda(k-1)e^{\lambda(\chi_1 - 1)} & (1-k)e^{\lambda(\chi_1 - 1)} \end{bmatrix}, \tag{4.14}$$

the eigenvalues are $\lambda_1 = 1/\chi_1$ and $\lambda_2 = (1-k)e^{\lambda(\chi_1 - 1)}$; as we are examining the case $\chi_1 > 1$, λ_1 is always in absolute value less than 1; $|\lambda_2| < 1$ when $(1-k)e^{\lambda(\chi_1 - 1)} < 1$ that implies $\chi_1 < (\lambda - \ln(k-1))/\lambda$. As we already observed, in the case $w > 0$ this inequality is always satisfied, as coherent with the condition of market consistency.

With similar calculations

$$J(Q_2) = \begin{bmatrix} \dfrac{1}{\chi_1^2} & 0 \\ -\lambda + \lambda(k-1)e^{\lambda(-\chi_1+1)} & (1-k)e^{\lambda(-\chi_1+1)} \end{bmatrix}, \tag{4.15}$$

with eigenvalues $\lambda_1 = 1/\chi_1$ and $\lambda_2 = (1-k)e^{\lambda(-\chi_1+1)}$; the first one is always less than 1 in absolute value whereas $|\lambda_2| < 1$ when $(1-k)e^{\lambda(-\chi_1+1)} < 1$, which implies $\chi_1 > (\lambda - \ln(1-k))/\lambda$ that is always satisfied.

Points Q_1 and Q_2 correspond to two nonfundamental asymptotically stable equilibria.

Finally, if $\chi_1 = -1$ a flip bifurcation occurs and when $\chi_1 < -1$ a two-period cycle appears in the phase plane (w, r_s), whose elements are $\{w_3^*, w_4^*\}, w_3^* = \chi_1 + 1, w_4^* = -\chi_1 - 1$. These elements can be identified through a double iteration of the system (4.11):

$$\begin{aligned} w'' &= \chi_1^2 \frac{w}{(1 + |\chi_1(w/(1+w))|)(1+|w|)} \\ r_s'' &= \ln\left(\frac{(\exp r_s + k - 1)/\exp(r_s + \lambda w)}{((\exp r_s + k - 1)/\exp(r_s + \lambda w) - (k-1))\exp \lambda\chi_1(w/(1+|w|))} - (k-1) \right), \end{aligned} \tag{4.16}$$

yielding two fixed points $Q_3 = (-\chi_1 - 1, \ln((1 + (1-k)e^{\lambda(\chi_1+1)})/(e^{\lambda(\chi_1+1)} + 1 - k)))$ and $Q_4 = (\chi_1 + 1, \ln((1 + (1-k)e^{-\lambda(\chi_1+1)})/(e^{-\lambda(\chi_1+1)} + (1-k))))$.

As a consequence of the contrarian attitude of this market ($\chi_1 < 0$), positive excess demand in period t turns into negative demand in period $t+1$. In particular the values of the excess demand orbit are $w_3^* = -\chi_1 - 1$ and $w_4^* = \chi_1 + 1$. The pressure on the price will accordingly wave from up and down. Corresponding market prices oscillate between two values $P_3 < \overline{P} < P_4$. In particular we have

$$\begin{aligned} P_3 &= \frac{e^{\lambda(\chi_1+1)} + (1-k)}{2-k}\overline{P} < \overline{P}, \\ P_4 &= \frac{e^{-\lambda(\chi_1+1)} + 1 - k}{2-k}\overline{P} > \overline{P}. \end{aligned} \tag{4.17}$$

Appendix C shows all required calculations and other details of such results.

The analysis of the two-period orbit stability is not an easy task, as must be done by studying the stability of fixed points Q_3 and Q_4; however, simulation analysis reveals a stable orbit (see Figures 4 and 5).

Figure 6 shows the equilibria stability in an example with different values of parameters χ_1. Figure 6(a) shows the graph of $f(w, \chi_1)$ for two values of χ_1; from $\chi_1 = 0.8$ to $\chi_1 = 4$ a Pitchfork bifurcation occurs. Figure 6(b) shows the graph of $f(w, \chi_1)$ for two values of χ_1 (i.e., -0.8 and -4) and the equilibrium values w^* included in the two-periods orbit. The convergence path is clearly pointing to the origin when $\chi_1 = -0.8$; vice versa it draws a cycle of period two when $\chi_1 = -4$.

We summarize the results of this section in Table 1.

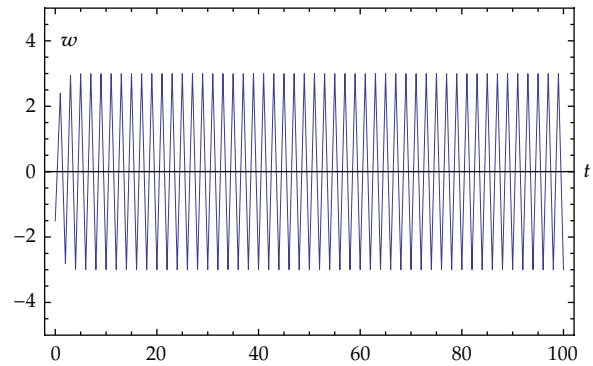


Figure 4: Trajectory of excess demand w_t when $\chi_1 = -4$.

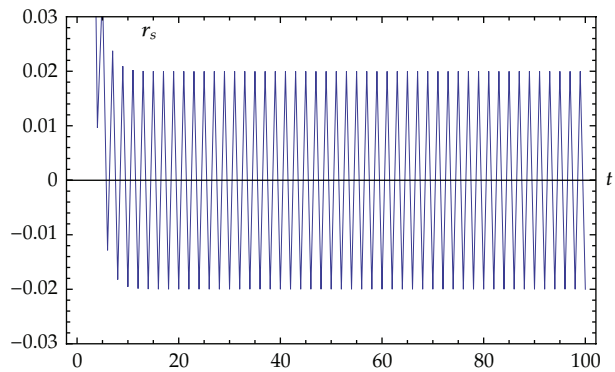


Figure 5: Trajectory of expected return r_s when $\chi_1 = -4$.

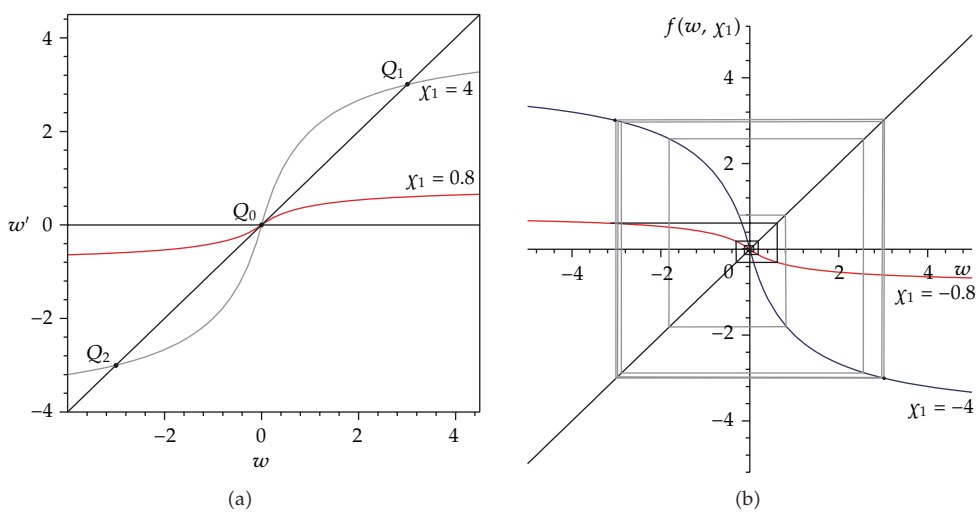


Figure 6: All investors uninformed scenario-equilibria. (a) $\chi_1 > 1$, (b) $\chi_1 < -1$.

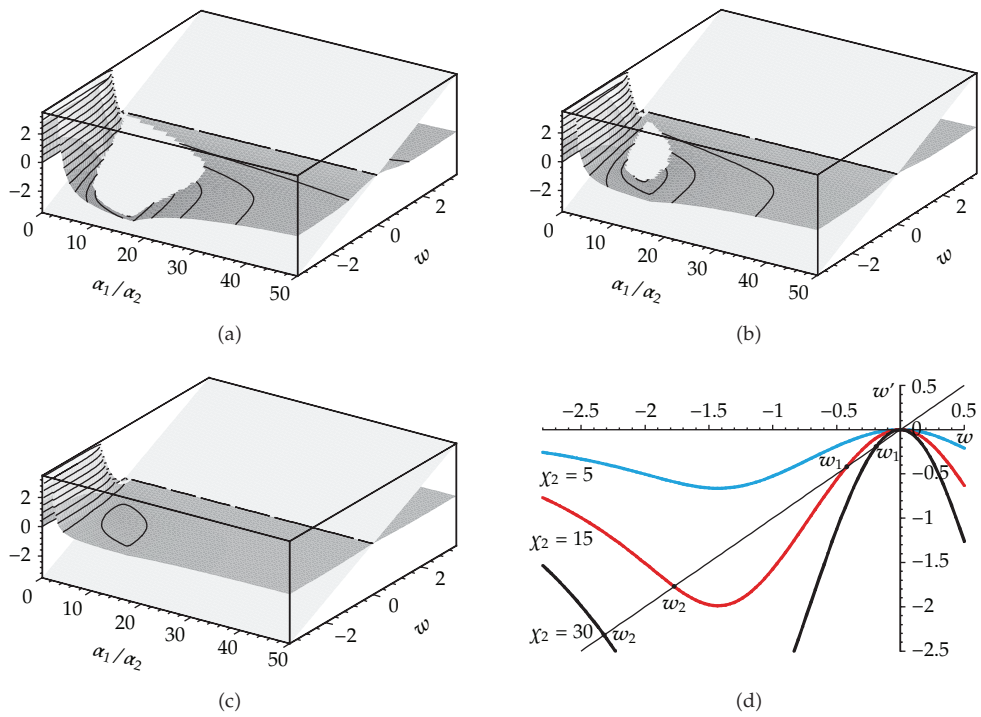


Figure 7: Fixed points in a rational market, for different values of parameter χ_2: 30 in (a), 15 in (b), and 5 in (c).

Table 1: Fixed point solutions depending on parameter χ_1.

Case	Asymptotic stability	Equilibrium price
$\chi_1 > 1$	$Q_0 = (0,0)$ unstable	$P_0 = \overline{P}$ unstable
	$Q_1 = (\chi_1 - 1, -\lambda(\chi_1 - 1))$ stable	$P_1 > \overline{P}$ stable
	$Q_2 = (-\chi_1 + 1, -\lambda(-\chi_1 + 1))$ stable	$P_2 < \overline{P}$ stable
$-1 \le \chi_1 \le 1$	$Q_0 = (0,0)$ stable	$P_0 = \overline{P}$ stable
$\chi_1 < -1$	2-period orbit $\{w_3^*, w_4^*\}$, $w_3^* = -\chi_1 - 1, w_4^* = \chi_1 + 1$	2-period orbit (P_3, P_4)

4.2.2. The Rational Effect

Let us suppose now that all the investors act rationally ($Y = 0$); in this case system (4.6) becomes

$$w' = \chi_2(q^* - \tilde{q}),$$
$$r_s' = \ln\left(\frac{\exp r_s + k - 1}{\exp(r_s + \lambda w)} - (k-1)\right), \tag{4.18}$$

where q^* is calculated as in (3.5) and $\tilde{q}$ as in (3.6).

The Jacobian matrix in $(0,0)$ is

$$J(0,0) = \begin{bmatrix} 0 & 0 \\ -k\lambda & 1-k \end{bmatrix}, \tag{4.19}$$

whose eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = 1 - k$. Since under standard assumptions $k \in (0,1)$, the eigenvalues are always less than 1 in absolute value, the solution $(0,0)$ is always asymptotically locally stable for every value of the parameter χ_2.

This can be easily seen also from system (4.18). When all rational investors start from a zero excess demand ($w_{t-1} = 0$) their optimal demand for the risky assets is $q^* = \tilde{q}$ (by the definition of $\tilde{q}$). As a consequence next period excess demand is again $w_t = 0$.

Inspecting again system (4.18) we obtain that $r_{s,t} = 0$, which implies that current price of the risky asset is equal to its fundamental value ($P_t = \overline{P}$).

Outside the fundamental solution the inspection of other fixed points is by far less simple. Actually some simulation and numerical analysis can be the only way to inspect other possible equilibria resulting in a market fully populated by informed agents. Figure 7 represents w' as a function of w and the parameter α_1, that is, the one-dimensional family of maps $w' = g(w, \alpha_1)$. In all three panels the plane $w' = w$ always intersects $g(w, \alpha_1)$ at $w = 0$, that is corresponding to the fundamental solution. However it is also possible to observe that for values of parameter α_1 less than about 0.05 (panel (a)) and 0.025 (panel (b)), $g(w, \alpha_1)$ intersects the plane $w' = w$ in other two points, call them w_1 and w_2. Corresponding to such intersections, we see in Figures 7(a) and 7(b), that the graph of $g(w, \alpha_1)$ "vanishes" below the bisecting plane $w' = w$. See also Figure 7(d) where these intersections (w_1 and w_2) are also shown for a fixed value of parameter α_1. For simple analytical properties of $g(w, \alpha_1)$ points w_1 can never be a stable solution. On the contrary w_2 can be both a stable or unstable solution. In the cases represented in Figure 7(d), where $\chi_2 = 30$ and $\chi_2 = 15$ we can observe that w_2 is, respectively, an unstable and stable solution.

Besides a stable non equilibrium fixed point, the full rational scenario can also generate stable orbits of various periods. The occurrence of such a situation can be attributed to particular combinations of the parameters and the initial values of the system. Markets can be hit occasionally by unexpected good (bad) news, shifting the fundamental value of the risky asset and attracting (chasing away) new significant portions of demand. Rational traders can then try to anticipate a possible demand rush and change the correlation estimate based on (3.3). Depending on the values of some parameters, even a market dominated by rational investors can be captured into dynamics keeping the system steadily out of equilibrium. Paradoxically at the origin of such imbalance is a "rational" yet myopic intent to prevent it. Indeed each agent optimizes rationally a private portfolio problem, without considering that many individuals, with an identical intent, are jointly swelling the order book on the same side.

In Figure 8 panels (a.1) and (a.2) show the trajectories of stable orbits of period 5. In particular panel (a.1) shows the corresponding time series of r_s and w while in panel (a.2) the trajectories are plotted in the phase plane. The emergence of orbits tend to occur especially when coefficient χ_2 is high (i.e., high impact on demand as a consequence of portfolio adjustments), when the ratio $\alpha_1/\alpha_2 \downarrow 1$ (i.e., small difference in volatility between the two traded assets), and coefficient μ is high (i.e., the correlation breakdown effect increases).

When χ_2 assumes intermediate values such as 20 and the ratio α_1/α_2 lays in an interval similar to that already discussed in Figure 7, the system allows the emergence of two stable equilibria (see Figure 8 panels (b.1) and (b.2)): one is the fundamental solution ($w = 0, r_s = 0, P = \overline{P}$), the other is a non-fundamental equilibrium (with $w < 0, r_s > 0$ and $P < \overline{P}$). Finally if the influence parameter is low enough such as $\chi_2 = 5$ (see Figure 8(c)), then the only fixed point appears to be the fundamental solution whatever the starting point.

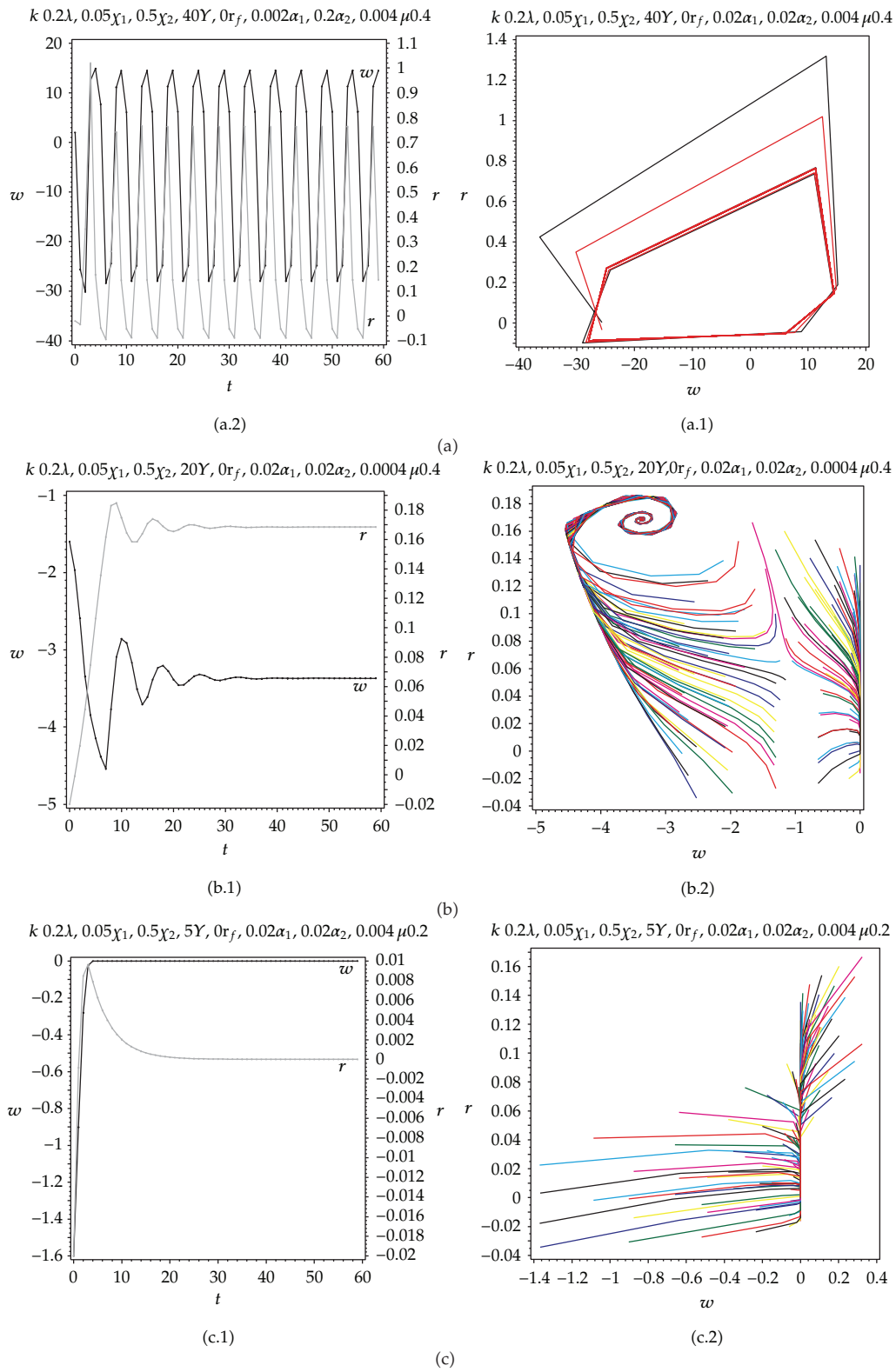


Figure 8: Trajectories w_t (excess demand) for $\chi_2 = 30$ (a), $\chi_2 = 15$ (b), and $\chi_2 = 5$ (c).

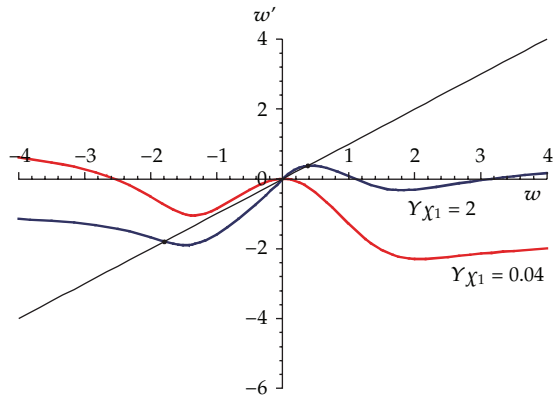


Figure 9: Pitchfork bifurcation: when $\Upsilon\chi_1 > 1$ $(0, 0)$ becomes unstable and two new fixed points appear.

4.2.3. Mixed Rational and Speculators Market

In the more general case the market is composed of a positive percentage of both informed and uninformed agents; in this case the market dynamics are described by the following system:

$$\begin{aligned} w' &= \Upsilon\chi_1 \frac{w}{1+|w|} + (1-\Upsilon)\chi_2\left(q_t^* - \tilde{q}\right), \\ r_s' &= \ln\left(\frac{\exp r_s + k - 1}{\exp(r_s + \lambda w)} - (k-1)\right). \end{aligned} \tag{4.20}$$

The Jacobian matrix in $(0,0)$ is

$$J(0,0) = \begin{bmatrix} \Upsilon\chi_1 & 0 \\ -k\lambda & 1-k \end{bmatrix}, \tag{4.21}$$

with eigenvalues $\lambda_1 = \Upsilon\chi_1$, $\lambda_2 = 1 - k$.

The situation we can observe is due to the convex combination (with coefficient Υ) of the two different effects (contagion and rational), thus the analysis of the fundamental equilibrium $(0,0)$ is similar to the analysis already done: taking $k \in (0,1)$, the condition $|\lambda_2| < 1$ is always satisfied, the solution $(0,0)$ is locally asymptotically stable if $|\Upsilon\chi_1| < 1$. When $|\Upsilon\chi_1| > 1$, the equilibrium point becomes unstable; $\Upsilon\chi_1 = -1$ and $\Upsilon\chi_1 = 1$ are bifurcation values. In particular, when $\Upsilon\chi_1 = 1$ a pitchfork bifurcation occurs and for $\Upsilon\chi_1 > 1$ the equilibrium $(0,0)$ loses stability whereas two new equilibria, having the form $(w, -\lambda w)$, appear. Figure 9 depicts this situation for $\chi_2 = 5$ and for suitable values of the other parameters.

When $\Upsilon\chi_1 = -1$ a flip bifurcation occurs, and a 2-period cycle appears as a solution of the system (4.20).

As pointed before, in the case $|\Upsilon\chi_1| < 1$ the analytical study allows us to conclude that the solution $(0,0)$ is always locally asymptotically stable, but we have no other information concerning the existence of eventual other fixed points; actually, some simulation study

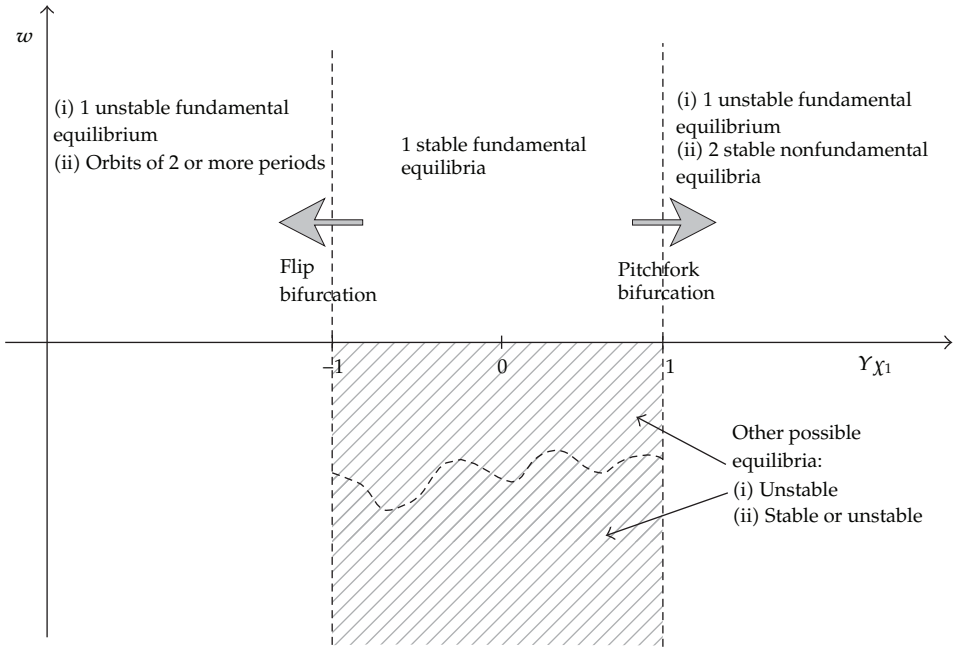


Figure 10: Diagram of stable/unstable equilibria (mixed rational and speculator).

shows that as $-1 < Y\chi_1 < 1$ two other solutions of system (4.20) can appear, both for negative values of the variable w (see Figure 10).

As one can imagine, the combination of rational and irrational agents will generate intermediate market conditions of the two already discussed. Given the partial analytical tractability, we can only extract some qualitative observations from Figure 11, where four representative cases are shown. In Figure 11 we plot the graph of excess demand as a function of the percentage Y of irrational agents and the excess demand level in $t-1$. Those graphs are intersected by the plane $w = w'$, so the intersections represent the fixed points of our system. Parameter χ_2 is set to 15, that is, the intermediate value used in the rational agents subsection.

Starting from the case where $\chi_1 = 4$ (Figure 11(a)), we observe that the combination of highly responsive momentum speculator and rational agents generate a large variety of market dynamics. The fundamental solution is composed by the segments U_0 and S_0 which lay on the line of coordinates $(Y, 0, 0)$ and are separated by a value Y_0. This percentage of Y is low enough to let rational mean reversion contrast the momentum speculation and to turn as a result the fundamental solution from unstable to stable. Under the same setting of Figure 11(a) we also can obtain two stable non fundamental equilibria S_1 and S_2; with respect to S_1, this equilibrium consists of a price steadily higher than $\overline{P}$ and sustained by the speculators. On the contrary S_2 consists of a stable equilibrium where the price remains below $\overline{P}$; despite its geometrical continuity its origin is very different depending on the prevalence of speculators rather than rational investors. Finally it must be said that when there is a prevalence of rational agents we can suppose that a two-period orbit appears with excess demand waving from negative to positive, and prices alternatively switching from above and below the fundamental value.

Turning to Figure 11(b) we mix extremely contrarian speculators ($\chi_1 = -4$) along with rational agents. Similarly as in the case of Figure 11(a) the fundamental solution is stable

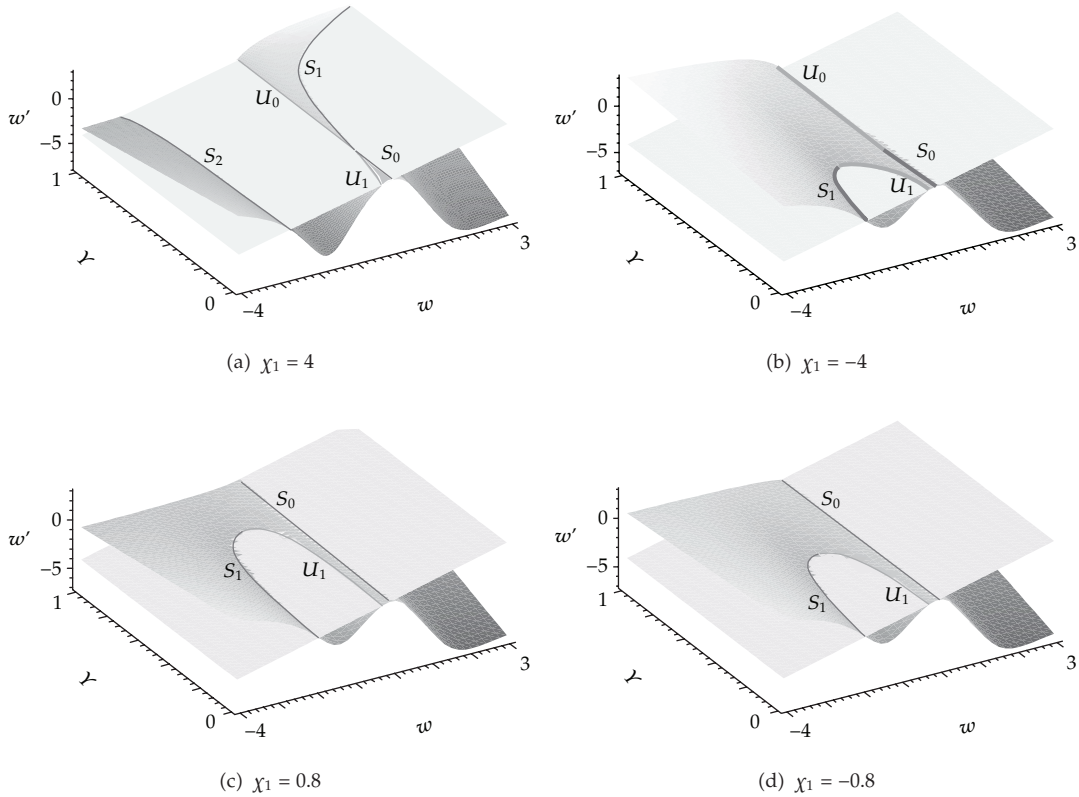


Figure 11: Different behaviours of agents for $\chi_2 = 15$ and many values of χ_1.

only if there is a large presence of rationals (segment S_0). Besides, the large presence of rationals can also determine non fundamental solutions U_1 (unstable) and S_1 (stable), as long as the excess demand goes lower than zero, which we have already discussed. If contrarian speculators prevail, we can suppose that the two-period orbit appears.

In Figure 11(c) we combine moderately momentum speculators ($\chi_1 = 0.8$) with rational agents. As we expect from the separate analysis of the two kinds of agents, the fundamental solution (S_0) is always stable; surprisingly enough, the only source of possible non equilibrium solutions is represented by the large presence of rational agents. As the excess demand lowers below zero, two equilibrium solutions appears, U_1 (unstable) and S_1 (stable). As in the previous cases, S_1 is characterized by a price of the risky asset lower than its fundamental value.

In Figure 11(d) ($\chi_1 = -0.8$) the market conditions are quite similar to those in Figure 11(c), even though speculation is moderately contrarian in this case. The fundamental solution S_0 is stable for any level of Y, the emergence of non fundamental equilibrium S_1 is still represented by a large presence of rational agents, even though less are required than in the case of Figure 11(c). The contrarian speculator, indeed, enhances the emergence of this non fundamental solution.

Figure 12 shows several interesting properties of our model. It is organized in two columns of five plots on the left (Figure 12(a)) and five on the right (Figure 12(b)). In particular, Figure 12(a) refers to a case where rational investors are more critical then speculators: parameter χ_2 ($\chi_2 = 30$) is much larger than χ_1 ($\chi_1 = 0.5$). In this way the decisions

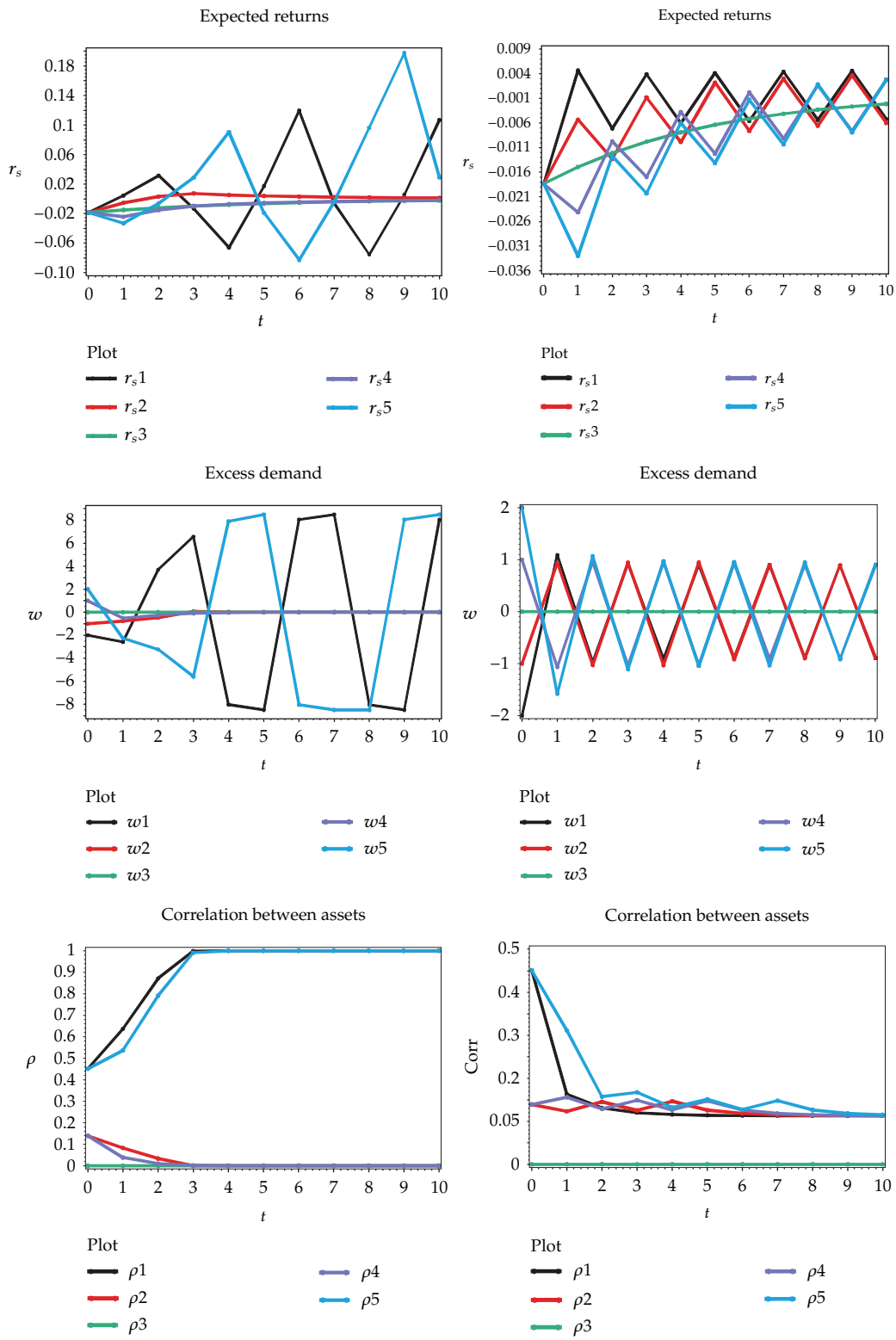
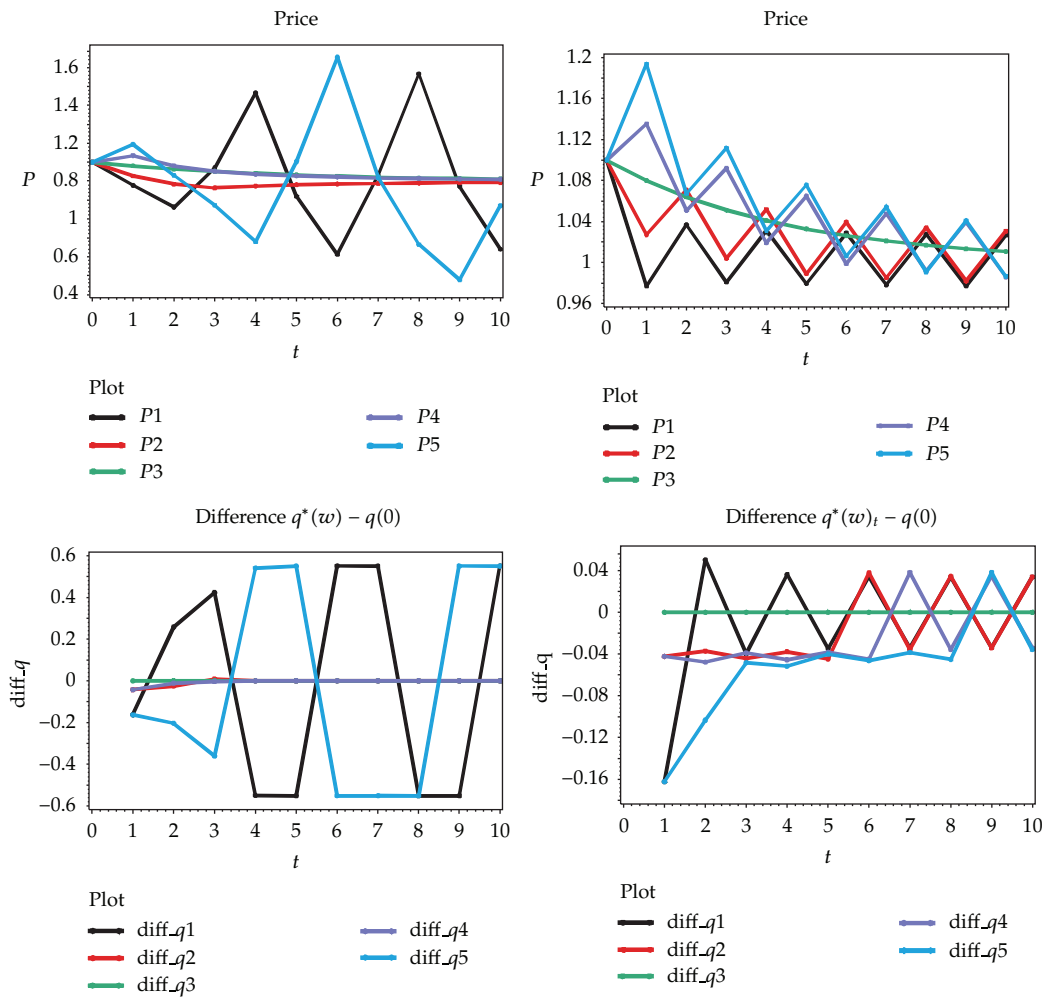


Figure 12: Continued.



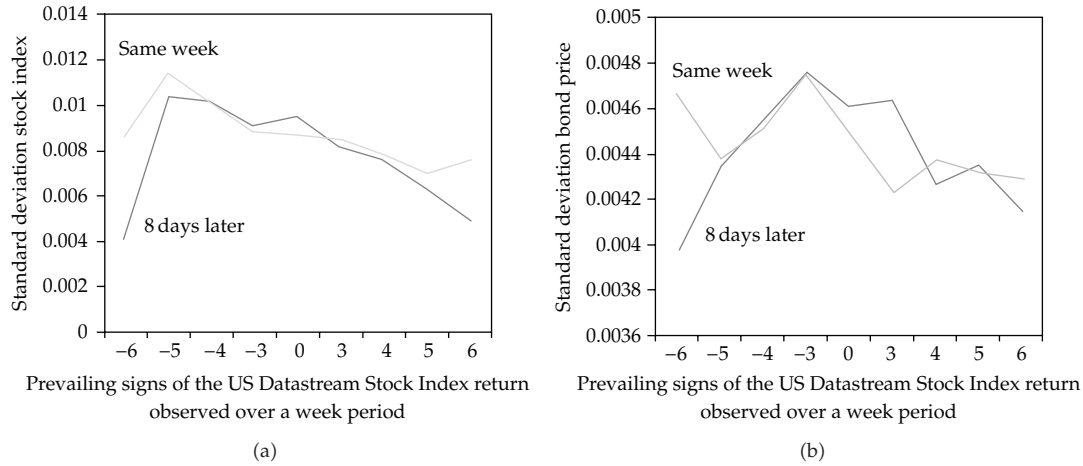
(a) α_1 0.02, α_2 0.06, λ 0.05, χ_1 0.5, χ_2 30, Y 0.5, r_0 −0.01835 P_{fond} 1, k0.20, μ 0.15

(b) α_1 0.02, α_2 0.06, λ 0.05, χ_{-4}1 0.5, χ_2 3, Y 0.5, r_0 −0.01835 P_{fond} 1, k0.20, μ 0.15

Figure 12: Plot of time evolution of some variables (mixed case).

of the rational investors have a larger impact on the market than those of speculators, although the numerical composition of the two kinds of agents is even ($Y = 0.5$). We can think of this case as one where the individuals in the rational group are on average wealthier and dominate the market. Besides, it is worth recalling that a parameter $\chi_1 = 0.5$ corresponds to the case where speculators are moderately "momentum." The five plots of Figure 12(b) represent a case which in some sense is opposed to (a). Indeed here speculators have a stronger impact than rational investors, as it can be deduced observing the sensitivity parameters $\chi_1 = -4$ and $\chi_2 = 3$.

Both panels show the dynamics of five key variables of our model: expected returns (r_s), excess demand (w), correlation between the two assets (ρ), price of the risky asset (P), and the difference between the optimal quantities of the risky asset in the portfolio of rational agents (i.e., $q_t^* - \tilde{q}_t = q_t^*(w_{t-1}, r_s) - \tilde{q}_t(0, r_s)$ which appears in the system (2.7)). The dynamics of each variable is computed starting from five different values of the excess



(a) (b)

Figure 13: The figure shows the dependency of the standard deviation of given returns from a "consensus" variable, which sums up the *prevailing sign* on a period of 6 days observed on the daily return series of the Datastream US Stock Index. We call prevailing sign that most frequently observed on a given number of returns. The cases of −3 and 3 verify when a period of 6 days reduces to 5 due to, for example, a market close. The case of 0 is that of no prevailing sign. (a) shows the weekly standard deviation of the daily returns of the Datastream US Stock Index. (b) shows the weekly standard deviation of the returns calculated on the daily price of the 25 yrs. Treasury bond issued on september 1986.

demand ($w_0 = -2, -1, 0, 1, 2$), so we distinguish five trajectories in each graph. The trajectory originating in $w_0 = 0$ respects only one of the conditions of the fundamental equilibrium. The other (i.e., $r_{s,0} = 0$) is not met, as in both panels the starting price (P_0) is fixed to 1.1, so, given a fundamental value $\overline{P} = 1$, the initial expected return of the risky asset is equal to −0.01835 for all five the trajectories. So none of them starts in an equilibrium.

It is possible to observe in Figure 12(a) that two trajectories are clearly unstable (trajectories number 1 and 5, resp., the darker and the lighter lines) in all the five plots. It is interesting to observe that these two trajectories correspond to the most extreme starting values of the excess demand (i.e., $w_0 = -2, w_0 = 2$, excess demand of Figure 12(a)). What explains such behavior is that excess demand influences price formation (i.e., the contagion effect) and when it takes large values, it moves prices away from their fundamental value. At the same time rational investors use the extreme values of excess demand to fix a correlation estimate close to 1 (see plot of correlation in Figure 12(a)). In this way they form their portfolios asking for percentages of the risky asset somewhat different from the case of small, or null value of excess demand (see plot of $q_t^* - \tilde{q}_t$, Figure 12(a)). However, their choice contributes significantly to next period excess demand, and so they reinforce next period contagion mechanism.

Quite an opposite case takes place in Figure 12(b), where speculators are dominant and are of contrarian type, that is they bet on the reversal of the excess demand. Such antithetic behavior forbids excess demand to deviate significantly from zero, so that also all other relevant variables do not deviate significantly from the fundamental equilibrium. As it is possible to observe in the first plot of Figure 12(b), the four expected returns trajectories corresponding to an initial nonzero value of excess demand oscillate around the one corresponding to the initial value $w_0 = 0$. Such fluctuations are well evident in the graph of excess demand (Figure 12(b)), with w switching symmetrically from −.89 to .89. These alternating values correspond to the strategy of contrarian investors, which keep recorrecting

their previous move. In this way they end up with neutralizing most of their influence on prices, which in the long run revert towards their fundamental value. Such reversion can be observed in the graph of price (Figure 12(b)). Though prices keep oscillating, the center of these trajectories tends indeed to $\overline{P} = 1$. In the end expected returns, prices, and excess demand enter into a 2-period orbit similarly to the case when only (contrarian) speculators populate the market. It is finally worth noticing that contrarian speculators and rational agents tend indeed to neutralize each other. This can be observed comparing $q_t^* - \tilde{q}_t$ and w which (after the first 8 periods) regularly take opposite signs.

4.3. Some Empirical Evidence

We report some empirical evidence which justify part of the assumptions of this work, with particular reference to the form of the variance-covariance estimate used by rational agents in (3.4). We collected the daily series of the of the US Stock Index calculated by Datastream and the daily series of prices of US Treasury bond, 25 years maturity, issued on September 1986, for the period from janurary-1-1993 to March-31-2008.

To the purpose of obtaining a proxy of the excess demand for the stock index, we elaborated a "consensus" variable based on the number of signs observed on the returns on a period of 6 days. In particular we considered the following summations:

$$\begin{aligned} \#_t^+ &= \sum_{i=0}^{5} I_{R^+}(r_{t-i}), \\ \#_t^- &= \sum_{i=0}^{5} I_{R^-}(r_{t-i}), \end{aligned} \tag{4.22}$$

where $I(\cdot)$ is the indicator function and r_t is the return observed on the Datastream US Stock Index. Those summations count, respectively, the number of positive and negative returns occurred in the six days from t back to $t-5$ (including t). Our consensus variable is then defined as

$$c_t = \begin{cases} \#_t^+ & \text{if } \#_t^+ > \#_t^-, \\ -\#_t^- & \text{if } \#_t^+ < \#_t^-, \\ 0 & \text{if } \#_t^+ = \#_t^-. \end{cases} \tag{4.23}$$

The reason why c_t can be considered as a proxy for the excess demand at time t is the it can be argued that when contagion mechanism is acting in a market during a given period, then a sign of the returns should clearly prevail, depending on the kind of sentiment. Moreover, it can be assumed that the stronger a sign is prevailing in a given period, the stronger is the contagion hitting a market.

Figure 13 shows the graph of the weekly standard deviation of the returns of the two series as a function of the consensus variable. More precisely letting c_t the consensus observed in t, the standard deviation has been calculated on two periods of time: the "same week" (i.e., between t and $t-4$) and "8 days later" (i.e., between $t+8$ and t). It can be noticed that the standard deviation of the returns of both the series in usually higher when there is no strong sign prevalence, that is, when $c_t = \{-4, -3.0, 3, 4\}$ (the cases -3 and 3 arise when one day out of six corresponds to a market close). The same statistic clearly lowers when

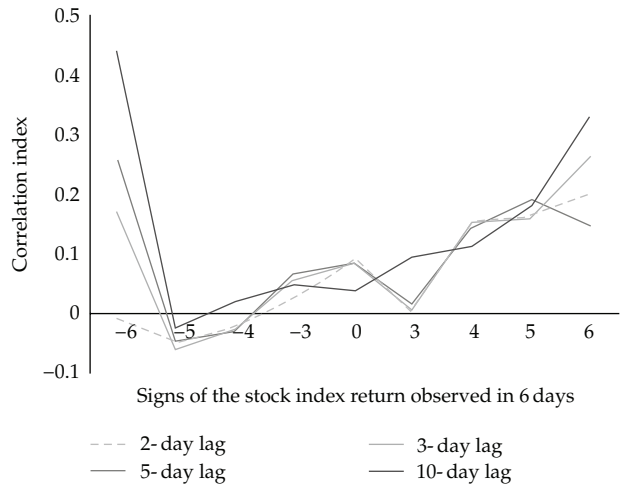


Figure 14: Correlation between the returns of the stock index and the bond price.

the consensus variable takes more extreme values. It is interesting to notice that the decrease in the standard deviation is more marked about a week later, especially for the extremely negative value of c_t (i.e., −6) and with particular reference to the bond series. This seems to show that in the US market agents "remember." In other words the effects of contagion (as long as they are captured by our consensus variable) tend to last longer than the week they first appeared. In the case of the bond series, the effects of negative contagion (on the stock market, which is what is measured by c_t) take about a week to appear. Figure 14 shows the correlation between the returns of the stock index and the bond price, calculated considering five consecutive working days dating back from $t+2$, $t+3$, $t+5$ and $t+10$ for different values of the consensus c_t. The four graphs are very similar, showing increasing positive correlation when c_t is extremely negative or positive. Again, such correlation seems stronger 10 days later than in the week where the consensus was calculated.

Such evidence is strongly consistent with the variance and the correlation functions which we assumed in (3.3). Also the hypothesis that rational investors anticipate the event of the correlation breakdown appears to be justified, given the persistence of contagion effects on the correlation and the standard deviation.

This evidence is also consistent with some of the results which we obtained through our model. The persistence of positive correlation between the stock index and the bond after that a marked pessimistic or optimistic contagion has spread in the market, is also predicted from our model. When the market is characterized by a significant portion of rational agents, our model can generate stable orbits where returns and excess demand swing from periods where they are markedly positive to periods where they are negative, so that predicted correlation keeps close to one for several more periods (infinitely many times).

5. Conclusions

We have developed a model where excess demand directly influence variance and correlation of asset returns. Informed investors follow rational portfolio decisions. Uninformed investors may act as momentum or contrarian speculators, based on their expectation of excess demand. We have found the conditions under which a market entirely driven by speculators converges towards a stable fundamental equilibrium (price reflecting its intrinsic value) or diverges to an orbit with prices oscillating around their fundamental value.

More surprising is the analysis of equilibrium when market is entirely based on rational informed investors. We have shown that when these agents anticipate the effect of excess demand they can generate both stable fundamental equilibrium as well as unstable price dynamics. The explanation of this (somehow) surprising result is due to the anticipation effect of excess demand on rational portfolio decision. When excess demand reaches (for external reasons) unusually high or low values, informed investors anticipate the "escape to unity" of returns correlation. However, following standard portfolio theory they will also alter their portfolio composition favoring extreme solutions (i.e., portfolio composed by all stocks or all bonds). In conclusion, anticipating events can be the origin of market instability, even when rational behavior dominates the market. The implication for the rational investors is to be cautious in their attempt to anticipate possible correlation breakdowns, since they can end up reinforcing the contagion effects more than protecting from it. The analysis of how rational investors should optimally use their information advantage can be the object of further research.

When finally the market is composed of both informed rational investors and uninformed speculators, the effects are mixed and prices can show various dynamics. Through simulation analysis we identify the minimal percentage of rational investors assuring a local fundamental market equilibrium for such a general case.

Appendices

A. Rational Agent Optimization: Solution of the Problem

Here we give the necessary calculations in order to solve the optimization problem examined in Section 3. We can solve the problem in a general case of a variance-covariance matrix $V = \begin{bmatrix} \alpha & \gamma \\ \gamma & \beta \end{bmatrix}$, where coefficients represent the variance (α and β) of two assets and γ their covariance.

It is well known that this matrix is symmetric and definite positive, and that $\mathbf{q}_t^T V \mathbf{q}_t$ represents the portfolio variance:

$$\mathbf{q}_t^T V \mathbf{q}_t = \alpha q_t^2 + 2\gamma q_t (1 - q_t) + \beta (1 - q_t)^2. \tag{A.1}$$

The problem of maximizing the performance indicator (3.1) is related to the solution of financial portfolio diversification depending on the variable q_t, then it can be solved searching the maximum value of the following one-variable function $f(q_t)$:

$$f(q_t) = \frac{q_t r_s + (1 - q_t) r_f}{\alpha q_t^2 + 2\gamma q_t (1 - q_t) + \beta (1 - q_t)^2}. \tag{A.2}$$

Being $f(q_t)$ a differentiable function, its extreme values are obtained by applying standard optimization methods; elementary calculations yield

$$f'(q_t) = \frac{(r_s - r_f)(-\alpha q_t^2 + 2\gamma q_t^2 - \beta q_t^2) - 2r_f(\alpha q_t - 2\gamma q_t + \beta q_t) + (r_s - r_f)\beta - 2r_f(\gamma - \beta)}{\left(\alpha q_t^2 + 2\gamma q_t (1 - q_t) + \beta (1 - q_t)^2\right)^2}. \tag{A.3}$$

When $r_s \neq r_f$, the equation $f'(q_t) = 0$ yields the two solutions $q^*_{1,t}$ and $q^*_{2,t}$ which can be seen as functions of the variable r_s

$$q^*_{1,t}(r_s) = \frac{r_f(\alpha - 2\gamma + \beta) - \sqrt{r_f^2(\alpha - 2\gamma + \beta)^2 - (r_s - r_f)(-\alpha + 2\gamma - \beta)((r_s - r_f)\beta - 2r_f(\gamma - \beta))}}{(r_s - r_f)(-\alpha + 2\gamma - \beta)},$$

$$q^*_{2,t}(r_s) = \frac{r_f(\alpha - 2\gamma + \beta) + \sqrt{r_f^2(\alpha - 2\gamma + \beta)^2 - (r_s - r_f)(-\alpha + 2\gamma - \beta)((r_s - r_f)\beta - 2r_f(\gamma - \beta))}}{(r_s - r_f)(-\alpha + 2\gamma - \beta)}, \tag{A.4}$$

observe that $(-\alpha+2\gamma-\beta) < 0$ under standard conditions. Two different cases may be discussed depending on $r_s \gtrless r_f$:

(i) $r_s < r_f$; in this case it is easy to verify that $q^*_{1,t} < q^*_{2,t}$, the function $f(q_t)$ is monotone decreasing in $(q^*_{1,t}, q^*_{2,t})$ and increasing otherwise, then $q^*_{1,t}$ is a point of maximum for $f(q_t)$;

(ii) $r_s > r_f$; in this case $q^*_{1,t} > q^*_{2,t}$, the function $f(q_t)$ is monotone increasing in $(q^*_{1,t}, q^*_{2,t})$ and decreasing otherwise, then again $q^*_{1,t}$ is a point of maximum for $f(q_t)$.

When $r_s = r_f$ the optimal solution $q^*_{1,t}(r_s)$ is not defined, as it is easily checked in (A.4); however (the authors will provide all the mathematical details should the reader request them):

$$\begin{aligned}
&\lim_{r_s \to r_f} q^*_{1,t}(r_s) \\
&\quad = \lim_{r_s \to r_f} \frac{r_f(\alpha - 2\gamma + \beta) - \sqrt{r_f^2(\alpha - 2\gamma + \beta)^2 - (r_s - r_f)(-\alpha + 2\gamma - \beta)((r_s - r_f)\beta - 2r_f(\gamma - \beta))}}{(r_s - r_f)(-\alpha + 2\gamma - \beta)} \\
&\quad = -\frac{(\gamma - \beta)}{\alpha - 2\gamma + \beta}.
\end{aligned} \tag{A.5}$$

Moreover when $r_s = r_f$ the equation $f'(q_t) = 0$ gives only one solution $q^*_t = -(\gamma - \beta)/(\alpha - 2\gamma + \beta)$, which coincides with the previous limit. So $\lim_{r_s \to r_f} q^*_{1,t} = q^*_t$ and the function $q^*_{1,t}(r_s)$ is continuous in r_f.

As an additional confirmation observe that the function $f(q_t)$ is monotone increasing in $q_t < -(\gamma - \beta)/(\alpha - 2\gamma + \beta)$ and decreasing otherwise, then $q^*_{1,t}(r_s)$ is overall maximum.

We are finally able to define the maximum value of $f(q_t)$ as a real-valued continuous function $q^*_t : R \to R$:

$$q^*_t(r_s) = \begin{cases} \dfrac{r_f(\alpha - 2\gamma + \beta) - \sqrt{r_f^2(\alpha - 2\gamma + \beta)^2 - (r_s - r_f)(-\alpha + 2\gamma - \beta)((r_s - r_f)\beta - 2r_f(\gamma - \beta))}}{(r_s - r_f)(-\alpha + 2\gamma - \beta)} & \text{if } r_s \neq r_f \\ \dfrac{-(\gamma - \beta)}{\alpha - 2\gamma + \beta} & \text{if } r_s = r_f. \end{cases} \tag{A.6}$$

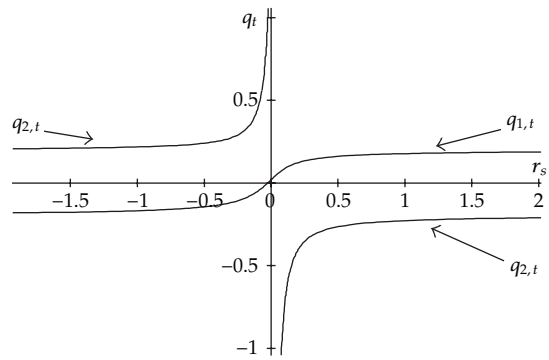


Figure 15: Zero-points curve of the equation $f'(q_t) = 0$.

Let us observe again that, being

$$\lim_{r_s \to \pm\infty} q_t^*(r_s) = \pm \frac{\beta}{\sqrt{\beta(\alpha + \beta - 2\gamma)}}, \tag{A.7}$$

$q_t^*(r_s)$ is a limited function between the values $-\beta/\sqrt{\beta(\alpha+\beta-2\gamma)}$ and $\beta/\sqrt{\beta(\alpha+\beta-2\gamma)}$. Figure 15 shows the graph of $q_{1,t}^*(r_s)$ and $q_{2,t}^*(r_s)$.

A.1. Specification of Variance and Covariance Matrix V_t

Now we apply the previous results to compute q_t^* in the case of the variance-covariance matrix V_{t-1} defined in the Section 3.

First, it is convenient to rewrite $q_t^*(r_s)$ as follows

$$q_t^*(r_s) = \begin{cases} -\dfrac{r_f}{r_s - r_f} - \dfrac{\sqrt{r_f^2(\alpha - 2\gamma + \beta)^2 - (r_s - r_f)(-\alpha + 2\gamma - \beta)((r_s - r_f)\beta - 2r_f(\gamma - \beta))}}{(r_s - r_f)(-\alpha + 2\gamma - \beta)} & \text{if } r_s \neq r_f, \\ \dfrac{-(\gamma - \beta)}{\alpha - 2\gamma + \beta} & \text{if } r_s = r_f. \end{cases} \tag{A.8}$$

Recall that the expression of the matrix V_{t-1} is

$$V_{t-1} = \begin{bmatrix} \alpha_1^2 e^{-2\mu\tau w_{t-1}^2} & \alpha_1\alpha_2 e^{-2\mu\tau w_{t-1}^2}\left(-e^{-\mu\tau w_{t-1}^2} + 1\right) \\ \alpha_1\alpha_2 e^{-2\mu\tau w_{t-1}^2}\left(-e^{-\mu\tau w_{t-1}^2} + 1\right) & \alpha_2^2 e^{-2\mu\tau w_{t-1}^2} \end{bmatrix}, \tag{A.9}$$

which can be decomposed in the following way to focus on the correlation between the two assets:

$$V_{t-1} = \begin{bmatrix} \alpha_1 e^{-\mu w_{t-1}^2} & 0 \\ 0 & \alpha_2 e^{-\mu w_{t-1}^2} \end{bmatrix} \begin{bmatrix} 1 & 1 - e^{-\mu w_{t-1}^2} \\ 1 - e^{-\mu w_{t-1}^2} & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 e^{-\mu w_{t-1}^2} & 0 \\ 0 & \alpha_2 e^{-\mu w_{t-1}^2} \end{bmatrix}, \tag{A.10}$$

when $\mu = 0$ we revert to the case of two uncorrelated assets with fixed variance α_1^2 and α_2^2, which is a standard case of Markowitz analysis:

$$V_{t-1} = \begin{bmatrix} \alpha_1^2 & 0 \\ 0 & \alpha_2^2 \end{bmatrix}. \tag{A.11}$$

Computing

$$\begin{aligned} (-\alpha + 2\gamma - \beta) &= e^{-2\mu w_{t-1}^2} \left[-\alpha_1^2 + 2\alpha_1\alpha_2 \left(-e^{-\mu w_{t-1}^2} + 1 \right) - \alpha_2^2 \right], \\ \beta - \gamma &= e^{-2\mu w_{t-1}^2} \left[\alpha_2^2 - \alpha_2\alpha_1 \left(-e^{-\mu w_{t-1}^2} + 1 \right) \right], \end{aligned} \tag{A.12}$$

we have, for $r_s \neq r_f$

$$q_t^*(w_{t-1}, r_s) = -\frac{r_f}{r_s - r_f} + \frac{\sqrt{\left(\alpha_1^2 + \alpha_2^2 + 2\alpha_1\alpha_2 \left(e^{-\mu w_{t-1}^2} - 1 \right) \right) \left(\left(r_s\alpha_2 - r_f\alpha_1 \right)^2 + 2\alpha_1\alpha_2 r_f r_s e^{-\mu w_{t-1}^2} \right)}}{\left(r_s - r_f \right) \left(\alpha_1^2 + \alpha_2^2 - 2\alpha_1\alpha_2 \left(-e^{-\mu w_{t-1}^2} + 1 \right) \right)}, \tag{A.13}$$

and, for $r_s = r_f$

$$q_t^*(w_{t-1}, r_s) = \frac{-(\gamma - \beta)}{\alpha - 2\gamma + \beta} = \frac{\alpha_1\alpha_2 \left(-e^{-\mu w_{t-1}^2} + 1 \right) - \alpha_2^2}{2\alpha_1\alpha_2 \left(-e^{-\mu w_{t-1}^2} + 1 \right) - \alpha_1^2 - \alpha_2^2}, \tag{A.14}$$

which are reported in (3.5).

When in particular $\mu = 0$ we obtain

$$q_t^*(w_{t-1}, r_s) = \begin{cases} -\dfrac{r_f}{r_s - r_f} + \dfrac{\sqrt{\left(\alpha_1^2 + \alpha_2^2 \right) \left(r_s^2\alpha_2^2 + r_f^2\alpha_1^2 \right)}}{\left(r_s - r_f \right) \left(\alpha_1^2 + \alpha_2^2 \right)} & \text{if } r_s \neq r_f \\ \dfrac{\alpha_2^2}{\alpha_1^2 + \alpha_2^2} & \text{if } r_s = r_f. \end{cases} \tag{A.15}$$

B. Jacobian Matrix: Computation of the Partial Derivatives

Here we report all the detailed calculations in order to construct the Jacobian matrix used in the local stability analysis.

In the general case (presence of both informed and uninformed agents) the market dynamics are described by the system:

$$w' = \Upsilon\chi_1 \frac{w}{1+|w|} + (1-\Upsilon)\chi_2(q^* - \tilde{q}),$$
$$r_s' = \ln\left(\frac{\exp r_s + k - 1}{\exp(r_s + \lambda w)} - (k-1)\right), \tag{B.1}$$

where the expression of $q^* = q^*(w, r_s)$ is calculated as in (3.5) and $\tilde{q}$ as in (3.6). Starting from the first equation, the partial derivatives of w' with respect to variables w and r_s, evaluated at point (w, r_s), are

(i)

$$\frac{\partial w'(w, r_s)}{\partial w} = \Upsilon\chi_1 \frac{1}{(1+|w|)^2} + (1-\Upsilon)\chi_2\left(\frac{\partial q*}{\partial w} - \frac{\partial \tilde{q}}{\partial w}\right), \tag{B.2}$$

observing that $\tilde{q}$ does not depends on w, $\partial\tilde{q}/\partial w = 0$, hence:

(ii)

$$\begin{aligned}\frac{\partial w'(w, r_s)}{\partial w} &= \Upsilon\chi_1 \frac{1}{(1+|w|)^2} \\ &+ (1-\Upsilon)\chi_2 \frac{w\mu e^{-w^2\mu}\left(2\alpha_1\alpha_2 r_f\left(2\alpha_1\alpha_2 - \left(\alpha_1^2+\alpha_2^2\right)\right) + \left(r_s\alpha_2 - r_f\alpha_1\right)^2\left(\alpha_1^2+\alpha_2^2\right)\right)}{\left(r_s - r_f\right)\left(\alpha_1^2+\alpha_2^2+2\alpha_1\alpha_2\left(e^{-\mu w^2}-1\right)\right)\sqrt{\left(\alpha_1^2+\alpha_2^2+2\alpha_1\alpha_2\left(e^{-\mu w^2}-1\right)\right)\left(\left(r_s\alpha_2 - r_f\alpha_1\right)^2 + 2\alpha_1\alpha_2 r_f r_s e^{-\mu w^2}\right)}},\end{aligned} \tag{B.3}$$

(iii)

$$\begin{aligned}\frac{\partial w'(w, r_s)}{\partial r_s} &= 0 + (1-\Upsilon)\chi_2\left(\frac{2\alpha_1\alpha_2 r_f e^{-\mu w^2} + 2\alpha_2\left(r_s\alpha_2 - r_f\alpha_1\right)}{2\left(r_s - r_f\right)\sqrt{\left(\alpha_1^2+\alpha_2^2+2\alpha_1\alpha_2\left(e^{-\mu w^2}-1\right)\right)\left(2\alpha_1\alpha_2 r_s r_f e^{-\mu w^2} + \left(r_s\alpha_2 - r_f\alpha_1\right)^2\right)}}\right. \\ &\left. - \frac{\sqrt{2\alpha_1\alpha r_s r_f e^{-\mu w^2} + \left(r_s\alpha_2 - r_f\alpha_1\right)^2}}{\left(\sqrt{\left(\alpha_1^2+\alpha_2^2+2\alpha_1\alpha_2\left(e^{-\mu w^2}-1\right)\right)}\left(r_s - r_f\right)^2\right)} + \frac{r_f^2\alpha_1^2 + r_s r_f \alpha_2^2}{\left(\left(r_s - r_f\right)^2\sqrt{\alpha_1^2+\alpha_2^2}\sqrt{r_f^2\alpha_1^2 + r_s^2\alpha_2^2}\right)}\right).\end{aligned} \tag{B.4}$$

Looking at the second equation of the system, the partial derivatives of r_s' with respect to variables w and r_s, evaluated at point (w, r_s), are

(i)

$$\frac{\partial r_s'}{\partial w} = \frac{k\lambda - \lambda + \lambda \exp r_s}{k\exp(r_s + w\lambda) - \exp r_s - \exp(r_s + w\lambda) - k + 1}, \tag{B.5}$$

(ii)

$$\frac{\partial r_s'}{\partial r_s} = \frac{k - 1}{k\exp(r_s + w\lambda) - \exp r_s - \exp(r_s + w\lambda) - k + 1}. \tag{B.6}$$

Evaluating the four derivatives at the equilibrium point $(0, 0)$ we have:

(i)

$$\frac{\partial w'(w, r_s)}{\partial w} = \Upsilon\chi_1 + (1 - \Upsilon)\chi_2\left(\frac{0}{-r_f(\alpha_1^2 + \alpha_2^2)\sqrt{(\alpha_1^2 + \alpha_2^2)r_f^2\alpha_1^2}}\right) = \Upsilon\chi_1, \tag{B.7}$$

(ii)

$$\begin{aligned}
&\frac{\partial w'(w, r_s)}{\partial r_s} \\
&= (1 - \Upsilon)\chi_2\left(\frac{2\alpha_1\alpha_2 r_f - 2\alpha_2\alpha_1 r_f}{2(-r_f)\sqrt{(\alpha_1^2 + \alpha_2^2)\left((-r_f\alpha_1)^2\right)}} - \frac{\sqrt{(-r_f\alpha_1)^2}}{\sqrt{(\alpha_1^2 + \alpha_2^2)}(-r_f)^2} + \frac{r_f^2\alpha_1^2}{(-r_f)^2\sqrt{\alpha_1^2 + \alpha_2^2}\sqrt{r_f^2\alpha_1^2}}\right) \\
&= (1 - \Upsilon)\chi_2\left(0 - \frac{\sqrt{r_f^2\alpha_1^2}}{r_f^2\sqrt{\alpha_1^2 + \alpha_2^2}} + \frac{\sqrt{r_f^2\alpha_1^2}}{r_f^2\sqrt{\alpha_1^2 + \alpha_2^2}}\right) = 0,
\end{aligned} \tag{B.8}$$

(iii)

$$\frac{\partial r_s'(0, 0)}{\partial w} = -k\lambda, \tag{B.9}$$

(iv)

$$\frac{\partial r_s'(0, 0)}{\partial r_s} = 1 - k. \tag{B.10}$$

So that the Jacobian matrix at point $(0,0)$ has the following form:

$$J(0,0) = \begin{bmatrix} \Upsilon\chi_1 & 0 \\ -k\lambda & 1-k \end{bmatrix}. \tag{B.11}$$

Observe that Jacobian matrices of the extreme cases, in which all agents are speculators or rational, can be easily obtained by the previous calculations simply setting $\Upsilon = 1$ and $\Upsilon = 0$ respectively.

C. Contagion Effect: Price in the Case of Contrarian Speculation

In a market only composed by speculators and in the case of $\chi_1 < -1$ (strongly contrarian attitude of investors) a two-period cycle appears in the phase plane (w, r_s). From an economic point of view in this case, the positive excess demand in period t ($w_3^* = -\chi_1 - 1$) turns into negative excess demand in period $t+1$ ($w_4^* = \chi_1 + 1$), and correspondingly market prices oscillate between two values $P_3 < \overline{P} < P_4$.

We can determine the two-period orbit and the exact values of P_3 and P_4. After two periods the system (4.5) becomes

$$\begin{aligned} w_{t+1} &= \chi_1^2 \frac{w_{t-1}}{(1+|w_{t-1}|)\left(1+\left|\chi_1(w_{t-1}/(1+|w_{t-1}|))\right|\right)} \\ r_{s,t+1} &= \ln\left(\frac{(\exp r_{s,t-1}+k-1)/\exp(r_{s,t-1}+\lambda w_{t-1})}{\left((\exp r_{s,t-1}+k-1)/(\exp(r_{s,t-1}+\lambda w_{t-1}))-(k-1)\right)\exp\lambda\chi_1(w_{t-1}/(1+|w_{t-1}|))} \right. \\ &\qquad \left. -(k-1)\right) \end{aligned} \tag{C.1}$$

recalling that $P_t = [P_{t-1} + k(\overline{P} - P_t)]\exp\lambda w_{t-1}$, the corresponding price equation is

$$P_{t+1} = \left\{(1-k)\left[k\overline{P} + (1-k)P_{t-1}\right]\exp\lambda w_{t-1} + k\overline{P}\right\}\exp\lambda\chi_1\left(\frac{w_{t-1}}{(1+|w_{t-1}|)}\right), \tag{C.2}$$

where the expression of P_{t+1} has been obtained by a double iteration.

The first equation of the system does not depends on r_s so, solving in a fixed point we have two solutions $w_3^* = -\chi_1 - 1$ and $w_4^* = \chi_2 + 1$. The corresponding price equation is

$$P = \left\{(1-k)\left[k\overline{P} + (1-k)P\right]\exp\lambda w + k\overline{P}\right\}\exp\lambda\chi_1\frac{w}{(1+|w|)}, \tag{C.3}$$

giving two different solutions for P depending on w_3^* and w_4^*.

When $w = w_3^* = -\chi_1 - 1$

$$P = \left\{(1-k)\left[k\overline{P} + (1-k)P\right]\exp\lambda(-\chi_1 - 1) + k\overline{P}\right\}\exp\lambda\chi_1\frac{-\chi_1 - 1}{1+|-\chi_1 - 1|}, \tag{C.4}$$

which simplifies as

$$P\left(1-(1-k)^2\right) = k\overline{P}(1-k) + k\overline{P}\exp\lambda(\chi_1+1), \tag{C.5}$$

giving the final price $P = ((e^{\lambda(\chi_1+1)} + 1 - k)/(2-k))\overline{P} \equiv P_3$, corresponding to the expected return (see (2.1)):

$$r_s^* = \ln\left(\frac{1+(1-k)\left(e^{\lambda(\chi_1+1)}\right)}{e^{\lambda(\chi_1+1)}+1-k}\right). \tag{C.6}$$

P_3 is always smaller than $\overline{P}$. Indeed $P_3 < \overline{P}$ if $(e^{\lambda(\chi_1+1)} + (1-k))/(2-k) < 1$, which simplifies to $\lambda(\chi_1+1) < 0$. This condition is always true under the hypothesis at hand $\chi_1 < -1$.

When $w = w_4^* = \chi_1 + 1$

$$P = \left\{(1-k)\left[k\overline{P} + (1-k)P\right]\exp\lambda(\chi_1+1) + k\overline{P}\right\}\exp\lambda\chi_1\frac{\chi_1+1}{1+|\chi_1+1|}, \tag{C.7}$$

which simplifies to

$$P\left(1-(1-k)^2\right) = (1-k)k\overline{P} + k\overline{P}\exp-\lambda(\chi_1+1). \tag{C.8}$$

We obtain the solution $P = ((e^{-\lambda(\chi_1+1)} + (1-k))/(2-k))\overline{P} \equiv P_4$, corresponding to an expected return:

$$r_s^* = \ln\left(\frac{1+(1-k)e^{-\lambda(\chi_1+1)}}{e^{-\lambda(\chi_1+1)}+(1-k)}\right). \tag{C.9}$$

We can observe that P_4 is always greater than $\overline{P}$. Indeed $P_4 > \overline{P}$ if $(e^{-\lambda(\chi_1+1)} + 1 - k)/(2-k) > 1$; this inequality is equivalent to $\lambda(\chi_1+1) < 0$ which is always satisfied under the hypothesis $\chi_1 < -1$.

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